Chapter 11

Analytic Geometry

11.5 Rotation of Axes; General Form of a Conic

1. $x^2 + 4x + y + 3 = 0$
   $A = 1$ and $C = 0$; $AC = (1)(0) = 0$. Since $AC = 0$, the equation defines a parabola.

2. $2y^2 - 3y + 3x = 0$
   $A = 0$ and $C = 2$; $AC = (0)(2) = 0$. Since $AC = 0$, the equation defines a parabola.

3. $6x^2 + 3y^2 - 12x + 6y = 0$
   $A = 6$ and $C = 3$; $AC = (6)(3) = 18$. Since $AC > 0$ and $A \neq C$, the equation defines an ellipse.

4. $2x^2 + y^2 - 8x + 4y + 2 = 0$
   $A = 2$ and $C = 1$; $AC = (2)(1) = 2$. Since $AC > 0$ and $A \neq C$, the equation defines an ellipse.

5. $3x^2 - 2y^2 + 6x + 4 = 0$
   $A = 3$ and $C = -2$; $AC = (3)(-2) = -6$. Since $AC < 0$, the equation defines a hyperbola.

6. $4x^2 - 3y^2 - 8x + 6y + 1 = 0$
   $A = 4$ and $C = -3$; $AC = (4)(-3) = -12$. Since $AC < 0$, the equation defines a hyperbola.

7. $2y^2 - x^2 - y + x = 0$
   $A = -1$ and $C = 2$; $AC = (-1)(2) = -2$. Since $AC < 0$, the equation defines a hyperbola.

8. $y^2 - 8x^2 - 2x - y = 0$
   $A = -8$ and $C = 1$; $AC = (-8)(1) = -8$. Since $AC < 0$, the equation defines a hyperbola.

9. $x^2 + y^2 - 8x + 4y = 0$
   $A = 1$ and $C = 1$; $AC = (1)(1) = 1$. Since $AC > 0$ and $A = C$, the equation defines a circle.

10. $2x^2 + 2y^2 - 8x + 8y = 0$
    $A = 2$ and $C = 2$; $AC = (2)(2) = 4$. Since $AC > 0$ and $A = C$, the equation defines a circle.

11. $x^2+4xy+y^2-3=0$

$A=1,\ B=4,$ and $C=1;\ \cot(2\theta)=\dfrac{A-C}{B}=\dfrac{1-1}{4}=\dfrac{0}{4}=0 \rightarrow 2\theta=\dfrac{\pi}{2} \rightarrow \theta=\dfrac{\pi}{4}$

$x=x'\cos\left(\dfrac{\pi}{4}\right)-y'\sin\left(\dfrac{\pi}{4}\right)=\dfrac{\sqrt{2}}{2}x'-\dfrac{\sqrt{2}}{2}y'=\dfrac{\sqrt{2}}{2}(x'-y')$

$y=x'\sin\left(\dfrac{\pi}{4}\right)+y'\cos\left(\dfrac{\pi}{4}\right)=\dfrac{\sqrt{2}}{2}x'+\dfrac{\sqrt{2}}{2}y'=\dfrac{\sqrt{2}}{2}(x'+y')$

12. $x^2-4xy+y^2-3=0$

$A=1,\ B=-4,$ and $C=1;\ \cot(2\theta)=\dfrac{A-C}{B}=\dfrac{1-1}{-4}=\dfrac{0}{-4}=0 \rightarrow 2\theta=\dfrac{\pi}{2} \rightarrow \theta=\dfrac{\pi}{4}$

$x=x'\cos\left(\dfrac{\pi}{4}\right)-y'\sin\left(\dfrac{\pi}{4}\right)=\dfrac{\sqrt{2}}{2}x'-\dfrac{\sqrt{2}}{2}y'=\dfrac{\sqrt{2}}{2}(x'-y')$

$y=x'\sin\left(\dfrac{\pi}{4}\right)+y'\cos\left(\dfrac{\pi}{4}\right)=\dfrac{\sqrt{2}}{2}x'+\dfrac{\sqrt{2}}{2}y'=\dfrac{\sqrt{2}}{2}(x'+y')$

13. $5x^2+6xy+5y^2-8=0$

$A=5,\ B=6,$ and $C=5;\ \cot(2\theta)=\dfrac{A-C}{B}=\dfrac{5-5}{6}=\dfrac{0}{6}=0 \rightarrow 2\theta=\dfrac{\pi}{2} \rightarrow \theta=\dfrac{\pi}{4}$

$x=x'\cos\left(\dfrac{\pi}{4}\right)-y'\sin\left(\dfrac{\pi}{4}\right)=\dfrac{\sqrt{2}}{2}x'-\dfrac{\sqrt{2}}{2}y'=\dfrac{\sqrt{2}}{2}(x'-y')$

$y=x'\sin\left(\dfrac{\pi}{4}\right)+y'\cos\left(\dfrac{\pi}{4}\right)=\dfrac{\sqrt{2}}{2}x'+\dfrac{\sqrt{2}}{2}y'=\dfrac{\sqrt{2}}{2}(x'+y')$

14. $3x^2-10xy+3y^2-32=0$

$A=3,\ B=-10,$ and $C=3;\ \cot(2\theta)=\dfrac{A-C}{B}=\dfrac{3-3}{-10}=\dfrac{0}{-10}=0 \rightarrow 2\theta=\dfrac{\pi}{2} \rightarrow \theta=\dfrac{\pi}{4}$

