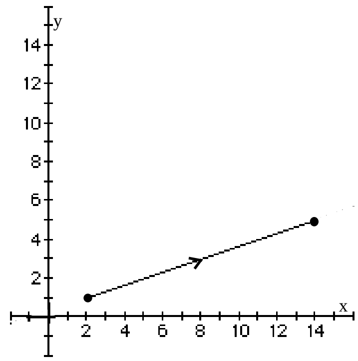
Chapter 11

Analytic Geometry

11.7 Plane Curves and Parametric Equations

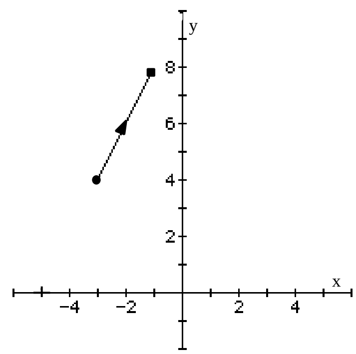
1. $x = 3t + 2, \quad y = t + 1, \quad 0 \le t \le 4$

$x = 3(y - 1) + 2$
$x = 3y - 3 + 2$
$x = 3y - 1$
$x - 3y + 1 = 0$



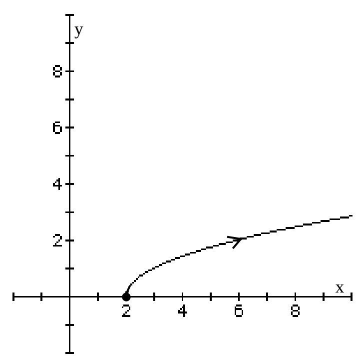
2. $x = t - 3, \quad y = 2t + 4, \quad 0 \le t \le 2$

$y = 2(x + 3) + 4$
$y = 2x + 6 + 4$
$y = 2x + 10$
$2x - y + 10 = 0$



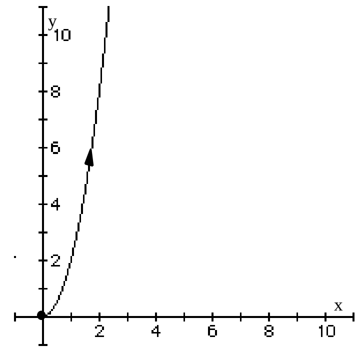
3. $x = t + 2, \quad y = \sqrt{t}, \quad t \ge 0$

$y = \sqrt{x - 2}$



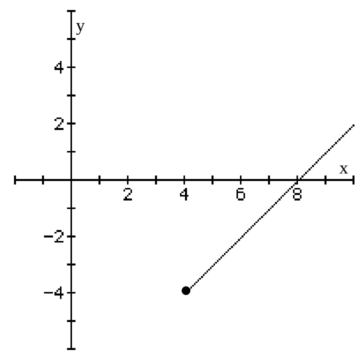
4. $x = \sqrt{2t}, \ \ y = 4t, \ \ t \geq 0$

$y = 4\left(\frac{x^2}{2}\right) = 2x^2$



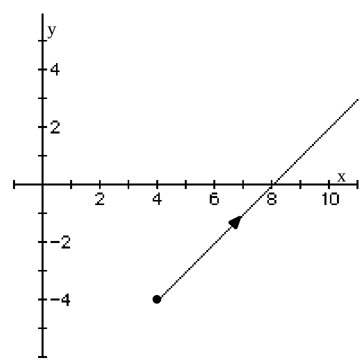
5. $x = t^2 + 4, \ \ y = t^2 - 4, \ \ -\infty < t < \infty$

$y = (x - 4) - 4$
$y = x - 8$
For $-\infty < t < 0$ the movement is to the left. For $0 < t < \infty$ the movement is to the right.



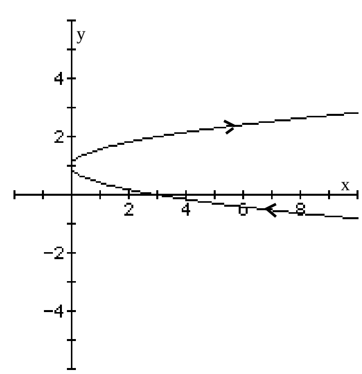
6. $x = \sqrt{t} + 4, \ \ y = \sqrt{t} - 4, \ \ t \geq 0$

$y = x - 4 - 4 = x - 8$



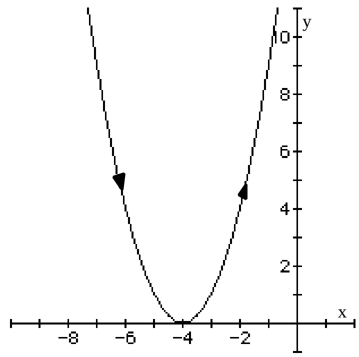
7. $x = 3t^2, \ \ y = t + 1, \ \ -\infty < t < \infty$

