Analytic Geometry

11.6 Polar Equations of Conics

1. $e=1$; $p=1$; parabola; directrix is perpendicular to the polar axis and 1 unit to the right of the pole.

2. $e=1$; $p=3$; parabola; directrix is parallel to the polar axis and 3 units below the pole.

3. $r=\dfrac{4}{2-3\sin\theta}=\dfrac{4}{2\left(1-\frac{3}{2}\sin\theta\right)}=\dfrac{2}{\left(1-\frac{3}{2}\sin\theta\right)}$; $ep=2$, $e=\dfrac{3}{2}$; $p=\dfrac{4}{3}$

   Hyperbola; directrix is parallel to the polar axis and $\frac{4}{3}$ units below the pole.

4. $r=\dfrac{2}{1+2\cos\theta}$; $ep=2$, $e=2$; $p=1$

   Hyperbola; directrix is perpendicular to the polar axis and 1 unit to the right of the pole.

5. $r=\dfrac{3}{4-2\cos\theta}=\dfrac{3}{4\left(1-\frac{1}{2}\cos\theta\right)}=\dfrac{\left(\frac{3}{4}\right)}{\left(1-\frac{1}{2}\cos\theta\right)}$; $ep=\dfrac{3}{4}$, $e=\dfrac{1}{2}$; $p=\dfrac{3}{2}$

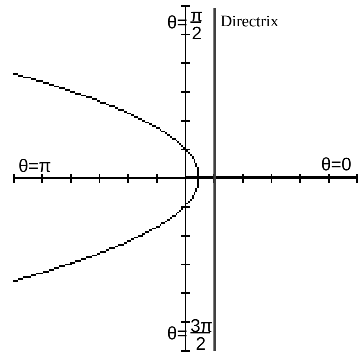
   Ellipse; directrix is perpendicular to the polar axis and $\frac{3}{2}$ units to the left of the pole.

6. $r=\dfrac{6}{8+2\sin\theta}=\dfrac{6}{8\left(1+\frac{1}{4}\sin\theta\right)}=\dfrac{\left(\frac{3}{4}\right)}{\left(1+\frac{1}{4}\sin\theta\right)}$; $ep=\dfrac{3}{4}$, $e=\dfrac{1}{4}$; $p=3$

   Ellipse; directrix is parallel to the polar axis and 3 units above the pole.

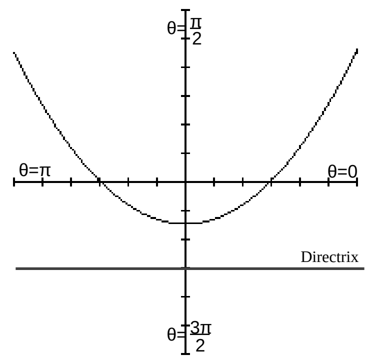
7. $r = \dfrac{1}{1+\cos\theta}$

$ep = 1,\ \ e = 1,\ \ p = 1$
Parabola; directrix is perpendicular to the polar axis 1 unit to the right of the pole; vertex is $\left(\dfrac{1}{2}, 0\right)$.



8. $r = \dfrac{3}{1-\sin\theta}$

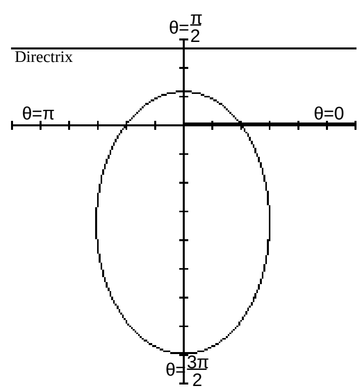
$ep = 3,\ \ e = 1,\ \ p = 3$
Parabola; directrix is parallel to the polar axis 3 units below the pole; vertex is $\left(\dfrac{3}{2}, \dfrac{3\pi}{2}\right)$.



9. $r = \dfrac{8}{4+3\sin\theta}$

$$r = \frac{8}{4\left(1+\frac{3}{4}\sin\theta\right)} = \frac{2}{\left(1+\frac{3}{4}\sin\theta\right)}$$

$ep = 2,\ \ e = \dfrac{3}{4},\ \ p = \dfrac{8}{3}$
Ellipse; directrix is parallel to the polar axis $\dfrac{8}{3}$ units above the pole; vertices are $\left(\dfrac{8}{7}, \dfrac{\pi}{2}\right)$ and $\left(8, \dfrac{3\pi}{2}\right)$.