$x=x'\cos\left(\dfrac{\pi}{4}\right)-y'\sin\left(\dfrac{\pi}{4}\right)=\dfrac{\sqrt{2}}{2}x'-\dfrac{\sqrt{2}}{2}y'=\dfrac{\sqrt{2}}{2}(x'-y')$

$y=x'\sin\left(\dfrac{\pi}{4}\right)+y'\cos\left(\dfrac{\pi}{4}\right)=\dfrac{\sqrt{2}}{2}x'+\dfrac{\sqrt{2}}{2}y'=\dfrac{\sqrt{2}}{2}(x'+y')$

15. $13x^2-6\sqrt{3}xy+7y^2-16=0$

$A=13, B=-6\sqrt{3},$ and $C=7;\ \cot(2\theta)=\dfrac{A-C}{B}=\dfrac{13-7}{-6\sqrt{3}}=\dfrac{6}{-6\sqrt{3}}=-\dfrac{\sqrt{3}}{3}$

$2\theta=\dfrac{2\pi}{3} \rightarrow \theta=\dfrac{\pi}{3}$

$$x = x'\cos\left(\frac{\pi}{3}\right) - y'\sin\left(\frac{\pi}{3}\right) = \frac{1}{2}x' - \frac{\sqrt{3}}{2}y' = \frac{1}{2}\left(x' - \sqrt{3}y'\right)$$

$$y = x'\sin\left(\frac{\pi}{3}\right) + y'\cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}x' + \frac{1}{2}y' = \frac{1}{2}\left(\sqrt{3}x' + y'\right)$$

16. $11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0$

$A = 11,\; B = 10\sqrt{3},$ and $C = 1;\quad \cot(2\theta) = \dfrac{A-C}{B} = \dfrac{11-1}{10\sqrt{3}} = \dfrac{10}{10\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

$$2\theta = \frac{\pi}{3} \;\rightarrow\; \theta = \frac{\pi}{6}$$

$$x = x'\cos\left(\frac{\pi}{6}\right) - y'\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y' = \frac{1}{2}\left(\sqrt{3}x' - y'\right)$$

$$y = x'\sin\left(\frac{\pi}{6}\right) + y'\cos\left(\frac{\pi}{6}\right) = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y' = \frac{1}{2}\left(x' + \sqrt{3}y'\right)$$

17. $4x^2 - 4xy + y^2 - 8\sqrt{5}\,x - 16\sqrt{5}\,y = 0$

$A = 4, B = -4,$ and $C = 1;\; \cot(2\theta) = \dfrac{A-C}{B} = \dfrac{4-1}{-4} = -\dfrac{3}{4};\quad \cos 2\theta = -\dfrac{3}{5}$

$$\sin\theta = \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5};\quad \cos\theta = \sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$x = x'\cos\theta - y'\sin\theta = \frac{\sqrt{5}}{5}x' - \frac{2\sqrt{5}}{5}y' = \frac{\sqrt{5}}{5}\left(x' - 2y'\right)$$

$$y = x'\sin\theta + y'\cos\theta = \frac{2\sqrt{5}}{5}x' + \frac{\sqrt{5}}{5}y' = \frac{\sqrt{5}}{5}\left(2x' + y'\right)$$

18. $x^2 + 4xy + 4y^2 + 5\sqrt{5}\,y + 5 = 0$

$A = 1,\; B = 4,$ and $C = 4;\; \cot(2\theta) = \dfrac{A-C}{B} = \dfrac{1-4}{4} = -\dfrac{3}{4};\quad \cos 2\theta = -\dfrac{3}{5}$

$$\sin\theta = \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5};\quad \cos\theta = \sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$x = x'\cos\theta - y'\sin\theta = \frac{\sqrt{5}}{5}x' - \frac{2\sqrt{5}}{5}y' = \frac{\sqrt{5}}{5}\left(x' - 2y'\right)$$

$$y = x'\sin\theta + y'\cos\theta = \frac{2\sqrt{5}}{5}x' + \frac{\sqrt{5}}{5}y' = \frac{\sqrt{5}}{5}\left(2x' + y'\right)$$

19. $25x^2 - 36xy + 40y^2 - 12\sqrt{13}\,x - 8\sqrt{13}\,y = 0$

$A = 25, B = -36, \text{and } C = 40;\ \cot(2\theta) = \dfrac{A-C}{B} = \dfrac{25-40}{-36} = \dfrac{5}{12};\quad \cos 2\theta = \dfrac{5}{13}$

$\sin\theta = \sqrt{\dfrac{\left(1-\frac{5}{13}\right)}{2}} = \sqrt{\dfrac{4}{13}} = \dfrac{2}{\sqrt{13}} = \dfrac{2\sqrt{13}}{13};\quad \cos\theta = \sqrt{\dfrac{\left(1+\frac{5}{13}\right)}{2}} = \sqrt{\dfrac{9}{13}} = \dfrac{3}{\sqrt{13}} = \dfrac{3\sqrt{13}}{13}$

$x = x'\cos\theta - y'\sin\theta = \dfrac{3\sqrt{13}}{13}x' - \dfrac{2\sqrt{13}}{13}y' = \dfrac{\sqrt{13}}{13}(3x' - 2y')$

$y = x'\sin\theta + y'\cos\theta = \dfrac{2\sqrt{13}}{13}x' + \dfrac{3\sqrt{13}}{13}y' = \dfrac{\sqrt{13}}{13}(2x' + 3y')$

