Chapter 10

Polar Coordinates; Vectors

10.5 The Dot Product

1. $\mathbf{v} = \mathbf{i} - \mathbf{j}, \quad \mathbf{w} = \mathbf{i} + \mathbf{j}$
   (a) $\mathbf{v} \cdot \mathbf{w} = 1(1) + (-1)(1) = 1 - 1 = 0$
   (b) $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{0}{\sqrt{1^2 + (-1)^2}\sqrt{1^2 + 1^2}} = \dfrac{0}{\sqrt{2}\sqrt{2}} = \dfrac{0}{2} = 0 \quad \to \quad \theta = 90^\circ$
   (c) The vectors are orthogonal.

2. $\mathbf{v} = \mathbf{i} + \mathbf{j}, \quad \mathbf{w} = -\mathbf{i} + \mathbf{j}$
   (a) $\mathbf{v} \cdot \mathbf{w} = 1(-1) + 1(1) = -1 + 1 = 0$
   (b) $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{0}{\sqrt{1^2 + 1^2}\sqrt{(-1)^2 + 1^2}} = \dfrac{0}{\sqrt{2}\sqrt{2}} = \dfrac{0}{2} = 0 \quad \to \quad \theta = 90^\circ$
   (c) The vectors are orthogonal.

3. $\mathbf{v} = 2\mathbf{i} + \mathbf{j}, \quad \mathbf{w} = \mathbf{i} + 2\mathbf{j}$
   (a) $\mathbf{v} \cdot \mathbf{w} = 2(1) + 1(2) = 2 + 2 = 4$
   (b) $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{4}{\sqrt{2^2 + 1^2}\sqrt{1^2 + 2^2}} = \dfrac{4}{\sqrt{5}\sqrt{5}} = \dfrac{4}{5} = 0.8 \quad \to \quad \theta \approx 36.87^\circ$
   (c) The vectors are neither parallel nor orthogonal.

4. $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j}, \quad \mathbf{w} = \mathbf{i} + 2\mathbf{j}$
   (a) $\mathbf{v} \cdot \mathbf{w} = 2(1) + 2(2) = 2 + 4 = 6$
   (b) $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{6}{\sqrt{2^2 + 2^2}\sqrt{1^2 + 2^2}} = \dfrac{6}{2\sqrt{2}\sqrt{5}} = \dfrac{3}{\sqrt{10}} = \dfrac{3\sqrt{10}}{10}$
   $\theta \approx 18.43^\circ$
   (c) The vectors are neither parallel nor orthogonal.

5. $\mathbf{v} = \sqrt{3}\,\mathbf{i} - \mathbf{j}, \quad \mathbf{w} = \mathbf{i} + \mathbf{j}$
   (a) $\mathbf{v} \cdot \mathbf{w} = \sqrt{3}(1) + (-1)(1) = \sqrt{3} - 1$
   (b) $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{\sqrt{3} - 1}{\sqrt{\left(\sqrt{3}\right)^2 + (-1)^2}\sqrt{1^2 + 1^2}} = \dfrac{\sqrt{3} - 1}{\sqrt{4}\sqrt{2}} = \dfrac{\sqrt{3} - 1}{2\sqrt{2}} = \dfrac{\sqrt{6} - \sqrt{2}}{4}$
   $\theta \approx 75^\circ$
   (c) The vectors are neither parallel nor orthogonal.

6. $\mathbf{v} = \mathbf{i} + \sqrt{3}\mathbf{j}, \quad \mathbf{w} = \mathbf{i} - \mathbf{j}$

(a) $\mathbf{v} \cdot \mathbf{w} = 1(1) + \sqrt{3}(-1) = 1 - \sqrt{3}$

(b) $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{1-\sqrt{3}}{\sqrt{1^2 + \left(\sqrt{3}\right)^2}\sqrt{1^2 + (-1)^2}} = \dfrac{1-\sqrt{3}}{\sqrt{4}\sqrt{2}} = \dfrac{1-\sqrt{3}}{2\sqrt{2}} = \dfrac{\sqrt{2}-\sqrt{6}}{4}$

$\theta \approx 105°$

(c) The vectors are neither parallel nor orthogonal.

7. $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}, \quad \mathbf{w} = 4\mathbf{i} + 3\mathbf{j}$

(a) $\mathbf{v} \cdot \mathbf{w} = 3(4) + 4(3) = 12 + 12 = 24$

(b) $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{24}{\sqrt{3^2 + 4^2}\sqrt{4^2 + 3^2}} = \dfrac{24}{\sqrt{25}\sqrt{25}} = \dfrac{24}{25} = 0.96 \rightarrow \theta \approx 16.26°$

(c) The vectors are neither parallel nor orthogonal.