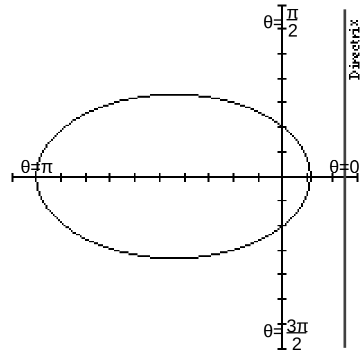


10. $r = \dfrac{10}{5+4\cos\theta}$

$r = \dfrac{10}{5\left(1+\frac{4}{5}\cos\theta\right)} = \dfrac{2}{1+\frac{4}{5}\cos\theta}$

$ep = 2,\ \ e = \dfrac{4}{5},\ \ p = \dfrac{5}{2}$

Ellipse; directrix is perpendicular to the polar axis $\dfrac{5}{2}$ units to the right of the pole; vertices are $\left(\dfrac{10}{9}, 0\right)$ and $(10, \pi)$.

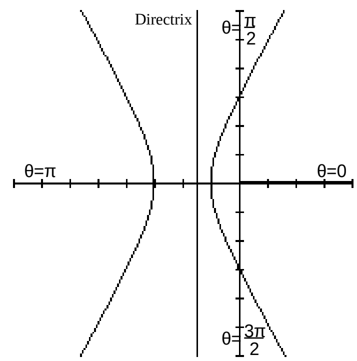


11. $r = \dfrac{9}{3-6\cos\theta}$

$r = \dfrac{9}{3(1-2\cos\theta)} = \dfrac{3}{1-2\cos\theta}$

$ep = 3,\ \ e = 2,\ \ p = \dfrac{3}{2}$

Hyperbola; directrix is perpendicular to the polar axis $\dfrac{3}{2}$ units to the left of the pole; vertices are $(-3, 0)$ and $(1, \pi)$.

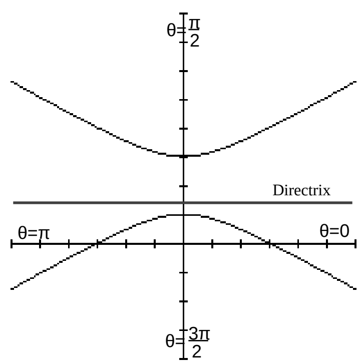


12. $r = \dfrac{12}{4+8\sin\theta}$

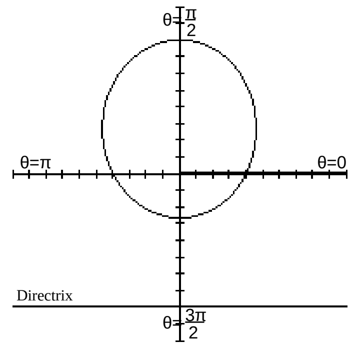
$r = \dfrac{12}{4(1+2\sin\theta)} = \dfrac{3}{1+2\sin\theta}$

$ep = 3,\ \ e = 2,\ \ p = \dfrac{3}{2}$

Hyperbola; directrix is parallel to the polar axis $\dfrac{3}{2}$ units above the pole; vertices are $\left(1, \dfrac{\pi}{2}\right)$ and $\left(-3, \dfrac{3\pi}{2}\right)$.



13. $r = \dfrac{8}{2 - \sin\theta}$

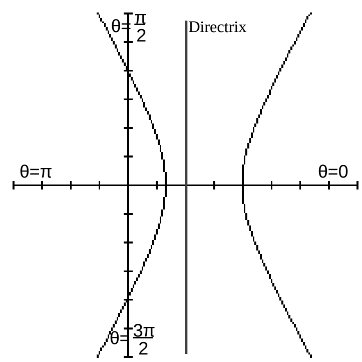


$$r = \frac{8}{2\left(1 - \frac{1}{2}\sin\theta\right)} = \frac{4}{1 - \frac{1}{2}\sin\theta}$$

$ep = 4,\ \ e = \dfrac{1}{2},\ \ p = 8$

Ellipse; directrix is parallel to the polar axis 8 units below the pole; vertices are $\left(8, \dfrac{\pi}{2}\right)$ and $\left(\dfrac{8}{3}, \dfrac{3\pi}{2}\right)$.