20. $34x^2 - 24xy + 41y^2 - 25 = 0$

$A = 34, B = -24, \text{and } C = 41;\ \cot(2\theta) = \dfrac{A-C}{B} = \dfrac{34-41}{-24} = \dfrac{7}{24};\quad \cos(2\theta) = \dfrac{7}{25}$

$\sin\theta = \sqrt{\dfrac{\left(1-\frac{7}{25}\right)}{2}} = \sqrt{\dfrac{9}{25}} = \dfrac{3}{5};\quad \cos\theta = \sqrt{\dfrac{\left(1+\frac{7}{25}\right)}{2}} = \sqrt{\dfrac{16}{25}} = \dfrac{4}{5}$

$x = x'\cos\theta - y'\sin\theta = \dfrac{4}{5}x' - \dfrac{3}{5}y' = \dfrac{1}{5}(4x' - 3y')$

$y = x'\sin\theta + y'\cos\theta = \dfrac{3}{5}x' + \dfrac{4}{5}y' = \dfrac{1}{5}(3x' + 4y')$

21. $x^2 + 4xy + y^2 - 3 = 0;\quad \theta = 45^\circ$ (see Problem 11)

$\left(\dfrac{\sqrt{2}}{2}(x'-y')\right)^2 + 4\left(\dfrac{\sqrt{2}}{2}(x'-y')\right)\left(\dfrac{\sqrt{2}}{2}(x'+y')\right) + \left(\dfrac{\sqrt{2}}{2}(x'+y')\right)^2 - 3 = 0$

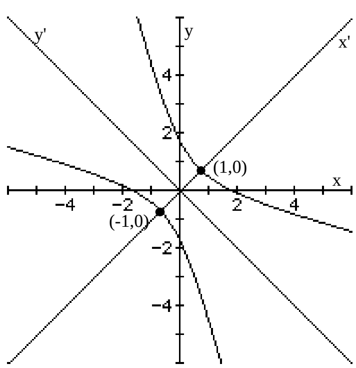
$\dfrac{1}{2}\left(x'^2 - 2x'y' + y'^2\right) + 2\left(x'^2 - y'^2\right) + \dfrac{1}{2}\left(x'^2 + 2x'y' + y'^2\right) - 3 = 0$

$\dfrac{1}{2}x'^2 - x'y' + \dfrac{1}{2}y'^2 + 2x'^2 - 2y'^2 + \dfrac{1}{2}x'^2 + x'y' + \dfrac{1}{2}y'^2 = 3$

$3x'^2 - y'^2 = 3$

$\dfrac{x'^2}{1} - \dfrac{y'^2}{3} = 1$

Hyperbola; center at the origin, transverse axis is the x'-axis, vertices at (±1, 0).



22. $x^2 - 4xy + y^2 - 3 = 0$; $\theta = 45^\circ$ (see Problem 12)

$$\left(\frac{\sqrt{2}}{2}(x'-y')\right)^2 - 4\left(\frac{\sqrt{2}}{2}(x'-y')\right)\left(\frac{\sqrt{2}}{2}(x'+y')\right) + \left(\frac{\sqrt{2}}{2}(x'+y')\right)^2 - 3 = 0$$

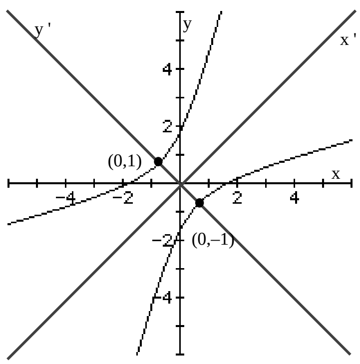
$$\frac{1}{2}\left(x'^2 - 2x'y' + y'^2\right) - 2\left(x'^2 - y'^2\right) + \frac{1}{2}\left(x'^2 + 2x'y' + y'^2\right) - 3 = 0$$

$$\frac{1}{2}x'^2 - x'y' + \frac{1}{2}y'^2 - 2x'^2 + 2y'^2 + \frac{1}{2}x'^2 + x'y' + \frac{1}{2}y'^2 = 3$$

$$-x'^2 + 3y'^2 = 3$$

$$\frac{y'^2}{1} - \frac{x'^2}{3} = 1$$

Hyperbola; center at the origin, transverse axis is the y'-axis, vertices at (0, ±1).



23. $5x^2 + 6xy + 5y^2 - 8 = 0$; $\theta = 45^\circ$ (see Problem 13)

$$5\left(\frac{\sqrt{2}}{2}(x'-y')\right)^2 + 6\left(\frac{\sqrt{2}}{2}(x'-y')\right)\left(\frac{\sqrt{2}}{2}(x'+y')\right) + 5\left(\frac{\sqrt{2}}{2}(x'+y')\right)^2 - 8 = 0$$

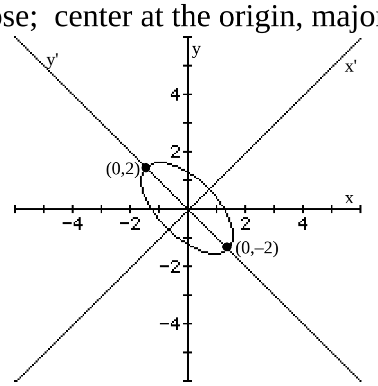
$$\frac{5}{2}\left(x'^2 - 2x'y' + y'^2\right) + 3\left(x'^2 - y'^2\right) + \frac{5}{2}\left(x'^2 + 2x'y' + y'^2\right) - 8 = 0$$

$$\frac{5}{2}x'^2 - 5x'y' + \frac{5}{2}y'^2 + 3x'^2 - 3y'^2 + \frac{5}{2}x'^2 + 5x'y' + \frac{5}{2}y'^2 = 8$$

$$8x'^2 + 2y'^2 = 8$$

$$\frac{x'^2}{1} + \frac{y'^2}{4} = 1$$

Ellipse; center at the origin, major axis is the y'-axis, vertices at (0, ±2).