$x = 3(y - 1)^2$



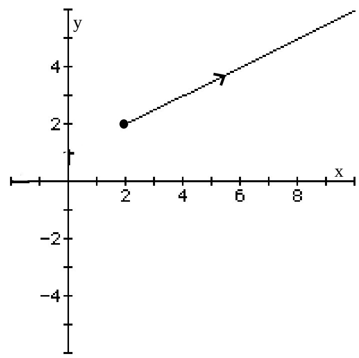
8. $x = 2t - 4,\quad y = 4t^2\ ,\ -\infty < t < \infty$

$$y = 4\left(\frac{x+4}{2}\right)^2 = (x+4)^2$$



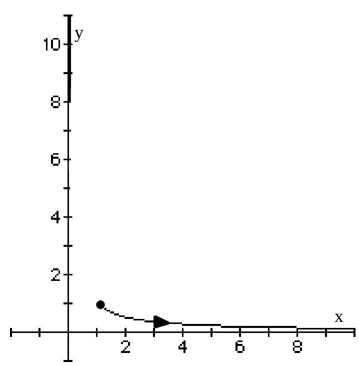
9. $x = 2e^t,\ \ y = 1 + e^t\ ,\ \ t \ge 0$

$$y = 1 + \frac{x}{2}$$
$$2y = 2 + x$$



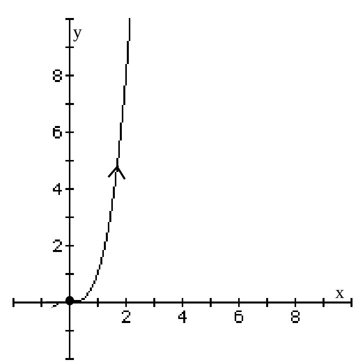
10. $x = e^t,\ \ y = e^{-t}\ ,\ \ t \ge 0$

$$y = x^{-1} = \frac{1}{x}$$



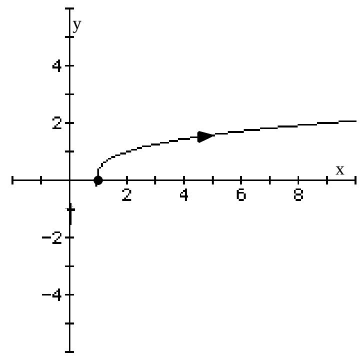
11. $x = \sqrt{t},\ \ y = t^{3/2}\ ,\ \ t \ge 0$

$$y = \left(x^2\right)^{3/2}$$
$$y = x^3$$



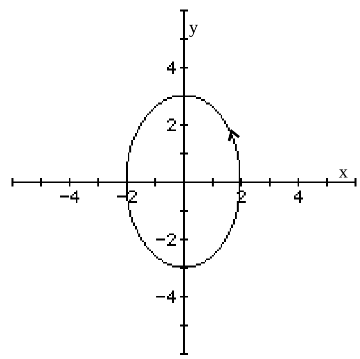
12. $x = t^{3/2} + 1, \; y = \sqrt{t}, \; t \geq 0$

$$x = \left(y^2\right)^{3/2} + 1$$
$$x = y^3 + 1$$



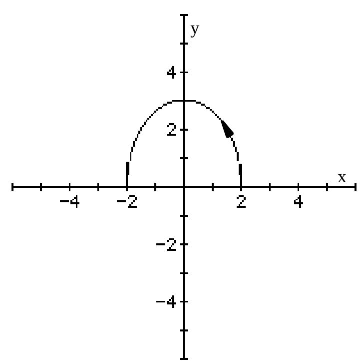
13. $x = 2\cos t, \; y = 3\sin t, \; 0 \leq t \leq 2\pi$

$$\frac{x}{2} = \cos t \qquad \frac{y}{3} = \sin t$$
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2 t + \sin^2 t = 1$$
$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$



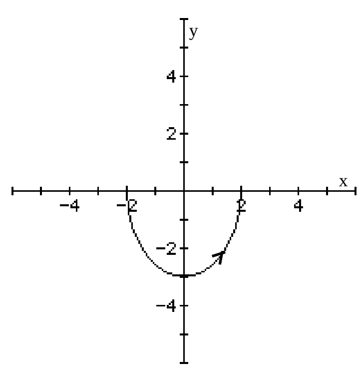
14. $x = 2\cos t, \; y = 3\sin t, \; 0 \leq t \leq \pi$

$$\frac{x}{2} = \cos t \qquad \frac{y}{3} = \sin t$$
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2 t + \sin^2 t = 1$$
$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \qquad y \geq 0$$



15. $x = 2\cos t, \; y = 3\sin t, \; -\pi \leq t \leq 0$

$$\frac{x}{2} = \cos t \qquad \frac{y}{3} = \sin t$$
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2 t + \sin^2 t = 1$$
$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