8. $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}, \quad \mathbf{w} = 4\mathbf{i} - 3\mathbf{j}$

(a) $\mathbf{v} \cdot \mathbf{w} = 3(4) + (-4)(-3) = 12 + 12 = 24$

(b) $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{24}{\sqrt{3^2 + (-4)^2}\sqrt{4^2 + (-3)^2}} = \dfrac{24}{\sqrt{25}\sqrt{25}} = \dfrac{24}{25} = 0.96$

$\theta \approx 16.26°$

(c) The vectors are neither parallel nor orthogonal.

9. $\mathbf{v} = 4\mathbf{i}, \quad \mathbf{w} = \mathbf{j}$

(a) $\mathbf{v} \cdot \mathbf{w} = 4(0) + 0(1) = 0 + 0 = 0$

(b) $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{0}{\sqrt{4^2 + 0^2}\sqrt{0^2 + 1^2}} = \dfrac{0}{4 \ 1} = \dfrac{0}{4} = 0 \quad \rightarrow \quad \theta = 90°$

(c) The vectors are orthogonal.

10. $\mathbf{v} = \mathbf{i}, \quad \mathbf{w} = -3\mathbf{j}$

(a) $\mathbf{v} \cdot \mathbf{w} = 1(0) + 0(-3) = 0 + 0 = 0$

(b) $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{0}{\sqrt{1^2 + 0^2}\sqrt{0^2 + (-3)^2}} = \dfrac{0}{1 \ 3} = \dfrac{0}{3} = 0 \quad \rightarrow \quad \theta = 90°$

(c) The vectors are orthogonal.

11. $\mathbf{v} = \mathbf{i} - a\mathbf{j}, \quad \mathbf{w} = 2\mathbf{i} + 3\mathbf{j}$

Two vectors are orthogonal if the dot product is zero. Solve for a:

$\mathbf{v} \cdot \mathbf{w} = 1(2) + (-a)(3) = 2 - 3a$

$2 - 3a = 0 \rightarrow 3a = 2 \rightarrow a = \dfrac{2}{3}$

12. $\mathbf{v} = \mathbf{i} + \mathbf{j}, \quad \mathbf{w} = \mathbf{i} + b\mathbf{j}$

Two vectors are orthogonal if the dot product is zero. Solve for b:

$\mathbf{v} \cdot \mathbf{w} = 1(1) + 1(b) = 1 + b$

$1 + b = 0 \rightarrow b = -1$

13. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}, \quad \mathbf{w} = \mathbf{i} - \mathbf{j}$

$$\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{2(1)+(-3)(-1)}{\left(\sqrt{1^2+(-1)^2}\right)^2}(\mathbf{i}-\mathbf{j}) = \frac{5}{2}(\mathbf{i}-\mathbf{j}) = \frac{5}{2}\mathbf{i} - \frac{5}{2}\mathbf{j}$$

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1 = (2\mathbf{i}-3\mathbf{j}) - \left(\frac{5}{2}\mathbf{i} - \frac{5}{2}\mathbf{j}\right) = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$$

14. $\mathbf{v} = -3\mathbf{i} + 2\mathbf{j}, \quad \mathbf{w} = 2\mathbf{i} + \mathbf{j}$

$$\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{-3(2)+2(1)}{\left(\sqrt{2^2+1^2}\right)^2}(2\mathbf{i}+\mathbf{j}) = -\frac{4}{5}(2\mathbf{i}+\mathbf{j}) = -\frac{8}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1 = (-3\mathbf{i}+2\mathbf{j}) - \left(-\frac{8}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = -\frac{7}{5}\mathbf{i} + \frac{14}{5}\mathbf{j}$$

15. $\mathbf{v} = \mathbf{i} - \mathbf{j}, \quad \mathbf{w} = \mathbf{i} + 2\mathbf{j}$

$$\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{1(1)+(-1)(2)}{\left(\sqrt{1^2+2^2}\right)^2}(\mathbf{i}+2\mathbf{j}) = -\frac{1}{5}(\mathbf{i}+2\mathbf{j}) = -\frac{1}{5}\mathbf{i} - \frac{2}{5}\mathbf{j}$$

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1 = (\mathbf{i}-\mathbf{j}) - \left(-\frac{1}{5}\mathbf{i} - \frac{2}{5}\mathbf{j}\right) = \frac{6}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$$

16. $\mathbf{v} = 2\mathbf{i} - \mathbf{j}, \quad \mathbf{w} = \mathbf{i} - 2\mathbf{j}$

$$\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{2(1)+(-1)(-2)}{\left(\sqrt{1^2+(-2)^2}\right)^2}(\mathbf{i}-2\mathbf{j}) = \frac{4}{5}(\mathbf{i}-2\mathbf{j}) = \frac{4}{5}\mathbf{i} - \frac{8}{5}\mathbf{j}$$