24. $3x^2 - 10xy + 3y^2 - 32 = 0$; $\theta = 45^\circ$ (see Problem 14)

$$3\left(\frac{\sqrt{2}}{2}(x'-y')\right)^2 - 10\left(\frac{\sqrt{2}}{2}(x'-y')\right)\left(\frac{\sqrt{2}}{2}(x'+y')\right) + 3\left(\frac{\sqrt{2}}{2}(x'+y')\right)^2 - 32 = 0$$

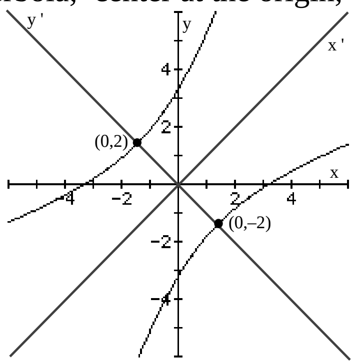
$$\frac{3}{2}\left(x'^2 - 2x'y' + y'^2\right) - 5\left(x'^2 - y'^2\right) + \frac{3}{2}\left(x'^2 + 2x'y' + y'^2\right) - 32 = 0$$

$$\frac{3}{2}x'^2 - 3x'y' + \frac{3}{2}y'^2 - 5x'^2 + 5y'^2 + \frac{3}{2}x'^2 + 3x'y' + \frac{3}{2}y'^2 = 32$$

$$-2x'^2 + 8y'^2 = 32$$

$$\frac{y'^2}{4} - \frac{x'^2}{16} = 1$$

Hyperbola; center at the origin, transverse axis is the y'-axis, vertices at (0, ±2).



25. $13x^2 - 6\sqrt{3}\,xy + 7y^2 - 16 = 0$; $\theta = 60^\circ$ (see Problem 15)

$$13\left(\frac{1}{2}(x' - \sqrt{3}y')\right)^2 - 6\sqrt{3}\left(\frac{1}{2}(x' - \sqrt{3}y')\right)\left(\frac{1}{2}(\sqrt{3}x' + y')\right) + 7\left(\frac{1}{2}(\sqrt{3}x' + y')\right)^2 - 16 = 0$$

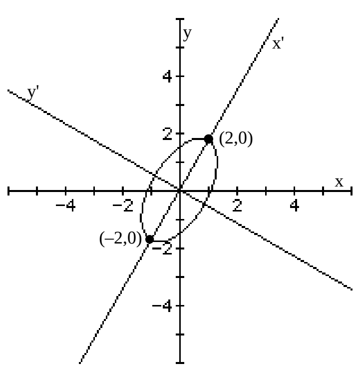
$$\frac{13}{4}\left(x'^2 - 2\sqrt{3}x'y' + 3y'^2\right) - \frac{3\sqrt{3}}{2}\left(\sqrt{3}x'^2 - 2x'y' - \sqrt{3}y'^2\right) + \frac{7}{4}\left(3x'^2 + 2\sqrt{3}x'y' + y'^2\right) = 16$$

$$\frac{13}{4}x'^2 - \frac{13\sqrt{3}}{2}x'y' + \frac{39}{4}y'^2 - \frac{9}{2}x'^2 + 3\sqrt{3}x'y' + \frac{9}{2}y'^2 + \frac{21}{4}x'^2 + \frac{7\sqrt{3}}{2}x'y' + \frac{7}{4}y'^2 = 16$$

$$4x'^2 + 16y'^2 = 16$$

$$\frac{x'^2}{4} + \frac{y'^2}{1} = 1$$

Ellipse; center at the origin, major axis is the x'-axis, vertices at (±2, 0).



26. $11x^2+10\sqrt{3}\,xy+y^2-4=0;\quad \theta=30^\circ$ (see Problem 16)

$$11\left(\frac{1}{2}\left(\sqrt{3}x'-y'\right)\right)^2+10\sqrt{3}\left(\frac{1}{2}\left(\sqrt{3}x'-y'\right)\right)\left(\frac{1}{2}\left(x'+\sqrt{3}\,y'\right)\right)+\left(\frac{1}{2}\left(x'+\sqrt{3}y'\right)\right)^2-4=0$$

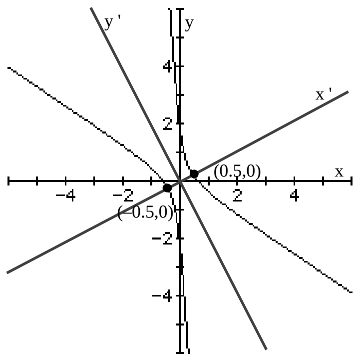
$$\frac{11}{4}\left(3x'^2-2\sqrt{3}x'y'+y'^2\right)+\frac{5\sqrt{3}}{2}\left(\sqrt{3}x'^2+2x'y'-\sqrt{3}y'^2\right)+\frac{1}{4}\left(x'^2+2\sqrt{3}x'y'+3y'^2\right)=4$$

$$\frac{33}{4}x'^2-\frac{11\sqrt{3}}{2}x'y'+\frac{11}{4}y'^2+\frac{15}{2}x'^2+5\sqrt{3}x'y'-\frac{15}{2}y'^2+\frac{1}{4}x'^2+\frac{\sqrt{3}}{2}x'y'+\frac{3}{4}y'^2=4$$

$$16x'^2-4y'^2=4$$

$$4x'^2-y'^2=1$$

Hyperbola; center at the origin, transverse axis is the x'-axis, vertices at (±0.5, 0).



