Chapter 11

Analytic Geometry

11.R Chapter Review

1. $y^2 = -16x$
   This is a parabola.
   $a = 4$
   Vertex: (0, 0)
   Focus: (–4, 0)
   Directrix: $x = 4$

2. $16x^2 = y \rightarrow x^2 = \frac{1}{16}y$
   This is a parabola.
   $a = \frac{1}{64}$
   Vertex: (0, 0)
   Focus: $\left(0, \frac{1}{64}\right)$
   Directrix: $y = -\frac{1}{64}$

3. $\frac{x^2}{25} - y^2 = 1$
   This is a hyperbola.
   $a = 5; \quad b = 1.$
   Find the value of c:
   $c^2 = a^2 + b^2 = 25 + 1 = 26$
   $c = \sqrt{26}$

   Center: (0, 0)
   Vertices: (5, 0), (–5, 0)
   Foci: $\left(\sqrt{26}, 0\right), \left(-\sqrt{26}, 0\right)$
   Asymptotes: $y = \frac{1}{5}x; \quad y = -\frac{1}{5}x$

4. $\frac{y^2}{25} - x^2 = 1$
   This is a hyperbola.
   $a = 5; \quad b = 1.$
   Find the value of c:
   $c^2 = a^2 + b^2 = 25 + 1 = 26$
   $c = \sqrt{26}$

   Center: (0, 0)
   Vertices: (0, 5), (0, –5)
   Foci: $\left(0, \sqrt{26}\right), \left(0, -\sqrt{26}\right)$
   Asymptotes: $y = 5x; \quad y = -5x$

5. $\dfrac{y^2}{25}+\dfrac{x^2}{16}=1$

This is an ellipse.

$a=5, \quad b=4.$

Find the value of c:

$c^2=a^2-b^2=25-16=9$

$c=3$

Center: (0, 0)
Vertices: (0, 5), (0, –5)
Foci: (0, 3), (0, –3)

6. $\dfrac{x^2}{9}+\dfrac{y^2}{16}=1$

This is an ellipse.

$a=4, \quad b=3.$

Find the value of c:

$c^2=a^2-b^2=16-9=7$

$c=\sqrt{7}$

Center: (0, 0)
Vertices: (0, 4), (0, –4)
Foci: $\left(0, \sqrt{7}\right), \left(0, -\sqrt{7}\right)$

7. $x^2+4y=4$

This is a parabola.

Write in standard form:

$$x^2=-4y+4$$
$$x^2=-4(y-1)$$

$a=1$

Vertex: (0, 1)
Focus: (0, 0)
Directrix: $y=2$

8. $3y^2-x^2=9$

This is a hyperbola.

Write in standard form:

$$\frac{y^2}{3}-\frac{x^2}{9}=1$$

$a=\sqrt{3}, \quad b=3$

Find the value of c:

$c^2=a^2+b^2=3+9=12$

$c=\sqrt{12}=2\sqrt{3}$

Center: (0, 0)
Vertices: $\left(0, \sqrt{3}\right), \left(0, -\sqrt{3}\right)$
Foci: $\left(0, 2\sqrt{3}\right), \left(0, -2\sqrt{3}\right)$
Asymptotes: $y=\dfrac{\sqrt{3}}{3}x; \quad y=-\dfrac{\sqrt{3}}{3}x$

9. $4x^2-y^2=8$

This is a hyperbola.

Write in standard form:

$$\frac{x^2}{2}-\frac{y^2}{8}=1$$

$a=\sqrt{2}, \quad b=\sqrt{8}=2\sqrt{2}.$

Find the value of c:

$c^2=a^2+b^2=2+8=10$

$c=\sqrt{10}$

Center: (0, 0)
Vertices: $\left(-\sqrt{2}, 0\right), \left(\sqrt{2}, 0\right)$
Foci: $\left(-\sqrt{10}, 0\right), \left(\sqrt{10}, 0\right)$
Asymptotes: $y=2x; \quad y=-2x$

10. $9x^2 + 4y^2 = 36$

This is an ellipse.

Write in standard form:

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$a = 3, \quad b = 2.$

Find the value of c:

$c^2 = a^2 - b^2 = 9 - 4 = 5$

$c = \sqrt{5}$

Center: (0, 0)

Vertices: (0, 3), (0, –3)

Foci:

$\left(0, \sqrt{5}\right), \left(0, -\sqrt{5}\right)$

11. $x^2 - 4x = 2y$

This is a parabola.

Write in standard form:

$$x^2 - 4x + 4 = 2y + 4$$

$$(x-2)^2 = 2(y+2)$$

$a = \frac{1}{2}$

Vertex: (2, –2)

Focus: $\left(2, -\frac{3}{2}\right)$

Directrix: $y = -\frac{5}{2}$

12. $2y^2 - 4y = x - 2$

This is a parabola.

Write in standard form:

$$2\left(y^2 - 2y + 1\right) = x - 2 + 2$$

$$(y-1)^2 = \frac{1}{2}x$$

$a = \frac{1}{8}$

Vertex: (0, 1)

Focus: $\left(\frac{1}{8}, 1\right)$

Directrix: $x = -\frac{1}{8}$

13. $y^2 - 4y - 4x^2 + 8x = 4$

This is a hyperbola.

Write in standard form:

$$(y^2 - 4y + 4) - 4(x^2 - 2x + 1) = 4 + 4 - 4$$

$$(y-2)^2 - 4(x-1)^2 = 4$$

$$\frac{(y-2)^2}{4} - \frac{(x-1)^2}{1} = 1$$

$a = 2, \quad b = 1.$

Find the value of c:

$c^2 = a^2 + b^2 = 4 + 1 = 5$

$c = \sqrt{5}$

Center: (1, 2)

Vertices: (1, 0), (1, 4)

Foci: $\left(1, 2 - \sqrt{5}\right), \left(1, 2 + \sqrt{5}\right)$

Asymptotes:

$y - 2 = 2(x-1); \quad y - 2 = -2(x-1)$

14. $4x^2+y^2+8x-4y+4=0$

This is an ellipse.

Write in standard form:

$$4(x^2+2x+1)+(y^2-4y+4)=-4+4+4$$
$$4(x+1)^2+(y-2)^2=4$$
$$\frac{(x+1)^2}{1}+\frac{(y-2)^2}{4}=1$$

$a=2,\ b=1.$

Find the value of c:

$c^2=a^2-b^2=4-1=3$

$c=\sqrt{3}$

Center: $(-1, 2)$
Vertices: $(-1, 0), (-1, 4)$
Foci: $\left(-1,\ 2-\sqrt{3}\right),\ \left(-1,\ 2+\sqrt{3}\right)$

15. $4x^2+9y^2-16x-18y=11$

This is an ellipse.

Write in standard form:

$$4x^2+9y^2-16x-18y=11$$
$$4(x^2-4x+4)+9(y^2-2y+1)=11+16+9$$
$$4(x-2)^2+9(y-1)^2=36$$
$$\frac{(x-2)^2}{9}+\frac{(y-1)^2}{4}=1$$

$a=3,\ b=2.$

Find the value of c:

$c^2=a^2-b^2=9-4=5$

$c=\sqrt{5}$

Center: $(2, 1)$
Vertices: $(-1, 1), (5, 1)$
Foci: $\left(2-\sqrt{5}, 1\right), \left(2+\sqrt{5}, 1\right)$

16. $4x^2+9y^2-16x+18y=11$

This is an ellipse.