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1 = (2\mathbf{i}-\mathbf{j}) - \left(\frac{4}{5}\mathbf{i} - \frac{8}{5}\mathbf{j}\right) = \frac{6}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$$

17. $\mathbf{v} = 3\mathbf{i} + \mathbf{j}, \quad \mathbf{w} = -2\mathbf{i} - \mathbf{j}$

$$\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{3(-2)+1(-1)}{\left(\sqrt{(-2)^2+(-1)^2}\right)^2}(-2\mathbf{i}-\mathbf{j}) = -\frac{7}{5}(-2\mathbf{i}-\mathbf{j}) = \frac{14}{5}\mathbf{i} + \frac{7}{5}\mathbf{j}$$

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1 = (3\mathbf{i}+\mathbf{j}) - \left(\frac{14}{5}\mathbf{i} + \frac{7}{5}\mathbf{j}\right) = \frac{1}{5}\mathbf{i} - \frac{2}{5}\mathbf{j}$$

18. $\mathbf{v} = \mathbf{i} - 3\mathbf{j}, \quad \mathbf{w} = 4\mathbf{i} - \mathbf{j}$

$$\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{1(4)+(-3)(-1)}{\left(\sqrt{4^2+(-1)^2}\right)^2}(4\mathbf{i}-\mathbf{j}) = \frac{7}{17}(4\mathbf{i}-\mathbf{j}) = \frac{28}{17}\mathbf{i} - \frac{7}{17}\mathbf{j}$$

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1 = (\mathbf{i}-3\mathbf{j}) - \left(\frac{28}{17}\mathbf{i} - \frac{7}{17}\mathbf{j}\right) = -\frac{11}{17}\mathbf{i} - \frac{44}{17}\mathbf{j}$$

19. Let $\mathbf{v_a}$ = the velocity of the plane in still air.

$\mathbf{v_w}$ = the velocity of the wind.

$\mathbf{v_g}$ = the velocity of the plane relative to the ground.

$$\mathbf{v_g} = \mathbf{v_a} + \mathbf{v_w}$$

$$\mathbf{v_a} = 550\left(\cos(225^\circ)\mathbf{i} + \sin(225^\circ)\mathbf{j}\right) = 550\left(-\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}\right) = -275\sqrt{2}\,\mathbf{i} - 275\sqrt{2}\,\mathbf{j}$$

$$\mathbf{v_w} = 80\mathbf{i}$$

$$\mathbf{v_g} = \mathbf{v_a} + \mathbf{v_w} = -275\sqrt{2}\,\mathbf{i} - 275\sqrt{2}\,\mathbf{j} + 80\mathbf{i} = \left(80 - 275\sqrt{2}\right)\mathbf{i} - 275\sqrt{2}\,\mathbf{j}$$

The speed of the plane relative to the ground is:

$$\|\mathbf{v_g}\| = \sqrt{\left(80 - 275\sqrt{2}\right)^2 + \left(-275\sqrt{2}\right)^2} = \sqrt{6400 - 44000\sqrt{2} + 151250 + 151250}$$

$$= \sqrt{246674.6} \approx 496.7 \text{ miles per hour}$$

To find the direction, find the angle between $\mathbf{v_g}$ and a convenient vector such as due south, $-\mathbf{j}$.

$$\cos\theta = \frac{\mathbf{v}_g \cdot -\mathbf{j}}{\|\mathbf{v}_g\|\|-\mathbf{j}\|} = \frac{\left(80 - 275\sqrt{2}\right)0 + \left(-275\sqrt{2}\right)(-1)}{496.7\sqrt{0^2 + (-1)^2}} = \frac{275\sqrt{2}}{496.7} \approx 0.7829$$

$$\theta \approx 38.5^\circ$$

The plane is traveling with a ground speed of about 496.7 miles per hour in a direction of 38.5° west of south.

20. Let $\mathbf{v_a}$ = the velocity of the plane in still air.

$\mathbf{v_w}$ = the velocity of the wind.