27. $4x^2-4xy+y^2-8\sqrt{5}\,x-16\sqrt{5}\,y=0;\quad \theta=63.4^\circ$ (see Problem 17)

$$4\left(\frac{\sqrt{5}}{5}(x'-2y')\right)^2-4\left(\frac{\sqrt{5}}{5}(x'-2y')\right)\left(\frac{\sqrt{5}}{5}(2x'+y')\right)+\left(\frac{\sqrt{5}}{5}(2x'+y')\right)^2$$

$$-8\sqrt{5}\left(\frac{\sqrt{5}}{5}(x'-2y')\right)-16\sqrt{5}\left(\frac{\sqrt{5}}{5}(2x'+y')\right)=0$$

$$\frac{4}{5}\left(x'^2-4x'y'+4y'^2\right)-\frac{4}{5}\left(2x'^2-3x'y'-2y'^2\right)+\frac{1}{5}\left(4x'^2+4x'y'+y'^2\right)$$

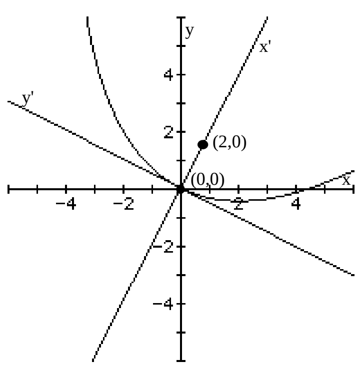
$$-8x'+16y'-32x'-16y'=0$$

$$\frac{4}{5}x'^2-\frac{16}{5}x'y'+\frac{16}{5}y'^2-\frac{8}{5}x'^2+\frac{12}{5}x'y'+\frac{8}{5}y'^2+\frac{4}{5}x'^2+\frac{4}{5}x'y'+\frac{1}{5}y'^2-40x'=0$$

$$5y'^2-40x'=0$$

$$y'^2=8x'$$

Parabola; vertex at the origin, focus at (2, 0).



28. $x^2 + 4xy + 4y^2 + 5\sqrt{5}\,y + 5 = 0$; $\theta = 63.4^\circ$ (see Problem 18)

$$\left(\frac{\sqrt{5}}{5}(x' - 2y')\right)^2 + 4\left(\frac{\sqrt{5}}{5}(x' - 2y')\right)\left(\frac{\sqrt{5}}{5}(2x' + y')\right) + 4\left(\frac{\sqrt{5}}{5}(2x' + y')\right)^2$$
$$+ 5\sqrt{5}\left(\frac{\sqrt{5}}{5}(2x' + y')\right) + 5 = 0$$

$$\frac{1}{5}\left(x'^2 - 4x'y' + 4y'^2\right) + \frac{4}{5}\left(2x'^2 - 3x'y' - 2y'^2\right) + \frac{4}{5}\left(4x'^2 + 4x'y' + y'^2\right)$$
$$+ 10x' + 5y' + 5 = 0$$

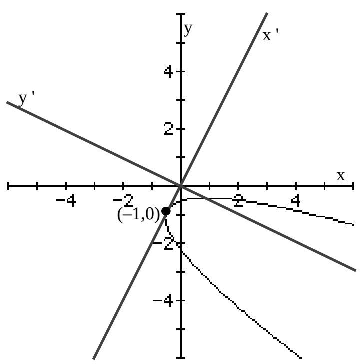
$$\frac{1}{5}x'^2 - \frac{4}{5}x'y' + \frac{4}{5}y'^2 + \frac{8}{5}x'^2 - \frac{12}{5}x'y' - \frac{8}{5}y'^2 + \frac{16}{5}x'^2 + \frac{16}{5}x'y' + \frac{4}{5}y'^2 + 10x' + 5y' + 5 = 0$$

$$5x'^2 + 10x' + 5y' + 5 = 0$$

$$x'^2 + 2x' + 1 = -y' - 1 + 1$$

$$y' = -(x' + 1)^2$$

Parabola; vertex at (–1, 0), axis of symmetry parallel to the y'-axis.



29. $25x^2 - 36xy + 40y^2 - 12\sqrt{13}x - 8\sqrt{13}y = 0$; $\theta \approx 33.7^\circ$ (see Problem 19)