Write in standard form:

$$4x^2+9y^2-16x+18y=11$$
$$4(x^2-4x+4)+9(y^2+2y+1)=11+16+9$$
$$4(x-2)^2+9(y+1)^2=36$$
$$\frac{(x-2)^2}{9}+\frac{(y+1)^2}{4}=1$$

$a=3,\ b=2.$

Find the value of c:

$c^2=a^2-b^2=9-4=5$

$c=\sqrt{5}$

Center: $(2, -1)$
Vertices: $(-1, -1), (5, -1)$
Foci: $\left(2-\sqrt{5}, -1\right), \left(2+\sqrt{5}, -1\right)$

17. $4x^2 - 16x + 16y + 32 = 0$
This is a parabola.
Write in standard form:
$4(x^2 - 4x + 4) = -16y - 32 + 16$
$4(x-2)^2 = -16(y+1)$
$(x-2)^2 = -4(y+1)$
$a = 1$

Vertex: (2, –1)
Focus: $(2, -2)$
Directrix: $y = 0$

18. $4y^2 + 3x - 16y + 19 = 0$
This is a parabola.
Write in standard form:
$4(y^2 - 4y + 4) = -3x - 19 + 16$
$4(y-2)^2 = -3(x+1)$
$(y-2)^2 = -\frac{3}{4}(x+1)$
$a = -\frac{3}{16}$

Vertex: (–1, 2)
Focus: $\left(-\frac{19}{16}, 2\right)$
Directrix: $x = -\frac{13}{16}$

19. $9x^2 + 4y^2 - 18x + 8y = 23$
This is an ellipse.
Write in standard form:
$9(x^2 - 2x + 1) + 4(y^2 + 2y + 1) = 23 + 9 + 4$
$9(x-1)^2 + 4(y+1)^2 = 36$
$\frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1$
$a = 3, \quad b = 2.$
Find the value of c:
$c^2 = a^2 - b^2 = 9 - 4 = 5$
$c = \sqrt{5}$

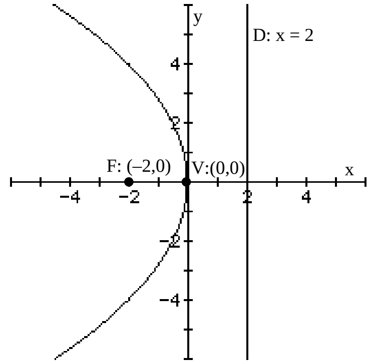
Center: (1, –1)
Vertices: (1, –4), (1, 2)
Foci: $\left(1, -1 - \sqrt{5}\right), \left(1, -1 + \sqrt{5}\right)$

20. $x^2 - y^2 - 2x - 2y = 1$
This is a hyperbola.
Write in standard form:
$(x^2 - 2x + 1) - (y^2 + 2y + 1) = 1 + 1 - 1$
$(x-1)^2 - (y+1)^2 = 1$
$a = 1, \quad b = 1.$
Find the value of c:
$c^2 = a^2 + b^2 = 1 + 1 = 2$
$c = \sqrt{2}$

Center: (1, –1)
Vertices: (0, –1), (2, –1)
Foci: $\left(1 + \sqrt{2}, -1\right), \left(1 - \sqrt{2}, -1\right)$
Asymptotes:
$y + 1 = x - 1; \quad y + 1 = -(x - 1)$

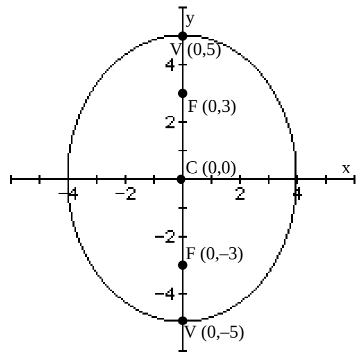
21. Parabola: The focus is (–2, 0) and the directrix is $x = 2$. The vertex is (0, 0). $a = 2$ and since (–2, 0) is to the left of (0, 0), the parabola opens to the left. The equation of the parabola is:

$$y^2 = -4ax$$
$$y^2 = -4(2)x$$
$$y^2 = -8x$$



22. Ellipse: The center is (0, 0), a focus is (0, 3), and a vertex is (0, 5). The major axis is $x = 0$. $a = 5$, $c = 3$. Find b: $b^2 = a^2 - c^2 = 25 - 9 = 16$. So, $b = 4$. The equation of the ellipse is:

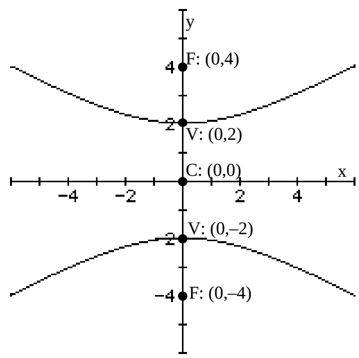
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$
$$\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1$$
$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$



23. Hyperbola: Center: (0, 0); Focus: (0, 4); Vertex: (0, –2); Transverse axis is the y-axis; $a = 2$; $c = 4$. Find b:

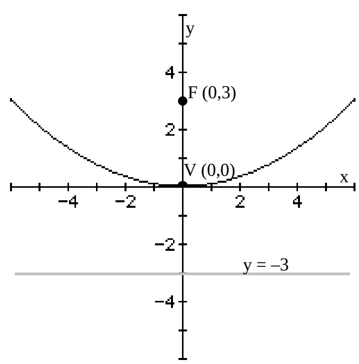
$$b^2 = c^2 - a^2 = 16 - 4 = 12$$
$$b = \sqrt{12} = 2\sqrt{3}$$

Write the equation: $\frac{y^2}{4} - \frac{x^2}{12} = 1$



24. Parabola: Vertex: (0, 0); Directrix: $y = -3$; $a = 3$. The graph opens up. The equation of the parabola is:

$$x^2 = 4ay$$
$$x^2 = 4(3)y$$
$$x^2 = 12y$$

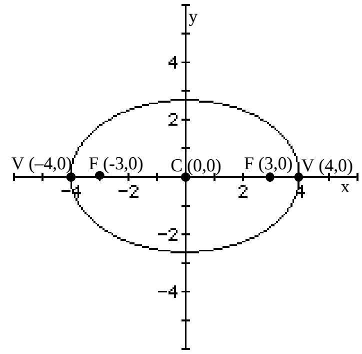


25. Ellipse: Foci: (–3, 0), (3, 0); Vertex: (4, 0); Center: (0, 0); Major axis is the x-axis; $a = 4$; $c = 3$. Find b:

$$b^2 = a^2 - c^2 = 16 - 9 = 7$$

$$b = \sqrt{7}$$

Write the equation: $\dfrac{x^2}{16} + \dfrac{y^2}{7} = 1$

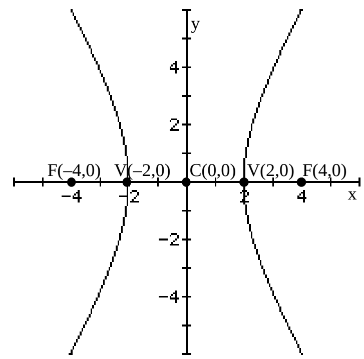


26. Hyperbola: Vertices: (–2, 0), (2, 0); Focus: (4, 0); Center: (0, 0); Transverse axis is the x-axis; $a = 2$; $c = 4$. Find b:

$$b^2 = c^2 - a^2 = 16 - 4 = 12$$

$$b = \sqrt{12} = 2\sqrt{3}$$

Write the equation: $\dfrac{x^2}{4} - \dfrac{y^2}{12} = 1$

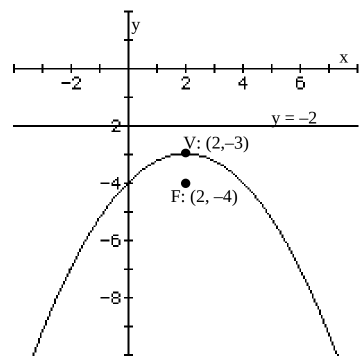


27. Parabola: The focus is (2, –4) and the vertex is (2, –3). Both lie on the vertical line $x = 2$. $a = 1$ and since (2, –4) is below (2, –3), the parabola opens down. The equation of the parabola is:

$$(x-h)^2 = -4a(y-k)$$

$$(x-2)^2 = -4\;1\;(y-(-3))$$

$$(x-2)^2 = -4(y+3)$$

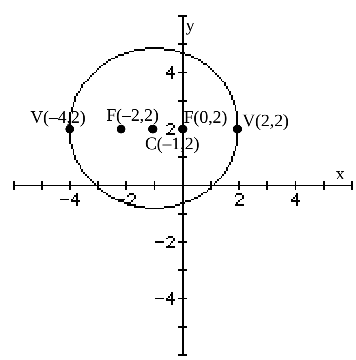


28. Ellipse: Center: (–1, 2); Focus: (0, 2); Vertex: (2, 2). Major axis: $y = 2$. $a = 3$; $c = 1$. Find b:

$$b^2 = a^2 - c^2 = 9 - 1 = 8$$

$$b = \sqrt{8} = 2\sqrt{2}$$

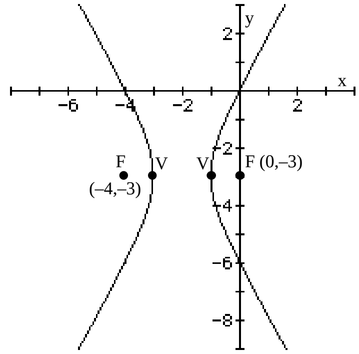
Write the equation: $\dfrac{(x+1)^2}{9} + \dfrac{(y-2)^2}{8} = 1$