$\mathbf{v_g}$ = the velocity of the plane relative to the ground.

$$\mathbf{v_g} = \mathbf{v_a} + \mathbf{v_w}$$

$$\mathbf{v_a} = 250\left(\cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}\right)$$

$$\mathbf{v_w} = 40\left(\frac{\mathbf{i} - \mathbf{j}}{\|\mathbf{i} - \mathbf{j}\|}\right) = 40\left(\frac{\mathbf{i} - \mathbf{j}}{\sqrt{1+1}}\right) = \frac{40}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$$

$$\mathbf{v_g} = \mathbf{v_a} + \mathbf{v_w} = 250\cos\alpha\,\mathbf{i} + 250\sin\alpha\,\mathbf{j} + 20\sqrt{2}\,\mathbf{i} - 20\sqrt{2}\,\mathbf{j} = a\,\mathbf{i}$$

Examining the $\mathbf{j}$ components:

$$250\sin\alpha - 20\sqrt{2} = 0 \rightarrow 250\sin\alpha = 20\sqrt{2}$$

$$\sin\alpha = \frac{20\sqrt{2}}{250} \approx 0.1131 \rightarrow \alpha \approx 6.5^\circ$$

The heading of the plane should be N83.5˚E.

Examining the $\mathbf{i}$ components:

$$250\cos 6.5^\circ\,\mathbf{i} + 20\sqrt{2}\,\mathbf{i} = a\,\mathbf{i}$$

$$276.7 = a$$

The speed of the plane relative to the ground is 276.7 miles per hour.

21. Let the positive x-axis point downstream, so that the velocity of the current is $\mathbf{v_c} = 3\mathbf{i}$.

Let $\mathbf{v_w}$ = the velocity of the boat in the water.

Let $\mathbf{v_g}$ = the velocity of the boat relative to the land.

Then $\mathbf{v_g} = \mathbf{v_w} + \mathbf{v_c}$

The speed of the boat is $\|\mathbf{v_w}\| = 20$; we need to find the direction.

Let $\mathbf{v_w} = a\mathbf{i} + b\,\mathbf{j}$ so $\|\mathbf{v_w}\| = \sqrt{a^2+b^2} = 20 \quad \rightarrow \quad a^2 + b^2 = 400$.

Let $\mathbf{v_g} = k\,\mathbf{j}$.

Since $\mathbf{v_g} = \mathbf{v_w} + \mathbf{v_c}$, $\quad k\,\mathbf{j} = a\mathbf{i} + b\,\mathbf{j} + 3\mathbf{i} \quad \rightarrow \quad k\,\mathbf{j} = (a+3)\mathbf{i} + b\,\mathbf{j}$

$a + 3 = 0$ and $k = b \quad \rightarrow \quad a = -3$

$a^2 + b^2 = 400 \quad \rightarrow \quad 9 + b^2 = 400 \quad \rightarrow \quad b^2 = 391 \quad \rightarrow \quad k = b \approx 19.77$

$\mathbf{v_w} = -3\mathbf{i} + 19.77\,\mathbf{j}$ and $\mathbf{v_g} = 19.77\,\mathbf{j}$

Find the angle between $\mathbf{v}_w$ and $\mathbf{j}$:

$$\cos\theta = \frac{\mathbf{v}_w \cdot \mathbf{j}}{\|\mathbf{v}_w\|\|\mathbf{j}\|} = \frac{-3\ \ 0 + 19.77(1)}{20\sqrt{0^2+1^2}} = \frac{19.77}{20} \approx 0.9885$$

$\theta \approx 8.7^{o}$

The heading of the boat needs to be 8.7° upstream.

The velocity of the boat directly across the river is 19.77 kilometers per hour. The time to cross the river is: $t = \frac{0.5}{19.77} \approx 0.025$ hours or $t \approx 1.5$ minutes.

22. Let the positive x-axis point downstream, so that the velocity of the current is $\mathbf{v_c} = 5\mathbf{i}$.

Let $\mathbf{v_w}$ = the velocity of the boat in the water.

Let $\mathbf{v_g}$ = the velocity of the boat relative to the land.

Then $\mathbf{v_g} = \mathbf{v_w} + \mathbf{v_c}$

The speed of the boat is $\|\mathbf{v_w}\| = 20$; we need to find the direction.

Let $\mathbf{v_w} = a\mathbf{i} + b\,\mathbf{j}$ so $\|\mathbf{v_w}\| = \sqrt{a^2+b^2} = 20 \quad \rightarrow \quad a^2 + b^2 = 400$.

Let $\mathbf{v_g} = k\,\mathbf{j}$.