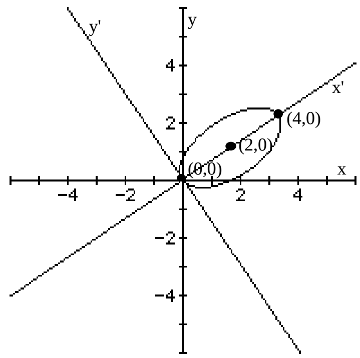
$$25\left(\frac{\sqrt{13}}{13}(3x' - 2y')\right)^2 - 36\left(\frac{\sqrt{13}}{13}(3x' - 2y')\right)\left(\frac{\sqrt{13}}{13}(2x' + 3y')\right) + 40\left(\frac{\sqrt{13}}{13}(2x' + 3y')\right)^2$$
$$-12\sqrt{13}\left(\frac{\sqrt{13}}{13}(3x' - 2y')\right) - 8\sqrt{13}\left(\frac{\sqrt{13}}{13}(2x' + 3y')\right) = 0$$

$$\frac{25}{13}\left(9x'^2 - 12x'y' + 4y'^2\right) - \frac{36}{13}\left(6x'^2 + 5x'y' - 6y'^2\right) + \frac{40}{13}\left(4x'^2 + 12x'y' + 9y'^2\right)$$
$$-36x' + 24y' - 16x' - 24y' = 0$$

$$\frac{225}{13}x'^2 - \frac{300}{13}x'y' + \frac{100}{13}y'^2 - \frac{216}{13}x'^2 - \frac{180}{13}x'y' + \frac{216}{13}y'^2$$
$$+ \frac{160}{13}x'^2 + \frac{480}{13}x'y' + \frac{360}{13}y'^2 - 52x' = 0$$

$$13x'^2 + 52y'^2 - 52x' = 0$$
$$x'^2 - 4x' + 4y'^2 = 0$$
$$(x' - 2)^2 + 4y'^2 = 4$$
$$\frac{(x' - 2)^2}{4} + \frac{y'^2}{1} = 1$$

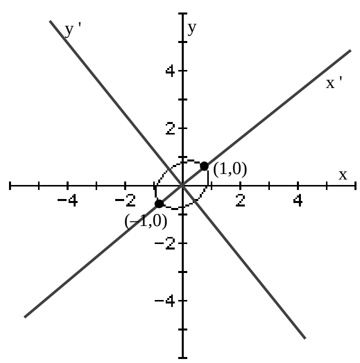
Ellipse; center at (2, 0), major axis is the x'-axis, vertices at (4, 0) and (0, 0).



30. $34x^2 - 24xy + 41y^2 - 25 = 0$; $\theta \approx 36.9^\circ$ (see Problem 20)

$$34\left(\frac{1}{5}(4x' - 3y')\right)^2 - 24\left(\frac{1}{5}(4x' - 3y')\right)\left(\frac{1}{5}(3x' + 4y')\right) + 41\left(\frac{1}{5}(3x' + 4y')\right)^2 - 25 = 0$$
$$\frac{34}{25}\left(16x'^2 - 24x'y' + 9y'^2\right) - \frac{24}{25}\left(12x'^2 + 7x'y' - 12y'^2\right) + \frac{41}{25}\left(9x'^2 + 24x'y' + 16y'^2\right) = 25$$
$$\frac{544}{25}x'^2 - \frac{816}{25}x'y' + \frac{306}{25}y'^2 - \frac{288}{25}x'^2 - \frac{168}{25}x'y' + \frac{288}{25}y'^2$$
$$+ \frac{369}{25}x'^2 + \frac{984}{25}x'y' + \frac{656}{25}y'^2 = 25$$
$$25x'^2 + 50y'^2 = 25$$
$$x'^2 + 2y'^2 = 1$$

Ellipse; center at the origin, major axis is the x'-axis, vertices at (±1, 0).



31. $16x^2 + 24xy + 9y^2 - 130x + 90y = 0$

$A = 16, B = 24, \text{ and } C = 9;\ \cot(2\theta) = \dfrac{A - C}{B} = \dfrac{16 - 9}{24} = \dfrac{7}{24} \quad \to \quad \cos(2\theta) = \dfrac{7}{25}$

$\sin\theta = \sqrt{\dfrac{\left(1 - \frac{7}{25}\right)}{2}} = \sqrt{\dfrac{9}{25}} = \dfrac{3}{5};\quad \cos\theta = \sqrt{\dfrac{\left(1 + \frac{7}{25}\right)}{2}} = \sqrt{\dfrac{16}{25}} = \dfrac{4}{5} \quad \to \quad \theta \approx 36.9$

$x = x'\cos\theta - y'\sin\theta = \dfrac{4}{5}x' - \dfrac{3}{5}y' = \dfrac{1}{5}(4x' - 3y')$

$y = x'\sin\theta + y'\cos\theta = \dfrac{3}{5}x' + \dfrac{4}{5}y' = \dfrac{1}{5}(3x' + 4y')$

$$16\left(\frac{1}{5}(4x' - 3y')\right)^2 + 24\left(\frac{1}{5}(4x' - 3y')\right)\left(\frac{1}{5}(3x' + 4y')\right) + 9\left(\frac{1}{5}(3x' + 4y')\right)^2$$
$$-130\left(\frac{1}{5}(4x' - 3y')\right) + 90\left(\frac{1}{5}(3x' + 4y')\right) = 0$$

$$\frac{16}{25}\left(16x'^2 - 24x'y' + 9y'^2\right) + \frac{24}{25}\left(12x'^2 + 7x'y' - 12y'^2\right) + \frac{9}{25}\left(9x'^2 + 24x'y' + 16y'^2\right)$$
$$-104x' + 78y' + 54x' + 72y' = 0$$

$$\frac{256}{25}x'^2 - \frac{384}{25}x'y' + \frac{144}{25}y'^2 + \frac{288}{25}x'^2 + \frac{168}{25}x'y' - \frac{288}{25}y'^2$$
$$+ \frac{81}{25}x'^2 + \frac{216}{25}x'y' + \frac{144}{25}y'^2 - 50x' + 150y' = 0$$