29. Hyperbola: Center: (–2, –3); Focus: (–4, –3); Vertex: (–3, –3); Transverse axis is parallel to the x-axis; $a = 1$; $c = 2$. Find b:

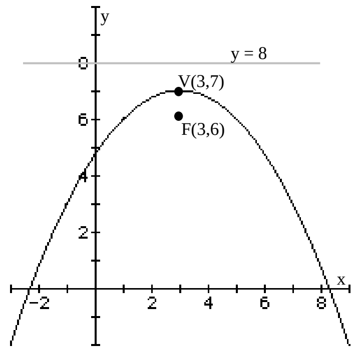
$$b^2 = c^2 - a^2 = 4 - 1 = 3$$
$$b = \sqrt{3}$$

Write the equation: $\dfrac{(x+2)^2}{1} - \dfrac{(y+3)^2}{3} = 1$



30. Parabola: Focus: (3, 6); Directrix: $y = 8$; Parabola opens down. Vertex: (3, 7) $a = 1$. The equation of the parabola is:

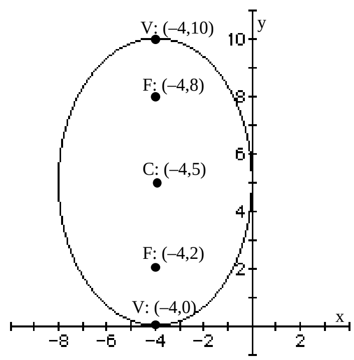
$$(x-h)^2 = -4a(y-k)$$
$$(x-3)^2 = -4(1)(y-7)$$
$$(x-3)^2 = -4(y-7)$$



31. Ellipse: Foci: (–4, 2), (–4, 8); Vertex: (–4, 10); Center: (–4, 5); Major axis is parallel to the y-axis; $a = 5$; $c = 3$. Find b:

$$b^2 = a^2 - c^2 = 25 - 9 = 16$$
$$b = 4$$

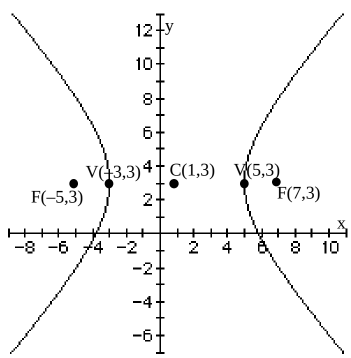
Write the equation: $\dfrac{(x+4)^2}{16} + \dfrac{(y-5)^2}{25} = 1$



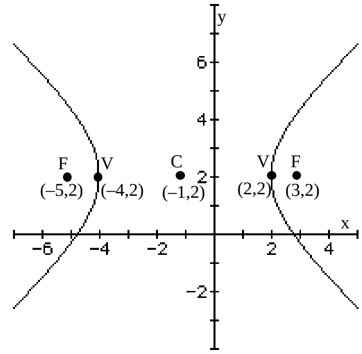
32. Hyperbola: Vertices: (–3, 3), (5, 3); Focus: (7, 3); Center: (1, 3); Major axis is parallel to the x-axis; $a = 4$; $c = 6$. Find b:

$$b^2 = c^2 - a^2 = 36 - 16 = 20$$
$$b = \sqrt{20} = 2\sqrt{5}$$

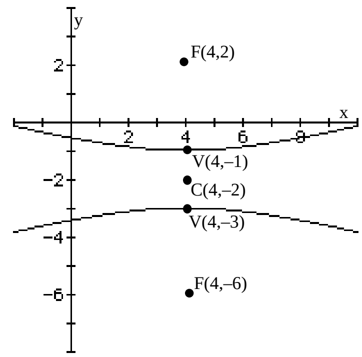
Write the equation: $\dfrac{(x-1)^2}{16} - \dfrac{(y-3)^2}{20} = 1$



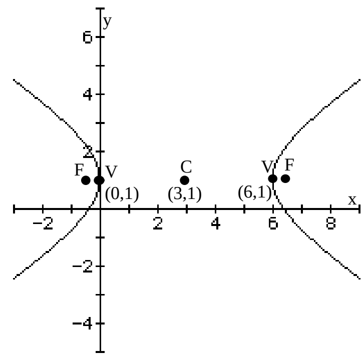
33. Hyperbola: Center: (–1, 2); $a = 3$; $c = 4$;
Transverse axis parallel to the x-axis; Find b:
$$b^2 = c^2 - a^2 = 16 - 9 = 7$$
$$b = \sqrt{7}$$
Write the equation: $\dfrac{(x+1)^2}{9} - \dfrac{(y-2)^2}{7} = 1$



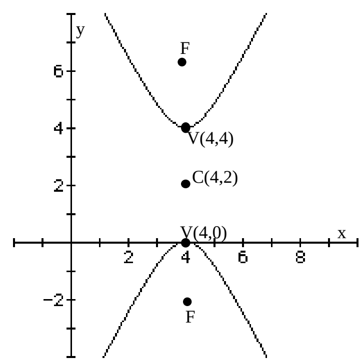
34. Hyperbola: Center: (4, –2); $a = 1$; $c = 4$;
Transverse axis parallel to the y-axis; Find b:
$$b^2 = c^2 - a^2 = 16 - 1 = 15$$
$$b = \sqrt{15}$$
Write the equation: $\dfrac{(y+2)^2}{1} - \dfrac{(x-4)^2}{15} = 1$



35. Hyperbola: Vertices: (0, 1), (6, 1);
Asymptote: $3y + 2x - 9 = 0$; Center: (3, 1);
Transverse axis is parallel to the x-axis; $a = 3$; The slope of the asymptote is $-\dfrac{2}{3}$; Find b:
$$\frac{-b}{a} = \frac{-b}{3} = \frac{-2}{3} \rightarrow -3b = -6 \rightarrow b = 2$$
Write the equation: $\dfrac{(x-3)^2}{9} - \dfrac{(y-1)^2}{4} = 1$



36. Hyperbola: Vertices: (4, 0), (4, 4);
Asymptote: $y + 2x - 10 = 0$; Center: (4, 2);
Transverse axis is parallel to the y-axis; $a = 2$;
The slope of the asymptote is -2; Find b:
$$\frac{-a}{b} = \frac{-2}{b} = -2 \rightarrow -2b = -2 \rightarrow b = 1$$
Write the equation: $\dfrac{(y-2)^2}{4} - \dfrac{(x-4)^2}{1} = 1$



37. $y^2 + 4x + 3y - 8 = 0$
$A = 0$ and $C = 1$; $AC = (0)(1) = 0$. Since $AC = 0$, the equation defines a parabola.

38. $2x^2 - y + 8x = 0$
$A = 2$ and $C = 0$; $AC = (2)(0) = 0$. Since $AC = 0$, the equation defines a parabola.

39. $x^2 + 2y^2 + 4x - 8y + 2 = 0$
$A = 1$ and $C = 2$; $AC = (1)(2) = 2$. Since $AC > 0$ and $A \neq C$, the equation defines an ellipse.