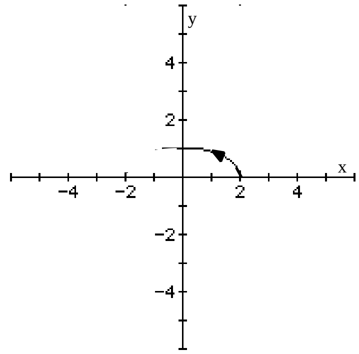


16. $x = 2\cos t,\ y = \sin t,\ 0 \le t \le \dfrac{\pi}{2}$

$\dfrac{x}{2} = \cos t \qquad y = \sin t$

$\left(\dfrac{x}{2}\right)^2 + (y)^2 = \cos^2 t + \sin^2 t = 1$

$\dfrac{x^2}{4} + y^2 = 1$

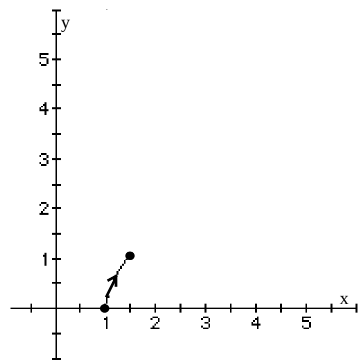


17. $x = \sec t,\ y = \tan t,\ 0 \le t \le \dfrac{\pi}{4}$

$\sec^2 t = 1 + \tan^2 t$

$x^2 = 1 + y^2$

$x^2 - y^2 = 1$

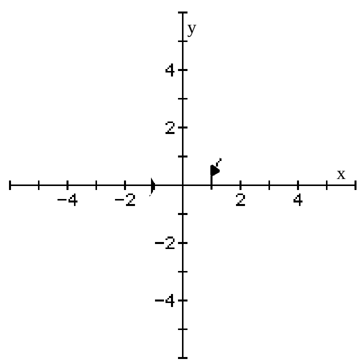


18. $x = \csc t,\ y = \cot t,\ \dfrac{\pi}{4} \le t \le \dfrac{\pi}{2}$

$\csc^2 t = 1 + \cot^2 t$

$x^2 = 1 + y^2$

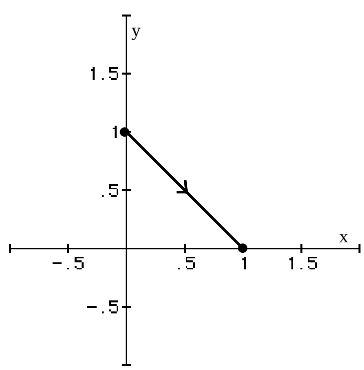
$x^2 - y^2 = 1$



19. $x = \sin^2 t,\ y = \cos^2 t,\ 0 \le t \le 2\pi$

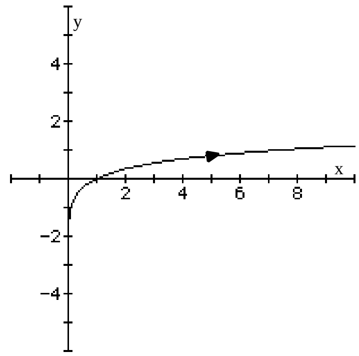
$\sin^2 t + \cos^2 t = 1$

$x + y = 1$



20. $x = t^2, \ y = \ln t, \ t > 0$

$$y = \ln \sqrt{x}$$
$$y = \frac{1}{2}\ln x$$



21. (a) Use equation (2):

$$x = 50\cos(90^{\circ}) = 0$$
$$y = -\frac{1}{2}(32)\,t^2 + (50\sin(90^{\circ}))\,t + 6 = -16t^2 + 50t + 6$$

(b) The ball is in the air until $y = 0$. Solve:

$$-16t^2 + 50t + 6 = 0$$
$$t = \frac{-50 \pm \sqrt{50^2 - 4(-16)(6)}}{2(-16)} = \frac{-50 \pm \sqrt{2884}}{-32} \approx -0.12 \text{ or } 3.24$$

The ball is in the air for about 3.24 seconds. (The negative solution is extraneous.)

(c) The maximum height occurs at the vertex of the quadratic function.

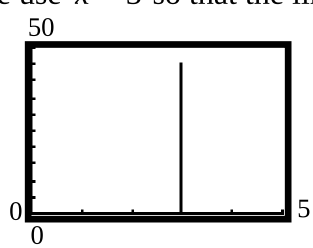
$$t = \frac{-b}{2a} = \frac{-50}{2(-16)} = 1.5625 \text{ seconds}$$

Evaluate the function to find the maximum height:

$$-16(1.5625)^2 + 50(1.5625) + 6 = 45.0625$$

The maximum height is 45.0625 feet.

(d) (We use $x = 3$ so that the line is not on top of the y-axis.)



22. (a) Use equation (2):

$$x = 40\cos(90^{\circ}) = 0$$
$$y = -\frac{1}{2}(32)\,t^2 + (40\sin(90^{\circ}))\,t + 5 = -16t^2 + 40t + 5$$

(b) The ball is in the air until $y = 0$. Solve:

$$-16t^2 + 40t + 5 = 0$$
$$t = \frac{-40 \pm \sqrt{40^2 - 4(-16)(5)}}{2(-16)} = \frac{-40 \pm \sqrt{1920}}{-32} \approx -0.12 \text{ or } 2.62$$

The ball is in the air for about 2.62 seconds. (The negative solution is extraneous.)