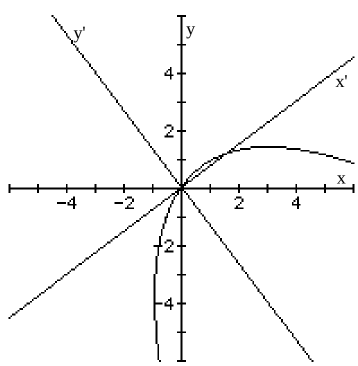
$25x'^2 - 50x' + 150y' = 0$

$x'^2 - 2x' = -6y'$

$(x' - 1)^2 = -6y' + 1$

$(x' - 1)^2 = -6\left(y' - \dfrac{1}{6}\right)$

Parabola; vertex at $\left(1, \dfrac{1}{6}\right)$, focus at $\left(1, -\dfrac{4}{3}\right)$.



32. $16x^2 + 24xy + 9y^2 - 60x + 80y = 0$

$A = 16, B = 24, \text{and } C = 9;\ \cot(2\theta) = \dfrac{A - C}{B} = \dfrac{16 - 9}{24} = \dfrac{7}{24} \ \rightarrow\ \cos(2\theta) = \dfrac{7}{25}$

$\sin\theta = \sqrt{\dfrac{\left(1 - \frac{7}{25}\right)}{2}} = \sqrt{\dfrac{9}{25}} = \dfrac{3}{5};\quad \cos\theta = \sqrt{\dfrac{\left(1 + \frac{7}{25}\right)}{2}} = \sqrt{\dfrac{16}{25}} = \dfrac{4}{5} \ \rightarrow\ \theta \approx 36.9$

$x = x'\cos\theta - y'\sin\theta = \dfrac{4}{5}x' - \dfrac{3}{5}y' = \dfrac{1}{5}(4x' - 3y')$

$y = x'\sin\theta + y'\cos\theta = \dfrac{3}{5}x' + \dfrac{4}{5}y' = \dfrac{1}{5}(3x' + 4y')$

$$16\left(\frac{1}{5}(4x' - 3y')\right)^2 + 24\left(\frac{1}{5}(4x' - 3y')\right)\left(\frac{1}{5}(3x' + 4y')\right) + 9\left(\frac{1}{5}(3x' + 4y')\right)^2$$
$$- 60\left(\frac{1}{5}(4x' - 3y')\right) + 80\left(\frac{1}{5}(3x' + 4y')\right) = 0$$

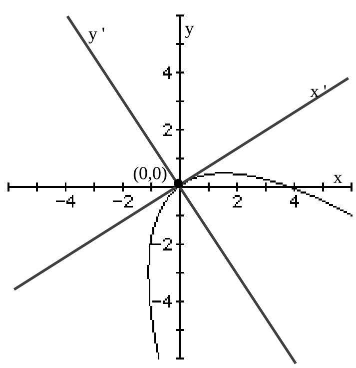
$$\frac{16}{25}\left(16x'^2 - 24x'y' + 9y'^2\right) + \frac{24}{25}\left(12x'^2 + 7x'y' - 12y'^2\right) + \frac{9}{25}\left(9x'^2 + 24x'y' + 16y'^2\right)$$
$$- 48x' + 36y' + 48x' + 64y' = 0$$

$$\frac{256}{25}x'^2 - \frac{384}{25}x'y' + \frac{144}{25}y'^2 + \frac{288}{25}x'^2 + \frac{168}{25}x'y' - \frac{288}{25}y'^2$$
$$+ \frac{81}{25}x'^2 + \frac{216}{25}x'y' + \frac{144}{25}y'^2 + 100y' = 0$$

$25x'^2 + 100y' = 0$

$x'^2 = -4y'$

Parabola; vertex at (0, 0), focus at (0, –1).



33. $A = 1.\ B = 3,\ C = -2 \qquad B^2 - 4AC = 3^2 - 4(1)(-2) = 17 > 0;$ hyperbola

34. $A = 2.\ B = -3,\ C = 4 \qquad B^2 - 4AC = (-3)^2 - 4(2)(4) = -23 < 0;$ ellipse

35. $A = 1.\ B = -7,\ C = 3 \qquad B^2 - 4AC = (-7)^2 - 4(1)(3) = 37 > 0;$ hyperbola

36. $A = 2.\ B = -3,\ C = 2 \qquad B^2 - 4AC = (-3)^2 - 4(2)(2) = -7 < 0;$ ellipse

37. $A = 9.\ B = 12,\ C = 4 \qquad B^2 - 4AC = 12^2 - 4(9)(4) = 0;$ parabola

38. $A=10.\ B=12,\ C=4 \qquad B^2-4AC=12^2-4(10)(4)=-16<0$; ellipse

39. $A=10.\ B=-12,\ C=4 \qquad B^2-4AC=(-12)^2-4(10)(4)=-16<0$; ellipse

40. $A=4.\ B=12,\ C=9 \qquad B^2-4AC=12^2-4(4)(9)=0$; parabola

41. $A=3.\ B=-2,\ C=1 \qquad B^2-4AC=(-2)^2-4(3)(1)=-8<0$; ellipse

42. $A=3.\ B=2,\ C=1 \qquad B^2-4AC=2^2-4(3)(1)=-8<0$; ellipse

43. See equation 6 on page 678.

$A' = A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta$
$B' = B(\cos^2\theta - \sin^2\theta) + 2(C-A)(\sin\theta\cos\theta)$
$C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta$
$D' = D\cos\theta + E\sin\theta$
$E' = -D\sin\theta + E\cos\theta$
$F' = F$