40. $x^2 - 8y^2 - x - 2y = 0$
$A = 1$ and $C = -8$; $AC = (1)(-8) = -8$. Since $AC < 0$, the equation defines a hyperbola.

41. $9x^2 - 12xy + 4y^2 + 8x + 12y = 0$
$A = 9$, $B = -12$, $C = 4$ $\quad B^2 - 4AC = (-12)^2 - 4(9)(4) = 0$; parabola

42. $4x^2 + 4xy + y^2 - 8\sqrt{5}\,x + 16\sqrt{5}\,y = 0$
$A = 4$, $B = 4$, $C = 1$ $\quad B^2 - 4AC = 4^2 - 4(4)(1) = 0$; parabola

43. $4x^2 + 10xy + 4y^2 - 9 = 0$
$A = 4$, $B = 10$, $C = 4$ $\quad B^2 - 4AC = 10^2 - 4(4)(4) = 36 > 0$; hyperbola

44. $4x^2 - 10xy + 4y^2 - 9 = 0$
$A = 4$, $B = -10$, $C = 4$ $\quad B^2 - 4AC = (-10)^2 - 4(4)(4) = 36 > 0$; hyperbola

45. $x^2 - 2xy + 3y^2 + 2x + 4y - 1 = 0$
$A = 1$, $B = -2$, $C = 3$ $\quad B^2 - 4AC = (-2)^2 - 4(1)(3) = -8 < 0$; ellipse

46. $4x^2 + 12xy - 10y^2 + x + y - 10 = 0$
$A = 4$, $B = 12$, $C = -10$ $\quad B^2 - 4AC = 12^2 - 4(4)(-10) = 304 > 0$; hyperbola

47. $2x^2 + 5xy + 2y^2 - \frac{9}{2} = 0$

$A = 2, B = 5,$ and $C = 2$; $\cot(2\theta) = \frac{A-C}{B} = \frac{2-2}{5} = 0 \rightarrow 2\theta = \frac{\pi}{2} \rightarrow \theta = \frac{\pi}{4}$

$$x = x'\cos\theta - y'\sin\theta = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y' = \frac{\sqrt{2}}{2}(x' - y')$$

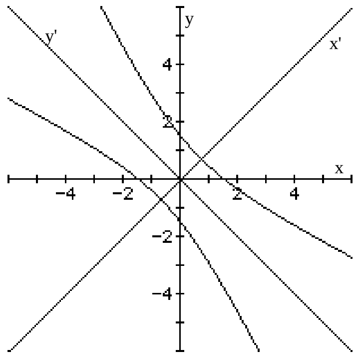
$$y = x'\sin\theta + y'\cos\theta = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' = \frac{\sqrt{2}}{2}(x' + y')$$

$$2\left(\frac{\sqrt{2}}{2}(x' - y')\right)^2 + 5\left(\frac{\sqrt{2}}{2}(x' - y')\right)\left(\frac{\sqrt{2}}{2}(x' + y')\right) + 2\left(\frac{\sqrt{2}}{2}(x' + y')\right)^2 - \frac{9}{2} = 0$$

$$\left(x'^2 - 2x'y' + y'^2\right) + \frac{5}{2}\left(x'^2 - y'^2\right) + \left(x'^2 + 2x'y' + y'^2\right) - \frac{9}{2} = 0$$

$$\frac{9}{2}x'^2 - \frac{1}{2}y'^2 = \frac{9}{2} \rightarrow 9x'^2 - y'^2 = 9 \rightarrow \frac{x'^2}{1} - \frac{y'^2}{9} = 1$$

Hyperbola; center at (0, 0), transverse axis is the x'-axis, vertices at (±1, 0).



48. $2x^2 - 5xy + 2y^2 - \frac{9}{2} = 0$

$A = 2,\ B = -5, \text{and } C = 2;\ \cot(2\theta) = \frac{A-C}{B} = \frac{2-2}{-5} = 0 \quad \rightarrow \quad 2\theta = \frac{\pi}{2} \quad \rightarrow \quad \theta = \frac{\pi}{4}$

$x = x'\cos\theta - y'\sin\theta = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y' = \frac{\sqrt{2}}{2}(x' - y')$

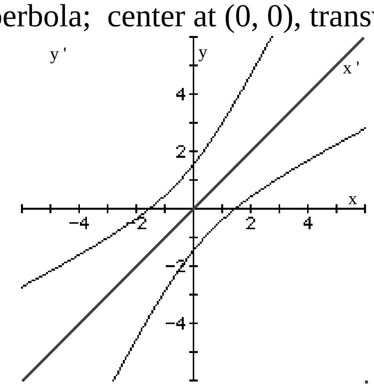
$y = x'\sin\theta + y'\cos\theta = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' = \frac{\sqrt{2}}{2}(x' + y')$

$2\left(\frac{\sqrt{2}}{2}(x'-y')\right)^2 - 5\left(\frac{\sqrt{2}}{2}(x'-y')\right)\left(\frac{\sqrt{2}}{2}(x'+y')\right) + 2\left(\frac{\sqrt{2}}{2}(x'+y')\right)^2 - \frac{9}{2} = 0$

$\left(x'^2 - 2x'y' + y'^2\right) - \frac{5}{2}\left(x'^2 - y'^2\right) + \left(x'^2 + 2x'y' + y'^2\right) - \frac{9}{2} = 0$

$-\frac{1}{2}x'^2 + \frac{9}{2}y'^2 = \frac{9}{2} \ \rightarrow \ -x'^2 + 9y'^2 = 9 \ \rightarrow \ \frac{y'^2}{1} - \frac{x'^2}{9} = 1$

Hyperbola; center at (0, 0), transverse axis is the y'-axis, vertices at (0, ±1).



49. $6x^2 + 4xy + 9y^2 - 20 = 0$

$A = 6, B = 4, \text{and } C = 9;\ \cot(2\theta) = \frac{A-C}{B} = \frac{6-9}{4} = -\frac{3}{4} \quad \rightarrow \quad \cos(2\theta) = -\frac{3}{5}$

$\sin\theta = \sqrt{\frac{\left(1 - \left(-\frac{3}{5}\right)\right)}{2}} = \sqrt{\frac{4}{5}} = \frac{2\sqrt{5}}{5};\quad \cos\theta = \sqrt{\frac{\left(1 + \left(-\frac{3}{5}\right)\right)}{2}} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5} \quad \rightarrow \quad \theta \approx 63.4^\circ$

$$x = x'\cos\theta - y'\sin\theta = \frac{\sqrt{5}}{5}x' - \frac{2\sqrt{5}}{5}y' = \frac{\sqrt{5}}{5}(x' - 2y')$$

$$y = x'\sin\theta + y'\cos\theta = \frac{2\sqrt{5}}{5}x' + \frac{\sqrt{5}}{5}y' = \frac{\sqrt{5}}{5}(2x' + y')$$

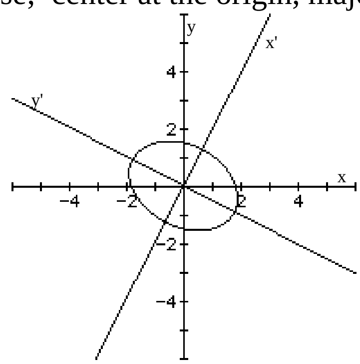
$$6\left(\frac{\sqrt{5}}{5}(x'-2y')\right)^2 + 4\left(\frac{\sqrt{5}}{5}(x'-2y')\right)\left(\frac{\sqrt{5}}{5}(2x'+y')\right) + 9\left(\frac{\sqrt{5}}{5}(2x'+y')\right)^2 - 20 = 0$$

$$\frac{6}{5}\left(x'^2 - 4x'y' + 4y'^2\right) + \frac{4}{5}\left(2x'^2 - 3x'y' - 2y'^2\right) + \frac{9}{5}\left(4x'^2 + 4x'y' + y'^2\right) - 20 = 0$$

$$\frac{6}{5}x'^2 - \frac{24}{5}x'y' + \frac{24}{5}y'^2 + \frac{8}{5}x'^2 - \frac{12}{5}x'y' - \frac{8}{5}y'^2 + \frac{36}{5}x'^2 + \frac{36}{5}x'y' + \frac{9}{5}y'^2 = 20$$

$$10x'^2 + 5y'^2 = 20 \rightarrow \frac{x'^2}{2} + \frac{y'^2}{4} = 1$$

Ellipse; center at the origin, major axis is the y'-axis, vertices at (0, ±2).