(c) The maximum height occurs at the vertex of the quadratic function.

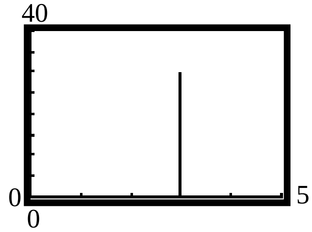
$$t = \frac{-b}{2a} = \frac{-40}{2(-16)} = 1.25 \text{ seconds}$$

Evaluate the function to find the maximum height:

$$-16(1.25)^2 + 40(1.25) + 5 = 30$$

The maximum height is 30 feet.

(d) (We use $x = 3$ so that the line is not on top of the y-axis.)



23. (a) Train: Use equation (2) with $g = 2$, $v_0 = 0$, $h = 0$

$$x_1 = \frac{1}{2}(2)t^2 + 0\ t + 0 = t^2$$
$$y_1 = 1$$

Bill:

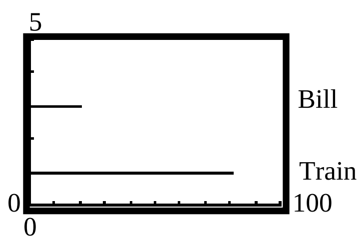
$$x_2 = 5(t - 5)$$
$$y_2 = 3$$

(b) Bill will catch the train if $x_1 = x_2$.

$$t^2 = 5(t - 5) \rightarrow t^2 = 5t - 25 \rightarrow t^2 - 5t + 25 = 0$$

Since $b^2 - 4ac = (-5)^2 - 4(1)(25) = 25 - 100 = -75 < 0$, the equation has no real solution. Thus, Bill will not catch the train.

(c)



24. (a) Bus: Use equation (2) with $g = 3$, $v_0 = 0$, $h = 0$

$$x_1 = \frac{1}{2}(3)t^2 + 0\ t + 0 = 1.5t^2$$
$$y_1 = 1$$

Jodi:

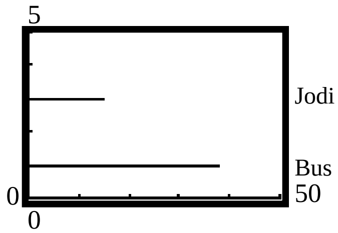
$$x_2 = 5(t - 2)$$
$$y_2 = 3$$

(b) Jodi will catch the bus if $x_1 = x_2$.

$$1.5t^2 = 5(t - 2) \rightarrow 1.5t^2 = 5t - 10 \rightarrow 1.5t^2 - 5t + 10 = 0$$

Since $b^2 - 4ac = (-5)^2 - 4(1.5)(10) = 25 - 60 = -35 < 0$, the equation has no real solution. Thus, Jodi will not catch the bus.

(c)



25. (a) Use equation (2):

$$x = (145\cos(20^\circ))\,t$$

$$y = -\frac{1}{2}(32)\,t^2 + (145\sin(20^\circ))\,t + 5$$

(b) The ball is in the air until $y = 0$. Solve:

$$-16t^2 + (145\sin 20^\circ)t + 5 = 0$$

$$t = \frac{-145\sin(20^\circ) \pm \sqrt{(145\sin(20^\circ))^2 - 4(-16)(5)}}{2(-16)}$$

$$\approx \frac{-49.59 \pm \sqrt{2779.46}}{-32} \approx -0.10 \text{ or } 3.20$$

The ball is in the air for about 3.20 seconds. (The negative solution is extraneous.)

(c) The maximum height occurs at the vertex of the quadratic function.

$$t = \frac{-b}{2a} = \frac{-145\sin(20^\circ)}{2(-16)} \approx 1.55 \text{ seconds}$$

Evaluate the function to find the maximum height:

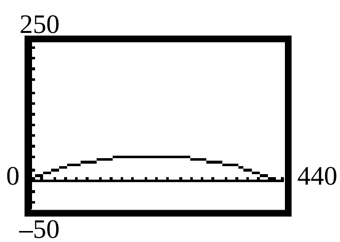
$$-16(1.55)^2 + 145\sin(20^\circ)(1.55) + 5 = 43.43$$

The maximum height is about 43.43 feet.

(d) Find the horizontal displacement:

$$x = (145\cos(20^\circ))(3.20) \approx 436 \text{ feet}$$

(e)



26. (a) Use equation (2):

$$x = (180\cos(40^\circ))\,t$$

$$y = -\frac{1}{2}(32)\,t^2 + (180\sin(40^\circ))\,t + 3$$

(b) The ball is in the air until $y = 0$. Solve:

$$-16t^2 + (180\sin(40^\circ))\,t + 3 = 0$$

$$t=\frac{-180\sin 40^\circ \pm\sqrt{\left(180\sin\left(40^\circ\right)\right)^2-4(-16)(3)}}{2(-16)}$$

$$\approx\frac{-115.7\pm\sqrt{13578.9}}{-32}\approx -0.03 \text{ or } 7.26$$

The ball is in the air for about 7.26 seconds. (The negative solution is extraneous.)