14. $r = \dfrac{8}{2 + 4\cos\theta}$

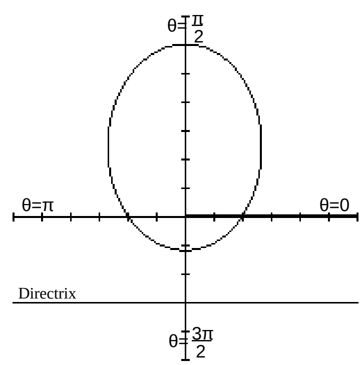


$$r = \frac{8}{2(1 + 2\cos\theta)} = \frac{4}{1 + 2\cos\theta}$$

$ep = 4,\ \ e = 2,\ \ p = 2$

Hyperbola; directrix is perpendicular to the polar axis 2 units to the right of the pole; vertices are $\left(\dfrac{4}{3}, 0\right)$ and $(-4, \pi)$.

15. $r(3 - 2\sin\theta) = 6 \quad \to \quad r = \dfrac{6}{3 - 2\sin\theta}$



$$r = \frac{6}{3\left(1 - \frac{2}{3}\sin\theta\right)} = \frac{2}{1 - \frac{2}{3}\sin\theta}$$

$ep = 2,\ \ e = \dfrac{2}{3},\ \ p = 3$

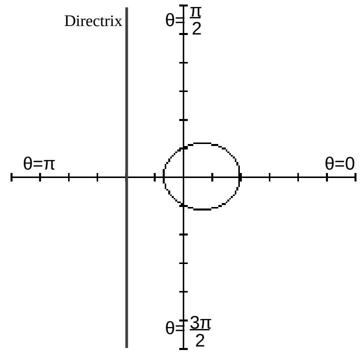
Ellipse; directrix is parallel to the polar axis 3 units below the pole; vertices are $\left(6, \dfrac{\pi}{2}\right)$ and $\left(\dfrac{6}{5}, \dfrac{3\pi}{2}\right)$.

16. $r(2-\cos\theta) = 2 \quad \rightarrow \quad r = \dfrac{2}{2-\cos\theta}$

$$r = \frac{2}{2\left(1-\frac{1}{2}\cos\theta\right)} = \frac{1}{\left(1-\frac{1}{2}\cos\theta\right)}$$

$ep = 1,\ e = \dfrac{1}{2},\ p = 2$

Ellipse; directrix is perpendicular to the polar axis 2 units to the left of the pole; vertices are $(2,\ 0)$ and $\left(\frac{2}{3},\ \pi\right)$.

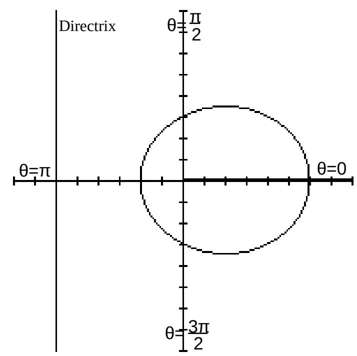


17. $r = \dfrac{6\sec\theta}{2\sec\theta - 1} = \dfrac{6}{2-\cos\theta}$

$$r = \frac{6}{2\left(1-\frac{1}{2}\cos\theta\right)} = \frac{3}{1-\frac{1}{2}\cos\theta}$$

$ep = 3,\ e = \dfrac{1}{2},\ p = 6$

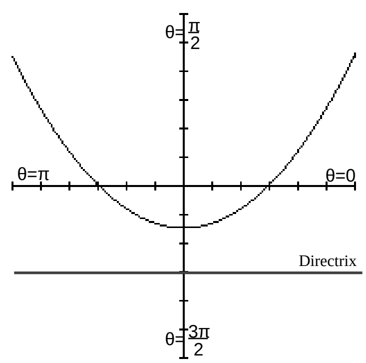
Ellipse; directrix is perpendicular to the polar axis 6 units to the left of the pole; vertices are $(6,\ 0)$ and $(2,\ \pi)$.



18. $r = \dfrac{3\csc\theta}{\csc\theta - 1} = \dfrac{3}{1-\sin\theta}$

$ep = 3,\ e = 1,\ p = 3$

Parabola; directrix is parallel to the polar axis 3 units below the pole; vertex is $\left(\frac{3}{2},\ \frac{3\pi}{2}\right)$.