50. $x^2 + 4xy + 4y^2 + 16\sqrt{5}x - 8\sqrt{5}y = 0$

$A = 1,\ B = 4,$ and $C = 4$; $\cot(2\theta) = \frac{A - C}{B} = \frac{1-4}{4} = -\frac{3}{4}$; $\cos(2\theta) = -\frac{3}{5}$

$$\sin\theta = \sqrt{\frac{1 - \left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{4}{5}} = \frac{2\sqrt{5}}{5};\quad \cos\theta = \sqrt{\frac{1 + \left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5} \quad \rightarrow \quad \theta = 63.4^\circ$$

$$x = x'\cos\theta - y'\sin\theta = \frac{\sqrt{5}}{5}x' - \frac{2\sqrt{5}}{5}y' = \frac{\sqrt{5}}{5}(x' - 2y')$$

$$y = x'\sin\theta + y'\cos\theta = \frac{2\sqrt{5}}{5}x' + \frac{\sqrt{5}}{5}y' = \frac{\sqrt{5}}{5}(2x' + y')$$

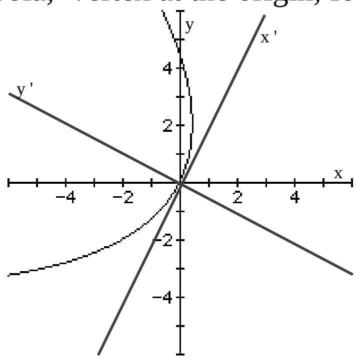
$$\left(\frac{\sqrt{5}}{5}(x'-2y')\right)^2 + 4\left(\frac{\sqrt{5}}{5}(x'-2y')\right)\left(\frac{\sqrt{5}}{5}(2x'+y')\right) + 4\left(\frac{\sqrt{5}}{5}(2x'+y')\right)^2$$
$$+16\sqrt{5}\left(\frac{\sqrt{5}}{5}(x'-2y')\right) - 8\sqrt{5}\left(\frac{\sqrt{5}}{5}(2x'+y')\right) = 0$$

$$\frac{1}{5}\left(x'^2 - 4x'y' + 4y'^2\right) + \frac{4}{5}\left(2x'^2 - 3x'y' - 2y'^2\right) + \frac{4}{5}\left(4x'^2 + 4x'y' + y'^2\right)$$
$$+16x' - 32y' - 16x' - 8y' = 0$$

$$\frac{1}{5}x'^2-\frac{4}{5}x'y'+\frac{4}{5}y'^2+\frac{8}{5}x'^2-\frac{12}{5}x'y'-\frac{8}{5}y'^2+\frac{16}{5}x'^2+\frac{16}{5}x'y'+\frac{4}{5}y'^2-40y'=0$$

$$5x'^2-40y'=0 \to x'^2=8y'$$

Parabola; vertex at the origin, focus at (0, 2).



51. $4x^2-12xy+9y^2+12x+8y=0$

$A=4, B=-12, \text{and } C=9;\ \cot(2\theta)=\dfrac{A-C}{B}=\dfrac{4-9}{-12}=\dfrac{5}{12} \quad \to \ \cos(2\theta)=\dfrac{5}{13}$

$$\sin\theta=\sqrt{\frac{\left(1-\frac{5}{13}\right)}{2}}=\sqrt{\frac{4}{13}}=\frac{2\sqrt{13}}{13};\quad \cos\theta=\sqrt{\frac{\left(1+\frac{5}{13}\right)}{2}}=\sqrt{\frac{9}{13}}=\frac{3\sqrt{13}}{13}\quad\to\quad \theta\approx 33.7^\circ$$

$$x=x'\cos\theta-y'\sin\theta=\frac{3\sqrt{13}}{13}x'-\frac{2\sqrt{13}}{13}y'=\frac{\sqrt{13}}{13}(3x'-2y')$$

$$y=x'\sin\theta+y'\cos\theta=\frac{2\sqrt{13}}{13}x'+\frac{3\sqrt{13}}{13}y'=\frac{\sqrt{13}}{13}(2x'+3y')$$

$$4\left(\frac{\sqrt{13}}{13}(3x'-2y')\right)^2-12\left(\frac{\sqrt{13}}{13}(3x'-2y')\right)\left(\frac{\sqrt{13}}{13}(2x'+3y')\right)+9\left(\frac{\sqrt{13}}{13}(2x'+3y')\right)^2$$
$$+12\left(\frac{\sqrt{13}}{13}(3x'-2y')\right)+8\left(\frac{\sqrt{13}}{13}(2x'+3y')\right)=0$$

$$\frac{4}{13}\left(9x'^2-12x'y'+4y'^2\right)-\frac{12}{13}\left(6x'^2+5x'y'-6y'^2\right)+\frac{9}{13}\left(4x'^2+12x'y'+9y'^2\right)$$
$$+\frac{36\sqrt{13}}{13}x'-\frac{24\sqrt{13}}{13}y'+\frac{16\sqrt{13}}{13}x'+\frac{24\sqrt{13}}{13}y'=0$$

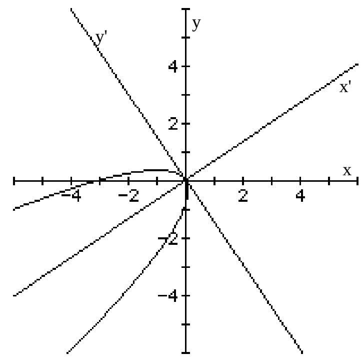
$$\frac{36}{13}x'^2-\frac{48}{13}x'y'+\frac{16}{13}y'^2-\frac{72}{13}x'^2-\frac{60}{13}x'y'+\frac{72}{13}y'^2$$
$$+\frac{36}{13}x'^2+\frac{108}{13}x'y'+\frac{81}{13}y'^2+4\sqrt{13}x'=0$$

$13y'^2 + 4\sqrt{13}x' = 0$

$y'^2 = -\dfrac{4\sqrt{13}}{13}x'$

Parabola; vertex at the origin,

focus at $\left(-\dfrac{\sqrt{13}}{13}, 0\right)$.



52. $9x^2 - 24xy + 16y^2 + 80x + 60y = 0$

$A = 9, B = -24, \text{and } C = 16;\ \cot(2\theta) = \dfrac{A-C}{B} = \dfrac{9-16}{-24} = \dfrac{7}{24} \quad \to \quad \cos(2\theta) = \dfrac{7}{25}$

$$\sin\theta = \sqrt{\frac{\left(1-\frac{7}{25}\right)}{2}} = \sqrt{\frac{9}{25}} = \frac{3}{5};\quad \cos\theta = \sqrt{\frac{\left(1+\frac{7}{25}\right)}{2}} = \sqrt{\frac{16}{25}} = \frac{4}{5} \quad \to \quad \theta \approx 36.9$$

$$x = x'\cos\theta - y'\sin\theta = \frac{4}{5}x' - \frac{3}{5}y' = \frac{1}{5}(4x' - 3y')$$

$$y = x'\sin\theta + y'\cos\theta = \frac{3}{5}x' + \frac{4}{5}y' = \frac{1}{5}(3x' + 4y')$$