44. $$A'+C' = \left(A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta\right) + \left(A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta\right)$$
$$= A\left(\cos^2\theta + \sin^2\theta\right) + C\left(\sin^2\theta + \cos^2\theta\right) = A(1) + C(1) = A + C$$

45. $$B'^2 - 4A'C'$$
$$= \left[B(\cos^2\theta - \sin^2\theta) + 2(C-A)\sin\theta\cos\theta\right]^2$$
$$- 4\left(A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta\right)\left(A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta\right)$$
$$= B^2\left(\cos^4\theta - 2\cos^2\theta\sin^2\theta + \sin^4\theta\right) + 4B(C-A)\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$$
$$+ 4(C-A)^2\sin^2\theta\cos^2\theta - 4\left[A^2\sin^2\theta\cos^2\theta - AB\sin\theta\cos^3\theta + AC\cos^4\theta\right.$$
$$+ AB\sin^3\theta\cos\theta - B^2\sin^2\theta\cos^2\theta + BC\sin\theta\cos^3\theta + AC\sin^4\theta$$
$$\left. - BC\sin^3\theta\cos\theta + C^2\sin^2\theta\cos^2\theta\right]$$
$$= B^2\left(\cos^4\theta - 2\cos^2\theta\sin^2\theta + \sin^4\theta + 4\sin^2\theta\cos^2\theta\right)$$
$$+ BC\left(4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) - 4\sin\theta\cos^3\theta + 4\sin^3\theta\cos\theta\right)$$
$$- AB\left(4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) - 4\sin\theta\cos^3\theta + 4\sin^3\theta\cos\theta\right)$$
$$+ 4C^2\left(\sin^2\theta\cos^2\theta - \sin^2\theta\cos^2\theta\right) - 4AC\left(2\sin^2\theta\cos^2\theta + \cos^4\theta + \sin^4\theta\right)$$
$$+ 4A^2\left(\sin^2\theta\cos^2\theta - \sin^2\theta\cos^2\theta\right)$$
$$= B^2\left(\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta\right) - 4AC\left(\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta\right)$$
$$= B^2\left(\cos^2\theta + \sin^2\theta\right)^2 - 4AC\left(\cos^2\theta + \sin^2\theta\right)^2 = B^2 - 4AC$$

46. Since $B^2 - 4AC = B'^2 - 4A'C'$ for any rotation θ (Problem 45), choose θ so that $B' = 0$. Then $B^2 - 4AC = -4A'C'$.

(a) If $B^2 - 4AC = -4A'C' = 0$ then $A'C' = 0$. Using the theorem for identifying conics without completing the square, the equation is a parabola.

(b) If $B^2 - 4AC = -4A'C' < 0$ then $A'C' > 0$. Thus, the equation is an ellipse (or circle).

(c) If $B^2 - 4AC = -4A'C' > 0$ then $A'C' < 0$. Thus, the equation is a hyperbola.

47.
$$\begin{aligned}
d^2 &= (y_2 - y_1)^2 + (x_2 - x_1)^2 \\
&= \left(x_2'\sin\theta + y_2'\cos\theta - x_1'\sin\theta - y_1'\cos\theta\right)^2 \\
&\qquad + \left(x_2'\cos\theta - y_2'\sin\theta - x_1'\cos\theta + y_1'\sin\theta\right)^2 \\
&= \left(\left(x_2' - x_1'\right)\sin\theta + \left(y_2' - y_1'\right)\cos\theta\right)^2 + \left(\left(x_2' - x_1'\right)\cos\theta - \left(y_2' - y_1'\right)\sin\theta\right)^2 \\
&= \left(x_2' - x_1'\right)^2\sin^2\theta + 2\left(x_2' - x_1'\right)\left(y_2' - y_1'\right)\sin\theta\cos\theta + \left(y_2' - y_1'\right)^2\cos^2\theta \\
&\qquad + \left(x_2' - x_1'\right)^2\cos^2\theta - 2\left(x_2' - x_1'\right)\left(y_2' - y_1'\right)\sin\theta\cos\theta + \left(y_2' - y_1'\right)^2\sin^2\theta \\
&= \left(x_2' - x_1'\right)^2\sin^2\theta + \left(x_2' - x_1'\right)^2\cos^2\theta + \left(y_2' - y_1'\right)^2\cos^2\theta + \left(y_2' - y_1'\right)^2\sin^2\theta \\
&= \left(x_2' - x_1'\right)^2 + \left(y_2' - y_1'\right)^2
\end{aligned}$$

48.
$$\begin{aligned}
x^{1/2} + y^{1/2} &= a^{1/2} \\
y^{1/2} &= a^{1/2} - x^{1/2} \\
y &= \left(a^{1/2} - x^{1/2}\right)^2 \\
y &= a - 2a^{1/2}x^{1/2} + x \\
2a^{1/2}x^{1/2} &= (a + x) - y \\
4ax &= (a + x)^2 - 2y(a + x) + y^2 \\
4ax &= a^2 + 2ax + x^2 - 2ay - 2xy + y^2 \\
0 &= x^2 - 2xy + y^2 - 2ax - 2ay + a^2
\end{aligned}$$

$B^2 - 4AC = (-2)^2 - 4(1)(1) = 4 - 4 = 0$

The equation is a parabola.