19.
$$\begin{aligned} r &= \frac{1}{1+\cos\theta} \\ r + r\cos\theta &= 1 \\ r &= 1 - r\cos\theta \\ r^2 &= (1 - r\cos\theta)^2 \\ x^2 + y^2 &= (1-x)^2 \\ x^2 + y^2 &= 1 - 2x + x^2 \\ y^2 + 2x - 1 &= 0 \end{aligned}$$

20.
$$\begin{aligned} r &= \frac{3}{1-\sin\theta} \\ r - r\sin\theta &= 3 \\ r &= 3 + r\sin\theta \\ r^2 &= (3 + r\sin\theta)^2 \\ x^2 + y^2 &= (3+y)^2 \\ x^2 + y^2 &= 9 + 6y + y^2 \\ x^2 - 6y - 9 &= 0 \end{aligned}$$

21.
$$
\begin{aligned}
r &= \frac{8}{4+3\sin\theta} \\
4r + 3r\sin\theta &= 8 \\
4r &= 8 - 3r\sin\theta \\
16r^2 &= (8 - 3r\sin\theta)^2 \\
16(x^2 + y^2) &= (8 - 3y)^2 \\
16x^2 + 16y^2 &= 64 - 48y + 9y^2 \\
16x^2 + 7y^2 + 48y - 64 &= 0
\end{aligned}
$$

22.
$$
\begin{aligned}
r &= \frac{10}{5+4\cos\theta} \\
5r + 4r\cos\theta &= 10 \\
5r &= 10 - 4r\cos\theta \\
25r^2 &= (10 - 4r\cos\theta)^2 \\
25(x^2 + y^2) &= (10 - 4x)^2 \\
25x^2 + 25y^2 &= 100 - 80x + 16x^2 \\
9x^2 + 25y^2 + 80x - 100 &= 0
\end{aligned}
$$

23.
$$
\begin{aligned}
r &= \frac{9}{3-6\cos\theta} \\
3r - 6r\cos\theta &= 9 \\
3r &= 9 + 6r\cos\theta \\
r &= 3 + 2r\cos\theta \\
r^2 &= (3 + 2r\cos\theta)^2 \\
x^2 + y^2 &= (3 + 2x)^2 \\
x^2 + y^2 &= 9 + 12x + 4x^2 \\
3x^2 - y^2 + 12x + 9 &= 0
\end{aligned}
$$

24.
$$
\begin{aligned}
r &= \frac{12}{4+8\sin\theta} \\
4r + 8r\sin\theta &= 12 \\
4r &= 12 - 8r\sin\theta \\
r &= 3 - 2r\sin\theta \\
r^2 &= (3 - 2r\sin\theta)^2 \\
x^2 + y^2 &= (3 - 2y)^2 \\
x^2 + y^2 &= 9 - 12y + 4y^2 \\
x^2 - 3y^2 + 12y - 9 &= 0
\end{aligned}
$$

25.
$$
\begin{aligned}
r &= \frac{8}{2-\sin\theta} \\
2r - r\sin\theta &= 8 \\
2r &= 8 + r\sin\theta \\
4r^2 &= (8 + r\sin\theta)^2 \\
4(x^2 + y^2) &= (8 + y)^2 \\
4x^2 + 4y^2 &= 64 + 16y + y^2 \\
4x^2 + 3y^2 - 16y - 64 &= 0
\end{aligned}
$$

26.
$$
\begin{aligned}
r &= \frac{8}{2+4\cos\theta} \\
2r + 4r\cos\theta &= 8 \\
2r &= 8 - 4r\cos\theta \\
r &= 4 - 2r\cos\theta \\
r^2 &= (4 - 2r\cos\theta)^2 \\
x^2 + y^2 &= (4 - 2x)^2 \\
x^2 + y^2 &= 16 - 16x + 4x^2 \\
3x^2 - y^2 - 16x + 16 &= 0
\end{aligned}
$$

27.
$$
\begin{aligned}
r(3 - 2\sin\theta) &= 6 \\
3r - 2r\sin\theta &= 6 \\
3r &= 6 + 2r\sin\theta \\
9r^2 &= (6 + 2r\sin\theta)^2 \\
9(x^2 + y^2) &= (6 + 2y)^2 \\
9x^2 + 9y^2 &= 36 + 24y + 4y^2 \\
9x^2 + 5y^2 - 24y - 36 &= 0
\end{aligned}
$$