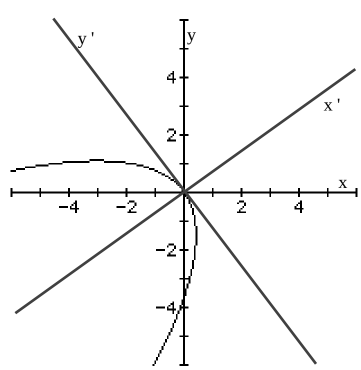
$$9\left(\frac{1}{5}(4x'-3y')\right)^2 - 24\left(\frac{1}{5}(4x'-3y')\right)\left(\frac{1}{5}(3x'+4y')\right) + 16\left(\frac{1}{5}(3x'+4y')\right)^2$$
$$+ 80\left(\frac{1}{5}(4x'-3y')\right) + 60\left(\frac{1}{5}(3x'+4y')\right) = 0$$

$$\frac{9}{25}\left(16x'^2 - 24x'y' + 9y'^2\right) - \frac{24}{25}\left(12x'^2 + 7x'y' - 12y'^2\right) + \frac{16}{25}\left(9x'^2 + 24x'y' + 16y'^2\right)$$
$$+ 64x' - 48y' + 36x' + 48y' = 0$$

$$\frac{144}{25}x'^2 - \frac{216}{25}x'y' + \frac{81}{25}y'^2 - \frac{288}{25}x'^2 - \frac{168}{25}x'y' + \frac{288}{25}y'^2$$
$$+ \frac{144}{25}x'^2 + \frac{384}{25}x'y' + \frac{256}{25}y'^2 + 100x' = 0$$

$25y'^2 + 100x' = 0 \to y'^2 = -4x'$

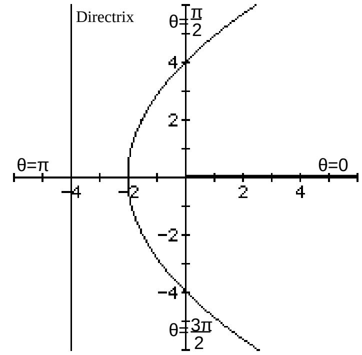
Parabola; vertex at (0, 0), focus at (–1, 0).



53. $r = \dfrac{4}{1-\cos\theta}$

$ep = 4, \quad e = 1, \quad p = 4$

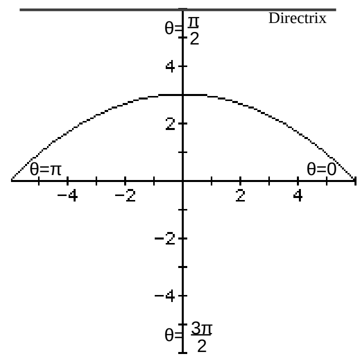
Parabola; directrix is perpendicular to the polar axis 4 units to the left of the pole; vertex is $(2, \pi)$.



54. $r = \dfrac{6}{1+\sin\theta}$

$ep = 6, \quad e = 1, \quad p = 6$

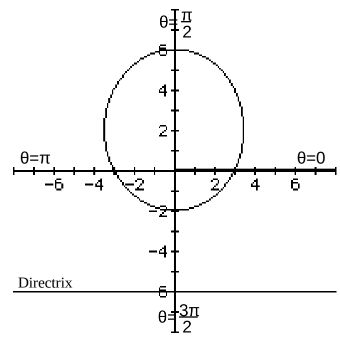
Parabola; directrix is parallel to the polar axis 6 units above the pole; vertex is $\left(3, \dfrac{\pi}{2}\right)$.



55. $r = \dfrac{6}{2-\sin\theta} = \dfrac{3}{1-\frac{1}{2}\sin\theta}$

$ep = 3, \quad e = \dfrac{1}{2}, \quad p = 6$

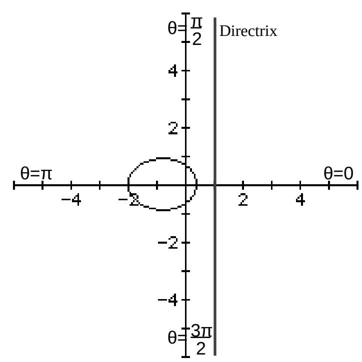
Ellipse; directrix is parallel to the polar axis 6 units below the pole; vertices are $\left(6, \dfrac{\pi}{2}\right)$ and $\left(2, \dfrac{3\pi}{2}\right)$.



56. $r = \dfrac{2}{3+2\cos\theta} = \dfrac{\frac{2}{3}}{1+\frac{2}{3}\cos\theta}$

$ep = \dfrac{2}{3}, \quad e = \dfrac{2}{3}, \quad p = 1$

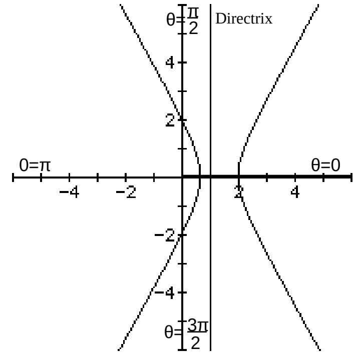
Ellipse; directrix is perpendicular to the polar axis 1 unit to the right of the pole; vertices are $\left(\dfrac{2}{5}, 0\right)$ and $(2, \pi)$.



57. $r = \dfrac{8}{4+8\cos\theta} = \dfrac{2}{1+2\cos\theta}$

$ep = 2, \quad e = 2, \quad p = 1$

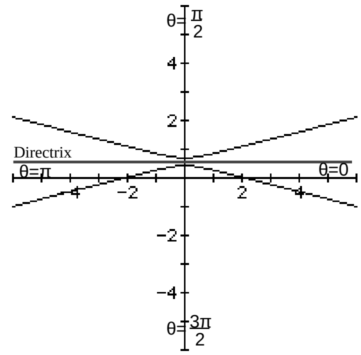
Hyperbola; directrix is perpendicular to the polar axis 1 unit to the right of the pole; vertices are $\left(\frac{2}{3}, 0\right)$ and $(-2, \pi)$.



58. $r = \dfrac{10}{5+20\sin\theta} = \dfrac{2}{1+4\sin\theta}$

$ep = 2, \quad e = 4, \quad p = \dfrac{1}{2}$

Hyperbola; directrix is parallel to the polar axis $\frac{1}{2}$ unit above the pole; vertices are $\left(\frac{2}{5}, \frac{\pi}{2}\right)$ and $\left(-\frac{2}{3}, \frac{3\pi}{2}\right)$.



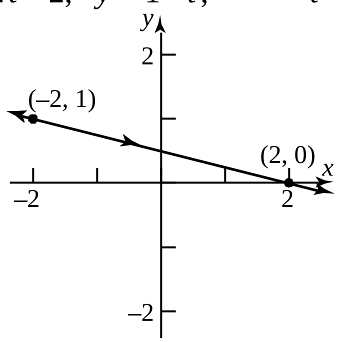
59.
$$\begin{aligned} r &= \frac{4}{1-\cos\theta} \\ r - r\cos\theta &= 4 \\ r &= 4 + r\cos\theta \\ r^2 &= (4 + r\cos\theta)^2 \\ x^2 + y^2 &= (4+x)^2 \\ x^2 + y^2 &= 16 + 8x + x^2 \\ y^2 - 8x - 16 &= 0 \end{aligned}$$

60.
$$\begin{aligned} r &= \frac{6}{2-\sin\theta} \\ 2r - r\sin\theta &= 6 \\ 2r &= 6 + r\sin\theta \\ 4r^2 &= (6 + r\sin\theta)^2 \\ 4\left(x^2+y^2\right) &= (6+y)^2 \\ 4x^2 + 4y^2 &= 36 + 12y + y^2 \\ 4x^2 + 3y^2 - 12y - 36 &= 0 \end{aligned}$$

61.
$$\begin{aligned} r &= \frac{8}{4+8\cos\theta} \\ 4r + 8r\cos\theta &= 8 \\ 4r &= 8 - 8r\cos\theta \\ r &= 2 - 2r\cos\theta \\ r^2 &= (2 - 2r\cos\theta)^2 \\ x^2 + y^2 &= (2-2x)^2 \\ x^2 + y^2 &= 4 - 8x + 4x^2 \\ 3x^2 - y^2 - 8x + 4 &= 0 \end{aligned}$$