Since $\mathbf{v_g} = \mathbf{v_w} + \mathbf{v_c}$, $\quad k\,\mathbf{j} = a\mathbf{i} + b\,\mathbf{j} + 5\mathbf{i} \quad \rightarrow \quad k\,\mathbf{j} = (a+5)\mathbf{i} + b\,\mathbf{j}$

$a + 5 = 0$ and $k = b \quad \rightarrow \quad a = -5$

$a^2 + b^2 = 400 \quad \rightarrow \quad 25 + b^2 = 400 \quad \rightarrow \quad b^2 = 375 \quad \rightarrow \quad k = b \approx 19.36$

$\mathbf{v_w} = -5\mathbf{i} + 19.36\,\mathbf{j}$ and $\mathbf{v_g} = 19.36\,\mathbf{j}$

Find the angle between $\mathbf{v}_w$ and $\mathbf{j}$:

$$\cos\theta = \frac{\mathbf{v}_w \cdot \mathbf{j}}{\|\mathbf{v}_w\|\|\mathbf{j}\|} = \frac{-5\ \ 0 + 19.36(1)}{20\sqrt{0^2+1^2}} = \frac{19.36}{20} \approx 0.9680$$

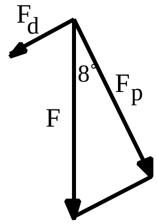
$\theta \approx 14.5^{o}$

The heading of the boat needs to be 14.5° upstream.

The velocity of the boat directly across the river is 19.36 kilometers per hour. The time to cross the river is: $t = \frac{0.5}{19.36} \approx 0.026$ hours or $t \approx 1.56$ minutes.

23. Split the force into the components going down the hill and perpendicular to the hill.

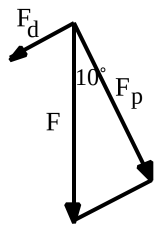
$$\mathbf{F_d} = \mathbf{F}\sin(8^\circ) = 5300\sin(8^\circ)$$
$$= 5300(0.1392) \approx 738 \text{ pounds}$$
$$\mathbf{F_p} = \mathbf{F}\cos(8^\circ) = 5300\cos(8^\circ)$$
$$= 5300(0.9903) \approx 5249 \text{ pounds}$$



The force required to keep the car from rolling down the hill is about 738 pounds. The force perpendicular to the hill is approximately 5249 pounds.

24. Split the force into the components going down the hill and perpendicular to the hill.

$$\mathbf{F_d} = \mathbf{F}\sin(10^\circ) = 4500\sin(10^\circ)$$
$$= 4500(0.1736) \approx 781.2 \text{ pounds}$$
$$\mathbf{F_p} = \mathbf{F}\cos(10^\circ) = 4500\cos(10^\circ)$$
$$= 4500(0.9848) \approx 4431.6 \text{ pounds}$$



The force required to keep the car from rolling down the hill is about 781.2 pounds. The force perpendicular to the hill is approximately 4431.6 pounds.

25. Let $\mathbf{v_a}$ = the velocity of the plane in still air.

$\mathbf{v_w}$ = the velocity of the wind.

$\mathbf{v_g}$ = the velocity of the plane relative to the ground.

$$\mathbf{v_g} = \mathbf{v_a} + \mathbf{v_w}$$

$$\mathbf{v_a} = 500\left(\cos(45^\circ)\mathbf{i} + \sin(45^\circ)\mathbf{j}\right) = 500\left(\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}\right) = 250\sqrt{2}\,\mathbf{i} + 250\sqrt{2}\,\mathbf{j}$$

$$\mathbf{v_w} = 60\left(\cos(120^\circ)\mathbf{i} + \sin(120^\circ)\mathbf{j}\right) = 60\left(-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right) = -30\mathbf{i} + 30\sqrt{3}\,\mathbf{j}$$

$$\mathbf{v_g} = \mathbf{v_a} + \mathbf{v_w} = 250\sqrt{2}\,\mathbf{i} + 250\sqrt{2}\,\mathbf{j} - 30\mathbf{i} + 30\sqrt{3}\,\mathbf{j}$$
$$= \left(-30 + 250\sqrt{2}\right)\mathbf{i} + \left(250\sqrt{2} + 30\sqrt{3}\right)\mathbf{j}$$

The speed of the plane relative to the ground is:

$$\|\mathbf{v_g}\| = \sqrt{\left(-30 + 250\sqrt{2}\right)^2 + \left(250\sqrt{2} + 30\sqrt{3}\right)^2}$$
$$= \sqrt{269129.1} \approx 518.8 \text{ kilometers per hour}$$

To find the direction, find the angle between $\mathbf{v_g}$ and a convenient vector such as due north, $\mathbf{j}$.

$$\cos\theta = \frac{\mathbf{v}_g \cdot \mathbf{j}}{\|\mathbf{v}_g\|\|\mathbf{j}\|} = \frac{\left(-30 + 250\sqrt{2}\right)0 + \left(250\sqrt{2} + 30\sqrt{3}\right)(1)}{518.8\sqrt{0^2 + 1^2}} = \frac{250\sqrt{2} + 30\sqrt{3}}{518.8}$$
$$= \frac{405.5}{518.8} \approx 0.7816 \quad \rightarrow \quad \theta \approx 38.6^\circ$$

The plane is traveling with a ground speed of about 518.8 kilometers per hour in a direction of 38.6˚ east of north.