28.
$$
\begin{aligned}
r(2 - \cos\theta) &= 2 \\
2r - r\cos\theta &= 2 \\
2r &= 2 + r\cos\theta \\
4r^2 &= (2 + r\cos\theta)^2 \\
4(x^2 + y^2) &= (2 + x)^2 \\
4x^2 + 4y^2 &= 4 + 4x + x^2 \\
3x^2 + 4y^2 - 4x - 4 &= 0
\end{aligned}
$$

29.
$$r=\frac{6\sec\theta}{2\sec\theta-1}$$
$$r=\frac{6}{2-\cos\theta}$$
$$2r-r\cos\theta=6$$
$$2r=6+r\cos\theta$$
$$4r^2=(6+r\cos\theta)^2$$
$$4(x^2+y^2)=(6+x)^2$$
$$4x^2+4y^2=36+12x+x^2$$
$$3x^2+4y^2-12x-36=0$$

30.
$$r=\frac{3\csc\theta}{\csc\theta-1}$$
$$r=\frac{3}{1-\sin\theta}$$
$$r-r\sin\theta=3$$
$$r=3+r\sin\theta$$
$$r^2=(3+r\sin\theta)^2$$
$$x^2+y^2=(3+y)^2$$
$$x^2+y^2=9+6y+y^2$$
$$x^2-6y-9=0$$

31. $r=\dfrac{ep}{1+e\sin\theta}$

$e=1;\quad p=1$

$$r=\frac{1}{1+\sin\theta}$$

32. $r=\dfrac{ep}{1-e\sin\theta}$

$e=1;\quad p=2$

$$r=\frac{2}{1-\sin\theta}$$

33. $r=\dfrac{ep}{1-e\cos\theta}$

$e=\dfrac{4}{5};\quad p=3$

$$r=\frac{\left(\frac{12}{5}\right)}{\left(1-\frac{4}{5}\cos\theta\right)}=\frac{12}{5-4\cos\theta}$$

34. $r=\dfrac{ep}{1+e\sin\theta}$

$e=\dfrac{2}{3};\quad p=3$

$$r=\frac{2}{\left(1+\frac{2}{3}\sin\theta\right)}=\frac{6}{3+2\sin\theta}$$

35. $r=\dfrac{ep}{1-e\sin\theta}$

$e=6;\quad p=2$

$$r=\frac{12}{1-6\sin\theta}$$

36. $r=\dfrac{ep}{1+e\cos\theta}$

$e=5;\quad p=5$

$$r=\frac{25}{1+5\cos\theta}$$

37. $d(F,P)=e\cdot d(D,P)\qquad d(D,P)=p-r\cos\theta$
$$r=e(p-r\cos\theta)$$
$$r=ep-er\cos\theta$$
$$r+er\cos\theta=ep\rightarrow r(1+e\cos\theta)=ep\rightarrow r=\frac{ep}{1+e\cos\theta}$$

38. $d(F,P)=e\cdot d(D,P)\qquad d(D,P)=p-r\sin\theta$
$$r=e(p-r\sin\theta)$$
$$r=ep-er\sin\theta$$
$$r+er\sin\theta=ep\rightarrow r(1+e\sin\theta)=ep\rightarrow r=\frac{ep}{1+e\sin\theta}$$

39. $d(F,\ P) = e\ \ d(D,\ P)$ $\qquad d(D,\ P) = p + r\sin\theta$

$$r = e(p + r\sin\theta)$$

$$r = ep + er\sin\theta$$

$$r - e\ r\sin\theta = ep \rightarrow r(1 - e\sin\theta) = ep \rightarrow r = \frac{ep}{1 - e\sin\theta}$$

40. $r = \dfrac{3.442\ \ 10^7}{1 - 0.206\cos\theta}$

At aphelion, the greatest distance from the sun, $\cos\theta = +1$.

$$r = \frac{(3.442)10^7}{1 - 0.206(1)} = \frac{(3.442)10^7}{0.794} \approx 4.335 \times 10^7 \text{ miles}$$

At perihelion, the shortest distance from the sun, $\cos\theta = -1$.

$$r = \frac{(3.442)10^7}{1 - 0.206(-1)} = \frac{(3.442)10^7}{1.206} \approx 2.854 \times 10^7 \text{ miles}$$