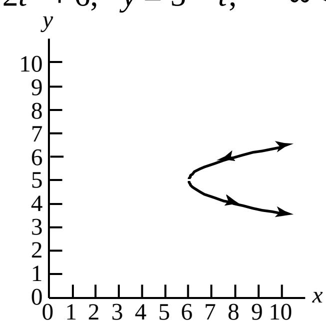
62.
$$\begin{aligned} r &= \frac{2}{3+2\cos\theta} \\ 3r + 2r\cos\theta &= 2 \\ 3r &= 2 - 2r\cos\theta \\ 9r^2 &= (2 - 2r\cos\theta)^2 \\ 9\left(x^2+y^2\right) &= (2-2x)^2 \\ 9x^2 + 9y^2 &= 4 - 8x + 4x^2 \\ 5x^2 + 9y^2 + 8x - 4 &= 0 \end{aligned}$$

63. $x = 4t - 2,\quad y = 1 - t,\quad -\infty < t < \infty$



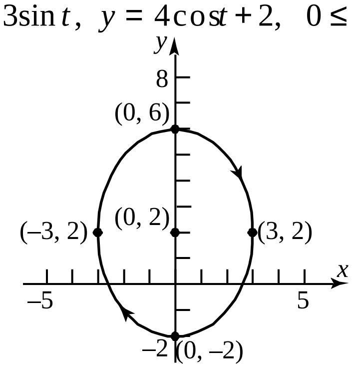
$$x = 4(1 - y) - 2$$
$$x = 4 - 4y - 2$$
$$x + 4y = 2$$

64. $x = 2t^2 + 6,\quad y = 5 - t,\quad -\infty < t < \infty$



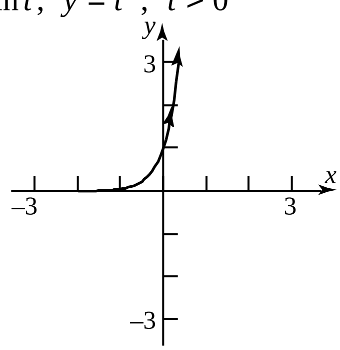
$$x = 2(5 - y)^2 + 6$$
$$x = 2\left(25 - 10y + y^2\right) + 6$$
$$x = 50 - 20y + 2y^2 + 6$$
$$x = 2y^2 - 20y + 56$$

65. $x = 3\sin t,\quad y = 4\cos t + 2,\quad 0 \le t \le 2\pi$



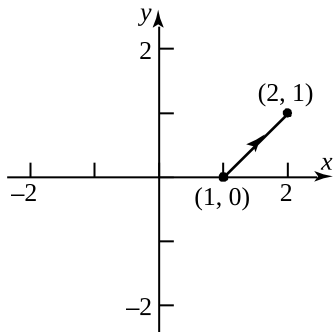
$$\frac{x}{3} = \sin t,\quad \frac{y-2}{4} = \cos t$$
$$\sin^2 t + \cos^2 t = 1$$
$$\left(\frac{x}{3}\right)^2 + \left(\frac{y-2}{4}\right)^2 = 1$$
$$\frac{x^2}{9} + \frac{(y-2)^2}{16} = 1$$

66. $x = \ln t,\quad y = t^3,\quad t > 0$



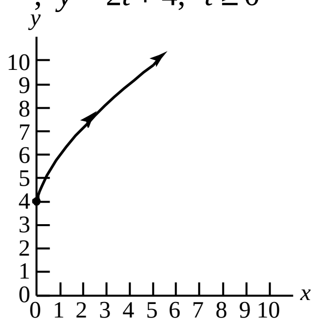
$$x = \ln y^{1/3}$$
$$x = \frac{1}{3}\ln y$$

67. $x = \sec^2 t,\ \ y = \tan^2 t,\ \ 0 \le t \le \dfrac{\pi}{4}$



$\tan^2 t + 1 = \sec^2 t$
$y + 1 = x$

68. $x = t^{3/2},\ \ y = 2t + 4,\ \ t \ge 0$



$y = 2x^{2/3} + 4$

69. Write the equation in standard form:

$$4x^2 + 9y^2 = 36 \ \rightarrow\ \frac{x^2}{9} + \frac{y^2}{4} = 1$$

The center of the ellipse is (0, 0). The major axis is the x-axis.
$a = 3;\ \ b = 2;\ \ c^2 = a^2 - b^2 = 9 - 4 = 5 \ \rightarrow\ c = \sqrt{5}.$
For the ellipse:
Vertices: (–3, 0), (3, 0)
Foci: $\left(-\sqrt{5},\ 0\right),\ \left(\sqrt{5},\ 0\right)$
For the hyperbola:
Foci: (–3, 0), (3, 0)
Vertices: $\left(-\sqrt{5},\ 0\right),\ \left(\sqrt{5},\ 0\right)$
Center: (0, 0)
$a = \sqrt{5};\ \ c = 3;\ \ b^2 = c^2 - a^2 = 9 - 5 = 4 \ \rightarrow\ b = 2$
The equation of the hyperbola is: $\dfrac{x^2}{5} - \dfrac{y^2}{4} = 1$

70. Write the equation in standard form:

$$x^2 - 4y^2 = 16 \ \rightarrow\ \frac{x^2}{16} - \frac{y^2}{4} = 1$$

The center of the hyperbola is (0, 0). The transverse axis is the x-axis.
$a = 4;\ \ b = 2;\ \ c^2 = a^2 + b^2 = 16 + 4 = 20 \ \rightarrow\ c = \sqrt{20} = 2\sqrt{5}.$
For the hyperbola:
Vertices: (–4, 0), (4, 0)
Foci: $\left(-2\sqrt{5},\ 0\right),\ \left(2\sqrt{5},\ 0\right)$

For the ellipse:

Foci: (–4, 0), (4, 0)

Vertices: $\left(-2\sqrt{5},\ 0\right), \left(2\sqrt{5},\ 0\right)$

Center: (0, 0)

$a = 2\sqrt{5};\ \ c = 4;\ \ b^2 = a^2 - c^2 = 20 - 16 = 4 \ \ \rightarrow\ \ b = 2$

The equation of the ellipse is: $\dfrac{x^2}{20} + \dfrac{y^2}{4} = 1$

71. Let (x, y) be any point in the collection of points.

The distance from (x, y) to $(3,\ 0) = \sqrt{(x-3)^2 + y^2}$.

The distance from (x, y) to the line $x = \dfrac{16}{3}$ is $\left|x - \dfrac{16}{3}\right|$.

Relating the distances, we have:

$$\sqrt{(x-3)^2 + y^2} = \frac{3}{4}\left|x - \frac{16}{3}\right|$$

$$(x-3)^2 + y^2 = \frac{9}{16}\left(x - \frac{16}{3}\right)^2$$

$$x^2 - 6x + 9 + y^2 = \frac{9}{16}\left(x^2 - \frac{32}{3}x + \frac{256}{9}\right)$$

$$16x^2 - 96x + 144 + 16y^2 = 9x^2 - 96x + 256$$

$$7x^2 + 16y^2 = 112 \rightarrow \frac{7x^2}{112} + \frac{16y^2}{112} = 1 \rightarrow \frac{x^2}{16} + \frac{y^2}{7} = 1$$

The set of points is an ellipse.

72. Let (x, y) be any point in the collection of points.

The distance from (x, y) to $(5,\ 0) = \sqrt{(x-5)^2 + y^2}$.

The distance from (x, y) to the line $x = \dfrac{16}{5}$ is $\left|x - \dfrac{16}{5}\right|$.