(c) The maximum height occurs at the vertex of the quadratic function.

$$t=\frac{-b}{2a}=\frac{-180\sin\left(40^\circ\right)}{2(-16)}\approx 3.62 \text{ seconds}$$

Evaluate the function to find the maximum height:

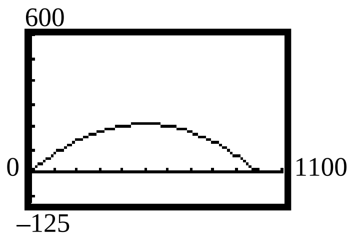
$$-16(3.62)^2+180\sin\left(40^\circ\right)(3.62)+3=212.17$$

The maximum height is about 212.17 feet.

(d) Find the horizontal displacement:

$$x=\left(180\cos\left(40^\circ\right)\right)(7.26)\approx 1001.1 \text{ feet}$$

(e)



27. (a) Use equation (2):

$$x=\left(40\cos\left(45^\circ\right)\right)t$$

$$y=-\frac{1}{2}(9.8)t^2+\left(40\sin\left(45^\circ\right)\right)t+300$$

(b) The ball is in the air until $y=0$. Solve:

$$-4.9t^2+\left(40\sin\left(45^\circ\right)\right)t+300=0$$

$$t=\frac{-20\sqrt{2}\pm\sqrt{\left(20\sqrt{2}\right)^2-4(-4.9)(300)}}{2(-4.9)}$$

$$=\frac{-20\sqrt{2}\pm\sqrt{6680}}{-9.8}\approx -5.45 \text{ or } 11.23$$

The ball is in the air for about 11.23 seconds. (The negative solution is extraneous.)

(c) The maximum height occurs at the vertex of the quadratic function.

$$t=\frac{-b}{2a}=\frac{-20\sqrt{2}}{2(-4.9)}\approx 2.89 \text{ seconds}$$

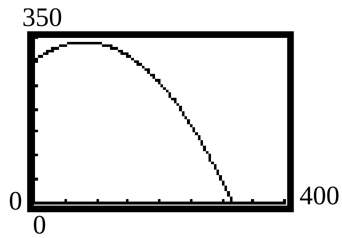
Evaluate the function to find the maximum height:

$$-4.9(2.89)^2+20\sqrt{2}(2.89)+300=340.8 \text{ meters}$$

(d) Find the horizontal displacement:

$$x=\left(40\cos\left(45^\circ\right)\right)(11.23)\approx 317.6 \text{ meters}$$

(e)



28. (a) Use equation (2):

$$x = (40\cos(45^\circ))\,t$$

$$y = -\frac{1}{2}\left(\frac{1}{6}\right)(9.8)t^2 + (40\sin(45^\circ))\,t + 300 = -\frac{9.8}{12}t^2 + (40\sin(45^\circ))\,t + 300$$

(b) The ball is in the air until $y = 0$. Solve:

$$-\frac{4.9}{6}t^2 + (40\sin(45^\circ))\,t + 300 = 0$$

$$t = \frac{-20\sqrt{2} \pm \sqrt{\left(20\sqrt{2}\right)^2 - 4\left(-\frac{4.9}{6}\right)(300)}}{2\left(-\frac{4.9}{6}\right)}$$

$$= \frac{-20\sqrt{2} \pm \sqrt{1780}}{\left(-\frac{4.9}{3}\right)} \approx -8.51 \text{ or } 43.15$$

The ball is in the air for about 43.15 seconds. (The negative solution is extraneous.)

(c) The maximum height occurs at the vertex of the quadratic function.

$$t = \frac{-b}{2a} = \frac{-20\sqrt{2}}{2\left(-\frac{4.9}{6}\right)} \approx 17.32 \text{ seconds}$$

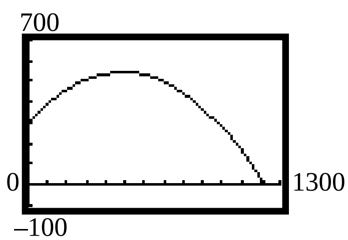
Evaluate the function to find the maximum height:

$$-\frac{4.9}{6}(17.32)^2 + 20\sqrt{2}(17.32) + 300 = 544.9 \text{ meters}$$

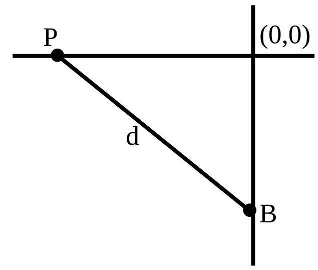
(d) Find the horizontal displacement:

$$x = (40\cos(45^\circ))(43.15) \approx 1220.5 \text{ meters}$$

(e)



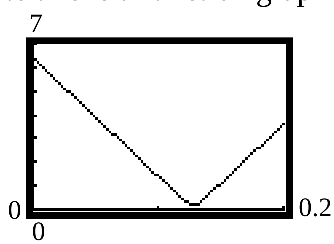
29\. (a) At $t = 0$, the Paseo is 5 miles from the intersection (at (0, 0)) traveling east (along the x-axis) at 40 mph. Thus, $x = 40t - 5$ describes the position of the Paseo as a function of time. The Bonneville, at $t = 0$, is 4 miles from the intersection traveling north (along the y-axis) at 30 mph. Thus, $y = 30t - 4$ describes the position of the Bonneville as a function of time.