26. Let $\mathbf{v_a}$ = the velocity of the plane in still air.

$\mathbf{v_w}$ = the velocity of the wind.

$\mathbf{v_g}$ = the velocity of the plane relative to the ground.

$\mathbf{v_g} = \mathbf{v_a} + \mathbf{v_w}$

$$\mathbf{v_a} = 600\left(\cos(60^\circ)\mathbf{i} - \sin(60^\circ)\mathbf{j}\right) = 600\left(\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}\right) = 300\mathbf{i} - 300\sqrt{3}\,\mathbf{j}$$

$$\mathbf{v_w} = 40\left(\cos(45^\circ)\mathbf{i} - \sin(45^\circ)\mathbf{j}\right) = 40\left(\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}\right) = 20\sqrt{2}\,\mathbf{i} - 20\sqrt{2}\,\mathbf{j}$$

$$\begin{aligned}\mathbf{v_g} &= \mathbf{v_a} + \mathbf{v_w} = 300\mathbf{i} - 300\sqrt{3}\,\mathbf{j} + 20\sqrt{2}\,\mathbf{i} - 20\sqrt{2}\,\mathbf{j} \\ &= \left(300 + 20\sqrt{2}\right)\mathbf{i} + \left(-300\sqrt{3} - 20\sqrt{2}\right)\mathbf{j}\end{aligned}$$

The speed of the plane relative to the ground is:

$$\begin{aligned}\|\mathbf{v_g}\| &= \sqrt{\left(300 + 20\sqrt{2}\right)^2 + \left(-300\sqrt{3} - 20\sqrt{2}\right)^2} \\ &= \sqrt{407964} \approx 639 \text{ kilometers per hour}\end{aligned}$$

To find the direction, find the angle between $\mathbf{v_g}$ and a convenient vector such as due north, $\mathbf{j}$.

$$\begin{aligned}\cos\theta &= \frac{\mathbf{v}_g \cdot \mathbf{j}}{\|\mathbf{v}_g\|\|\mathbf{j}\|} = \frac{\left(300 + 20\sqrt{2}\right)\ 0 + \left(-300\sqrt{3} - 20\sqrt{2}\right)(1)}{639\sqrt{0^2 + 1^2}} = \frac{-300\sqrt{3} - 20\sqrt{2}}{639} \\ &= \frac{-547.9}{639} \approx -0.8574 \\ \theta &\approx 149^\circ\end{aligned}$$

The plane is traveling with a ground speed of about 639 kilometers per hour in a direction of 31° east of south.

27. Let the positive x-axis point downstream, so that the velocity of the current is $\mathbf{v_c} = 3\mathbf{i}$.

Let $\mathbf{v_w}$ = the velocity of the boat in the water.

Let $\mathbf{v_g}$ = the velocity of the boat relative to the land.

Then $\mathbf{v_g} = \mathbf{v_w} + \mathbf{v_c}$

The speed of the boat is $\|\mathbf{v_w}\| = 20$; its direction is directly across the river, so

Let $\mathbf{v_w} = 20\mathbf{j}$.

$$\mathbf{v_g} = \mathbf{v_w} + \mathbf{v_c} = 20\mathbf{j} + 3\mathbf{i} = 3\mathbf{i} + 20\mathbf{j}$$

Let $\|\mathbf{v_g}\| = \sqrt{3^2 + 20^2} = \sqrt{409} \approx 20.2$ miles per hour

Find the angle between $\mathbf{v_g}$ and $\mathbf{j}$:

$$\begin{aligned}\cos\theta &= \frac{\mathbf{v}_g \cdot \mathbf{j}}{\|\mathbf{v}_g\|\|\mathbf{j}\|} = \frac{3\ 0 + 20(1)}{20.2\sqrt{0^2 + 1^2}} = \frac{20}{20.2} \approx 0.9901 \\ \theta &\approx 8.1^\circ\end{aligned}$$

The heading of the boat will be 8.1° downstream.

28. Let the positive x-axis point downstream, so that the velocity of the current is $\mathbf{v_c} = 4\mathbf{i}$.
Let $\mathbf{v_w}$ = the velocity of the boat in the water.
Let $\mathbf{v_g}$ = the velocity of the boat relative to the land.
Then $\mathbf{v_g} = \mathbf{v_w} + \mathbf{v_c}$
The speed of the boat is $\|\mathbf{v_w}\| = 10$; its direction is directly across the river, so

Let $\mathbf{v_w} = 10\mathbf{j}$.