Let d represent the distance between the cars. Use the Pythagorean Theorem to find the distance:

$$d = \sqrt{(40t - 5)^2 + (30t - 4)^2}$$

(b) (Note this is a function graph not a parametric graph.)

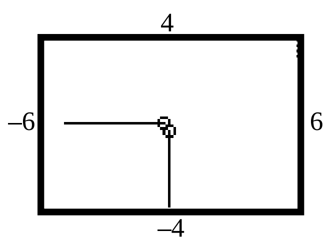


(c) The minimum distance between the cars is 0.2 miles and occurs at 0.128 seconds.

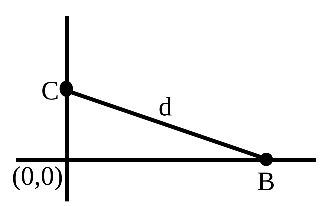
(d) From part (a):

Paseo: $x = 40t - 5$, $y = 0$ Bonneville: $x = 0$, $y = 30t - 4$

(e)



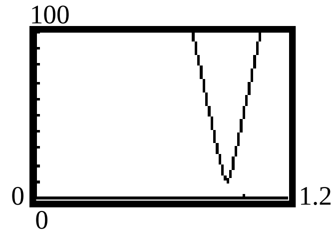
30\. (a) At $t = 0$, the Boeing 747 is 550 miles from the intersection (at (0, 0)) traveling west (along the x-axis) at 600 mph. Thus, $x = -600t + 550$ describes the position of the Boeing 747 as a function of time. The Cessna, at $t = 0$, is 100 miles from the intersection traveling south (along the y-axis) at 120 mph. Thus, $y = -120t + 100$ describes the position of the Cessna as a function of time.



Let d represent the distance between the planes. Use the Pythagorean Theorem to find the distance:

$$d = \sqrt{(-600t + 550)^2 + (-120t + 100)^2}$$

(b) (Note this is a function graph not a parametric graph.)

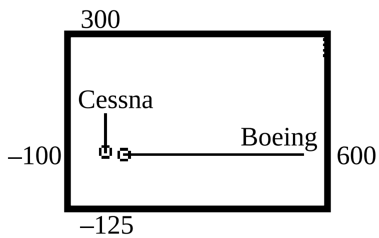


(c) The minimum distance between the planes is 9.8 miles and occurs at 0.91 hours.

(d) From part (a):

Boeing 747: $x = -600t + 550$, $y = 0$ Cessna: $x = 0$, $y = -120t + 100$

(e)



(The axes were turned off to get this graph.)

31. $x = t,\ y = 4t - 1$ $x = t + 1,\ y = 4t + 3$

32. $x = t,\ y = -8t + 3$ $x = t + 1,\ y = -8t - 5$

33. $x = t,\ y = t^2 + 1$ $x = t - 1,\ y = t^2 - 2t + 2$

34. $x = t,\ y = -2t^2 + 1$ $x = t - 1,\ y = -2t^2 + 4t - 1$

35. $x = t,\ y = t^3$ $x = \sqrt[3]{t},\ y = t$

36. $x = t,\ y = t^4 + 1$ $x = t^2,\ y = t^2 + 1$

37. $x = t^{3/2},\ y = t$ $x = t,\ y = t^{2/}$

38. $x = \sqrt{t},\ y = t$ $x = t,\ y = t^2$

39. $x = t + 2,\ y = t;\quad 0 \le t \le 5$

40. $x = t,\ y = -t + 1;\quad -1 \le t \le 3$

41. $x = 3\cos t,\ y = 2\sin t;\quad 0 \le t \le 2\pi$

42. $x = \cos t,\ y = 4\sin t;\quad -\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$

43. $x = 2\cos(\omega t),\ y = -3\sin(\omega t)$

$\dfrac{2\pi}{\omega} = 2 \quad \rightarrow \quad \omega = \pi$

$x = 2\cos(\pi t),\ y = -3\sin(\pi t), \quad 0 \le t \le 2$

44. $x = -2\sin \omega t,\ y = 3\cos \omega t$

$\dfrac{2\pi}{\omega} = 1 \quad \rightarrow \quad \omega = 2\pi$

$x = -2\sin 2\pi t,\ y = 3\cos 2\pi t, \quad 0 \le t \le 1$

45. $x = -2\sin(\omega)t,\ y = 3\cos(\omega t)$

$\dfrac{2\pi}{\omega} = 1 \quad \rightarrow \quad \omega = 2\pi$

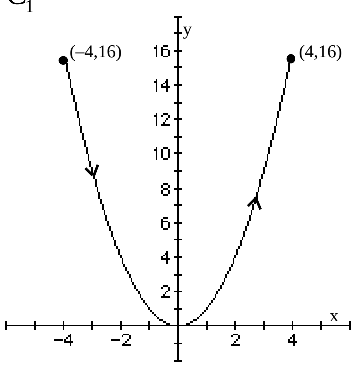
$x = -2\sin(2\pi t),\ y = 3\cos(2\pi t), \quad 0 \le t \le 1$