$\mathbf{v_g} = \mathbf{v_w} + \mathbf{v_c} = 10\mathbf{j} + 4\mathbf{i} = 4\mathbf{i} + 10\mathbf{j}$

Let $\|\mathbf{v_g}\| = \sqrt{4^2 + 10^2} = \sqrt{116} \approx 10.8$ miles per hour .

Find the angle between $\mathbf{v_g}$ and $\mathbf{j}$:

$$\cos\theta = \frac{\mathbf{v}_g \cdot \mathbf{j}}{\|\mathbf{v}_g\|\|\mathbf{j}\|} = \frac{4\ \ 0 + 10(1)}{10.8\sqrt{0^2 + 1^2}} = \frac{10}{10.8} \approx 0.9259$$

$$\theta \approx 22.2^\circ$$

The heading of the boat will be 22.2° downstream.

29. $\mathbf{F} = 3\left(\cos\left(60^\circ\right)\mathbf{i} + \sin\left(60^\circ\right)\mathbf{j}\right) = 3\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right) = \frac{3}{2}\mathbf{i} + \frac{3\sqrt{3}}{2}\mathbf{j}$

$W = \mathbf{F} \cdot AB = \left(\frac{3}{2}\mathbf{i} + \frac{3\sqrt{3}}{2}\mathbf{j}\right) \cdot 2\mathbf{i} = \frac{3}{2}(2) + \frac{3\sqrt{3}}{2}\ \ 0 = 3$ foot-pounds

30. $\mathbf{F} = 1\left(\cos\left(45^\circ\right)\mathbf{i} + \sin\left(45^\circ\right)\mathbf{j}\right) = 1\left(\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}\right) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

$W = \mathbf{F} \cdot AB = \left(\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}\right) \cdot 5\mathbf{i} = \frac{\sqrt{2}}{2}(5) + \frac{\sqrt{2}}{2}\ \ 0 = \frac{5\sqrt{2}}{2}$ foot-pounds

31. $\mathbf{F} = 20\left(\cos\left(30^\circ\right)\mathbf{i} + \sin\left(30^\circ\right)\mathbf{j}\right) = 20\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = 10\sqrt{3}\mathbf{i} + 10\mathbf{j}$

$W = \mathbf{F} \cdot AB = \left(10\sqrt{3}\mathbf{i} + 10\mathbf{j}\right) \cdot 100\mathbf{i} = 10\sqrt{3}(100) + 10\ \ 0 = 1732$ foot-pounds

32. $W = \mathbf{F} \cdot AB \qquad W = 2, \quad AB = 4\mathbf{i}$

$\mathbf{F} = \cos\alpha\,\mathbf{i} - \sin\alpha\,\mathbf{j}$

$2 = \left(\cos\alpha\,\mathbf{i} - \sin\alpha\,\mathbf{j}\right) \cdot 4\mathbf{i}$

$2 = 4\cos\alpha \rightarrow \frac{1}{2} = \cos\alpha \rightarrow \alpha = 60^\circ$

33. Let $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j}, \quad \mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j}, \quad \mathbf{w} = a_3\mathbf{i} + b_3\mathbf{j}$

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \left(a_1\mathbf{i} + b_1\mathbf{j}\right) \cdot \left(a_2\mathbf{i} + b_2\mathbf{j} + a_3\mathbf{i} + b_3\mathbf{j}\right) = \left(a_1\mathbf{i} + b_1\mathbf{j}\right) \cdot \left(a_2\mathbf{i} + a_3\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{j}\right) \\
&= \left(a_1\mathbf{i} + b_1\mathbf{j}\right) \cdot \left((a_2 + a_3)\mathbf{i} + (b_2 + b_3)\mathbf{j}\right) = a_1(a_2 + a_3) + b_1(b_2 + b_3) \\
&= a_1a_2 + a_1a_3 + b_1b_2 + b_1b_3 = a_1a_2 + b_1b_2 + a_1a_3 + b_1b_3 \\
&= \left(a_1\mathbf{i} + b_1\mathbf{j}\right) \cdot \left(a_2\mathbf{i} + b_2\mathbf{j}\right) + \left(a_1\mathbf{i} + b_1\mathbf{j}\right) \cdot \left(a_3\mathbf{i} + b_3\mathbf{j}\right) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}
\end{aligned}$$

34. $\mathbf{0} = 0\mathbf{i} + 0\,\mathbf{j}$ and $\mathbf{v} = \text{a}\,\mathbf{i} + \text{b}\,\mathbf{j}$. $\mathbf{0}\cdot\mathbf{v} = 0 \quad a + 0 \quad b = 0$

35. Let $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j}$.
Since $\mathbf{v}$ is a unit vector, $\|\mathbf{v}\| = \sqrt{a^2 + b^2} = 1$ or $a^2 + b^2 = 1$
If α is the angle between $\mathbf{v}$ and $\mathbf{i}$, then $\cos\alpha = \dfrac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\|\mathbf{i}\|}$ or $\cos\alpha = \dfrac{(\text{a}\mathbf{i} + \text{b}\mathbf{j})\cdot\mathbf{i}}{1\;1} = a$.

$$a^2 + b^2 = 1$$

$$\cos^2\alpha + b^2 = 1 \rightarrow b^2 = 1 - \cos^2\alpha \rightarrow b^2 = \sin^2\alpha \rightarrow b = \sin\alpha$$

Thus, $\mathbf{v} = \cos\alpha\;\mathbf{i} + \sin\alpha\;\mathbf{j}$

36. If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} = \cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} = \cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j}$, then

$$\cos(\alpha - \beta) = \mathbf{v}\cdot\mathbf{w} = a_1a_2 + b_1b_2 = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

37. Let $\mathbf{v} = a\,\mathbf{i} + b\,\mathbf{j}$.

$$\text{proj}_{\mathbf{i}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{i}\|^2}\,\mathbf{i} = \frac{(a\,\mathbf{i} + b\,\mathbf{j})\cdot\mathbf{i}}{\sqrt{1^2 + 0^2}}\,\mathbf{i} = \frac{a(1) + b(0)}{1}\mathbf{i} = a\,\mathbf{i}$$

$\mathbf{v}\cdot\mathbf{i} = a,\quad \mathbf{v}\cdot\mathbf{j} = b,$
$\mathbf{v} = (\mathbf{v}\cdot\mathbf{i})\mathbf{i} + (\mathbf{v}\cdot\mathbf{j})\mathbf{j}$

38. (a) Let $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j}$

$$\begin{aligned}(\mathbf{u} + \mathbf{v})\cdot(\mathbf{u} - \mathbf{v}) &= \left((a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j}\right)\cdot\left((a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j}\right)\\ &= (a_1 + a_2)(a_1 - a_2) + (b_1 + b_2)(b_1 - b_2) = a_1^2 - a_2^2 + b_1^2 - b_2^2\\ &= \left(a_1^2 + b_1^2\right) - \left(a_2^2 + b_2^2\right)\end{aligned}$$

Since the vectors have the same magnitudes and these two quantities represent the squares of the magnitudes of each, the difference is 0 and the vectors are orthogonal.

(b) Because the vectors $\mathbf{u}$ and $\mathbf{v}$ are radii of the circle, we know they have the same magnitude. Since $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are sides of the angle inscribed in the semicircle and we know that they are orthogonal (part a), then this angle must be a right angle.

39. $(\mathbf{v} - \alpha\mathbf{w})\cdot\mathbf{w} = \mathbf{v}\cdot\mathbf{w} - \alpha\mathbf{w}\cdot\mathbf{w} = \mathbf{v}\cdot\mathbf{w} - \alpha\|\mathbf{w}\|^2 = \mathbf{v}\cdot\mathbf{w} - \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\|\mathbf{w}\|^2 = 0$

Therefore the vectors are orthogonal.

40. $\left(\|\mathbf{w}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{w}\right)\cdot\left(\|\mathbf{w}\|\mathbf{v} - \|\mathbf{v}\|\mathbf{w}\right)$

$$\begin{aligned}&= \|\mathbf{w}\|^2\,\mathbf{v}\cdot\mathbf{v} - \|\mathbf{w}\|\|\mathbf{v}\|\mathbf{v}\cdot\mathbf{w} + \|\mathbf{w}\|\|\mathbf{v}\|\mathbf{v}\cdot\mathbf{w} - \|\mathbf{v}\|^2\,\mathbf{w}\cdot\mathbf{w}\\ &= \|\mathbf{w}\|^2\,\mathbf{v}\cdot\mathbf{v} - \|\mathbf{v}\|^2\,\mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2\|\mathbf{v}\|^2 - \|\mathbf{v}\|^2\|\mathbf{w}\|^2 = 0\end{aligned}$$

41. If $\mathbf{F}$ is orthogonal to $\mathbf{AB}$, then $W = \mathbf{F}\cdot\mathbf{AB} = (\mathbf{i} + \mathbf{j})(\mathbf{i} - \mathbf{j}) = 1\;\;1 + 1(-1) = 0$

42. $\|u + v\|^2 - \|u - v\|^2 = (u + v)\cdot(u + v) - (u - v)\cdot(u - v)$

$$\begin{aligned}&= (u\cdot u + u\cdot v + v\cdot u + v\cdot v) - (u\cdot u - u\cdot v - v\cdot u + v\cdot v)\\ &= 2\,(u\cdot v) + 2(u\cdot v) = 4(u\cdot v)\end{aligned}$$