46. $x = 2\cos \omega t,\ y = 3\sin \omega t$

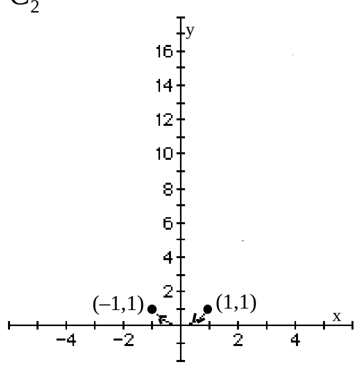
$\dfrac{2\pi}{\omega} = 3 \quad \rightarrow \quad \omega = \dfrac{2\pi}{3}$

$x = 2\cos\left(\dfrac{2\pi}{3}t\right),\ y = 3\sin\left(\dfrac{2\pi}{3}t\right), \quad 0 \le t \le 3$

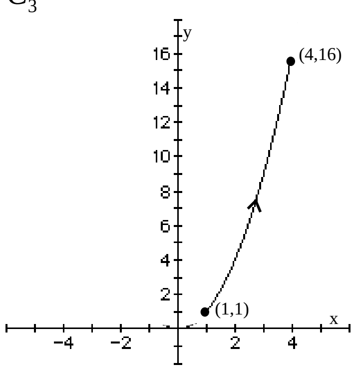
47. C_1



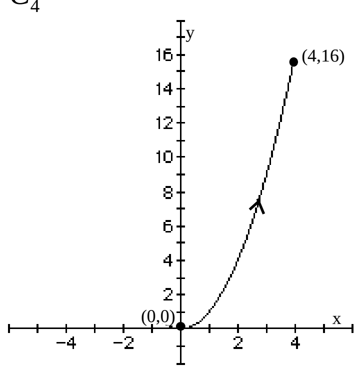
C_2



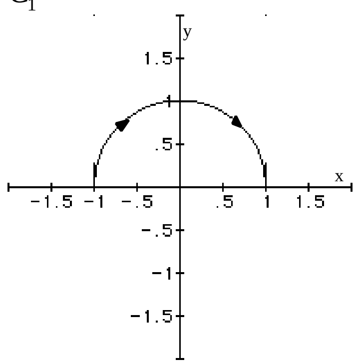
C_3



C_4



48. C_1



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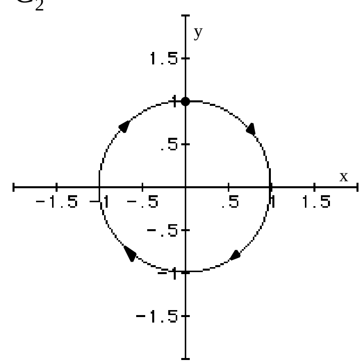

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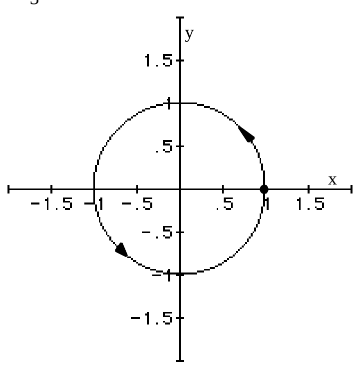
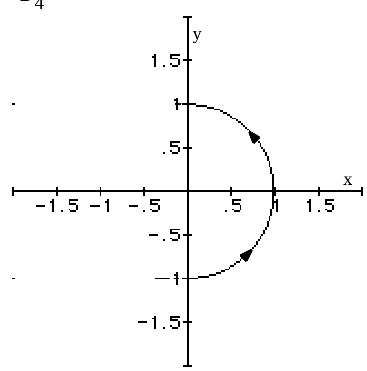


C_4



49. $x = (x_2 - x_1)t + x_1, \quad y = (y_2 - y_1)t + y_1, \quad -\infty < t < \infty$

$$\frac{x - x_1}{x_2 - x_1} = t$$

$$y = (y_2 - y_1)\left(\frac{x - x_1}{x_2 - x_1}\right) + y_1 \to y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

This is the two point form for the equation of a line. Its orientation is from (x_1, y_1) to (x_2, y_2).

50. (a) $x = (v_0 \cos\theta)t, \quad y = (v_0 \sin\theta)t - 16t^2$

$$t = \frac{x}{v_0 \cos\theta}$$

$$y = v_0 \sin\theta\left(\frac{x}{v_0\cos\theta}\right) - 16\left(\frac{x}{v_0\cos\theta}\right)^2 \to y = (\tan\theta)x - \frac{16}{v_0^{\,2}\cos^2\theta}x^2$$

y is a quadratic function of x; its graph is a parabola.

(b) $y = 0 \quad \to \quad (v_0 \sin\theta)t - 16t^2 = 0 \quad \to \quad t(v_0 \sin\theta - 16t) = 0$

$\to \quad t = 0$ or $v_0 \sin\theta - 16t = 0$

$$t = \frac{v_0 \sin\theta}{16}$$

(c) $x = (v_0\cos\theta)t = (v_0\cos\theta)\left(\frac{v_0\sin\theta}{16}\right) = \frac{v_0^{\,2}\sin 2\theta}{32}$ feet

(d) $x = y$

$$(v_0\cos\theta)t = (v_0\sin\theta)t - 16t^2$$

$$16t^2 + (v_0\cos\theta)t - (v_0\sin\theta)t = 0 \rightarrow t\left(16t + (v_0\cos\theta) - (v_0\sin\theta)\right) = 0$$

$$t = 0 \text{ or } 16t + (v_0\cos\theta) - (v_0\sin\theta) = 0$$

$$t = \frac{v_0\sin\theta - v_0\cos\theta}{16} \rightarrow t = \frac{v_0}{16}(\sin\theta - \cos\theta)$$

At $t = \frac{v_0}{16}(\sin\theta - \cos\theta)$:

$$x = v_0\cos\theta\left(\frac{v_0}{16}(\sin\theta - \cos\theta)\right) = \frac{v_0^{\,2}}{16}\cos\theta(\sin\theta - \cos\theta)$$

$$y = v_0\sin\theta\left(\frac{v_0}{16}(\sin\theta - \cos\theta)\right) - 16\left(\frac{v_0}{16}(\sin\theta - \cos\theta)\right)^2$$

$$= \frac{v_0^{\,2}}{16}\sin\theta(\sin\theta - \cos\theta) - \frac{v_0^{\,2}}{16}\left(\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta\right)$$

$$= \frac{v_0^{\,2}}{16}\sin\theta(\sin\theta - \cos\theta) - \frac{v_0^{\,2}}{16}(1 - 2\sin\theta\cos\theta)$$

$$= \frac{v_o^{\,2}}{16}\left(\sin^2\theta - \sin\theta\cos\theta - 1 + 2\sin\theta\cos\theta\right)$$

$$= \frac{v_o^{\,2}}{16}\left(-\cos^2\theta + \sin\theta\cos\theta\right)$$

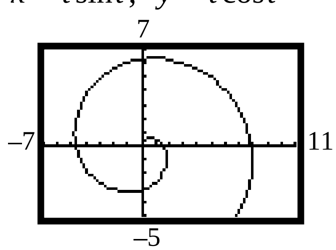
$$\sqrt{x^2 + y^2} = \sqrt{\left(\frac{v_0^{\,2}}{16}\cos\theta(\sin\theta - \cos\theta)\right)^2 + \left(\frac{v_o^{\,2}}{16}\left(-\cos^2\theta + \sin\theta\cos\theta\right)\right)^2}$$

$$= \sqrt{\left(\frac{v_0^{\,2}}{16}\cos\theta(\sin\theta - \cos\theta)\right)^2 + \left(\frac{v_o^{\,2}}{16}\cos\theta(\sin\theta - \cos\theta)\right)^2}$$

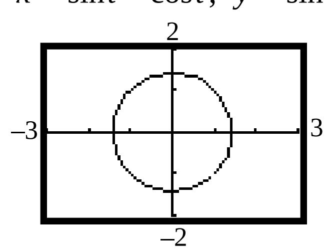
$$= \sqrt{2\left(\frac{v_o^{\,2}}{16}\cos\theta(\sin\theta - \cos\theta)\right)^2}$$

$$= \frac{v_0^{\,2}}{16}\sqrt{2}\cos\theta(\sin\theta - \cos\theta)$$

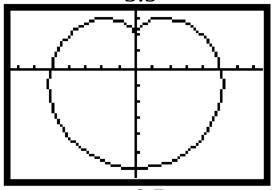
51. $x = t\sin t,\ y = t\cos t$



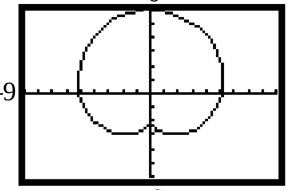
52. $x = \sin t + \cos t,\ y = \sin t - \cos t$



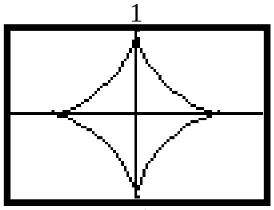
53. $x = 4\sin t - 2\sin(2t)$
$y = 4\cos t - 2\cos(2t)$



54. $x = 4\sin t + 2\sin(2t)$
$y = 4\cos t + 2\cos(2t)$



55. (a) $x(t) = \cos^3 t,\ \ y(t) = \sin^3 t, \quad 0 \le t \le 2\pi$



(b) $\cos t = x^{2/3},\ \ \sin t = y^{2/3}$

$\cos^2 t + \sin^2 t = \left(x^{1/3}\right)^2 + \left(y^{1/3}\right)^2$

$x^{2/3} + y^{2/3} = 1$