Relating the distances, we have:

$$\sqrt{(x-5)^2 + y^2} = \frac{5}{4}\left|x - \frac{16}{5}\right|$$

$$(x-5)^2 + y^2 = \frac{25}{16}\left(x - \frac{16}{5}\right)^2$$

$$x^2 - 10x + 25 + y^2 = \frac{25}{16}\left(x^2 - \frac{32}{5}x + \frac{256}{25}\right)$$

$$16x^2 - 160x + 400 + 16y^2 = 25x^2 - 160x + 256$$

$$9x^2 - 16y^2 = 144 \quad \rightarrow \quad \frac{9x^2}{144} - \frac{16y^2}{144} = 1 \quad \rightarrow \quad \frac{x^2}{16} - \frac{y^2}{9} = 1$$

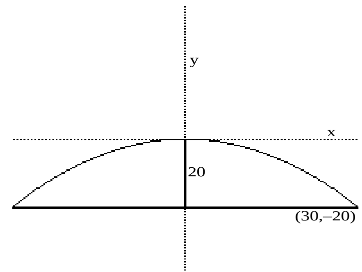
The set of points is a hyperbola.

73. Locate the parabola so that the vertex is at (0, 0) and opens up. It then has the equation: $x^2 = 4ay$. Since the light source is located at the focus and is 1 foot from the base, $a = 1$. The diameter is 2, so the point (1, y) is located on the parabola. Solve for y:

$$1^2 = 4(1)\,y \rightarrow 1 = 4y \rightarrow y = 0.25 \text{ feet}$$

The mirror should be 0.25 feet deep or 3 inches deep.

74. Set up the problem so that the vertex of the parabola is at (0, 0) and it opens down. Then the equation of the parabola has the form: $x^2 = cy$.
The point (30, –20) is a point on the parabola.
Solve for c and find the equation:



$$30^2 = c(-20)$$
$$c = -45$$
$$x^2 = -45y$$

To find the height of the bridge, 5 feet from the center, the point (5, y) is a point on the parabola. Solve for y:

$$5^2 = -45y \rightarrow 25 = -45y \rightarrow y \approx -0.56$$

The height of the bridge, 5 feet from the center, is 20 – 0.56 = 19.44 feet.

To find the height of the bridge, 10 feet from the center, the point (10, y) is a point on the parabola. Solve for y:

$$10^2 = -45y \rightarrow 100 = -45y \rightarrow y \approx -2.22$$

The height of the bridge, 10 feet from the center, is 20 – 2.22 = 17.78 feet.

To find the height of the bridge, 20 feet from the center, the point (20, y) is a point on the parabola. Solve for y:

$$20^2 = -45y \rightarrow 400 = -45y \rightarrow y \approx -8.89$$

The height of the bridge, 20 feet from the center, is 20 – 8.89 = 11.11 feet.

75. Place the semielliptical arch so that the x-axis coincides with the water and the y-axis passes through the center of the arch. Since the bridge has a span of 60 feet, the length of the major axis is 60, or $2a = 60$ or $a = 30$. The maximum height of the bridge is 20 feet, so $b = 20$. The equation is: $\dfrac{x^2}{900} + \dfrac{y^2}{400} = 1$.

The height 5 feet from the center:

$$\frac{5^2}{900} + \frac{y^2}{400} = 1 \rightarrow \frac{y^2}{400} = 1 - \frac{25}{900} \rightarrow y^2 = 400 \cdot \frac{875}{900} \rightarrow y \approx 19.72 \text{ feet}$$

The height 10 feet from the center:

$$\frac{10^2}{900} + \frac{y^2}{400} = 1 \rightarrow \frac{y^2}{400} = 1 - \frac{100}{900} \rightarrow y^2 = 400 \cdot \frac{800}{900} \rightarrow y \approx 18.86 \text{ feet}$$

The height 20 feet from the center:

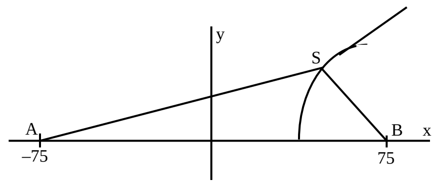
$$\frac{20^2}{900} + \frac{y^2}{400} = 1 \rightarrow \frac{y^2}{400} = 1 - \frac{400}{900} \rightarrow y^2 = 400 \cdot \frac{500}{900} \rightarrow y \approx 14.91 \text{ feet}$$

76. The major axis is 80 feet; therefore, $2a = 80$ or $a = 40$. The maximum height is 25 feet, so $b = 25$. To locate the foci, find c:

$$c^2 = a^2 - b^2 = 1600 - 625 = 975 \quad \rightarrow \quad c \approx 31.2$$

The foci are 31.2 feet from the center of the room or 8.8 feet from each wall.

77. (a) Set up a coordinate system so that the two stations lie on the x-axis and the origin is midway between them. The ship lies on a hyperbola whose foci are the locations of the two stations. Since the time difference is 0.00032 seconds and the speed of the signal is 186,000 miles per second, the difference in the distances of the ships from each station is:



distance = (186,000)(0.00032) = 59.52 miles

The difference of the distances from the ship to each station, 59.52, equals $2a$, so $a = 29.76$ and the vertex of the corresponding hyperbola is at (29.76, 0). Since the focus is at (75, 0), following this hyperbola, the ship would reach shore 45.24 miles from the master station.

(b) The ship should follow a hyperbola with a vertex at (60, 0). For this hyperbola, $a = 60$, so the constant difference of the distances from the ship to each station is 120. The time difference the ship should look for is:

$$\text{time} = \frac{120}{186,000} = 0.000645 \text{ seconds}$$

(c) Find the equation of the hyperbola with vertex at (60, 0) and a focus at (75, 0). The form of the equation of the hyperbola is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{where } a = 60.$$

Since $c = 75$ and $b^2 = c^2 - a^2 \quad \rightarrow \quad b^2 = 75^2 - 60^2 = 2025$.

The equation of the hyperbola is: $\dfrac{x^2}{3600} - \dfrac{y^2}{2025} = 1$.

Since the ship is 20 miles off shore, we have $y = 20$. Solve the equation for x:

$$\frac{x^2}{3600} - \frac{20^2}{2025} = 1 \quad \rightarrow \quad \frac{x^2}{3600} = 1 + \frac{400}{2025} = \frac{97}{81} \quad \rightarrow \quad x^2 = 3600\left(\frac{97}{81}\right)$$

$$x \approx 66 \text{ miles}$$

The ship's location is (66, 20).

78. (a) Train: Use equation (2) with $g = 3$, $v_0 = 0$, $h = 0$

$$x_1 = \frac{1}{2}(3)t^2 + 0\,t + 0 = \frac{3}{2}t^2$$

$$y_1 = 1$$

Mary:

$$x_2 = 6(t - 2)$$

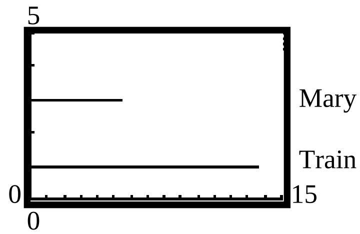
$$y_2 = 3$$

(b) Mary will catch the train if $x_1 = x_2$.

$$\frac{3}{2}t^2 = 6(t - 2) \rightarrow \frac{3}{2}t^2 = 6t - 12 \rightarrow t^2 - 4t + 8 = 0$$

Since $b^2 - 4ac = (-4)^2 - 4(1)(8) = 16 - 32 = -16 < 0$, the equation has no real solution. Thus, Mary will not catch the train.

(c)



79. (a) Use equation (2):

$$x = (100\cos(35^\circ))t$$

$$y = -\frac{1}{2}(32)t^2 + (100\sin(35^\circ))t + 6$$

(b) The ball is in the air until $y = 0$. Solve:

$$-16t^2 + (100\sin(35^\circ))t + 6 = 0$$

$$t = \frac{-100\sin 35^\circ \pm \sqrt{(100\sin(35^\circ))^2 - 4(-16)(6)}}{2(-16)}$$

$$\approx \frac{-57.36 \pm \sqrt{3673.9}}{-32} \approx -0.10 \text{ or } 3.69$$

The ball is in the air for about 3.69 seconds. (The negative solution is extraneous.)

(c) The maximum height occurs at the vertex of the quadratic function.

$$t = \frac{-b}{2a} = \frac{-100\sin(35^\circ)}{2(-16)} \approx 1.79 \text{ seconds}$$

Evaluate the function to find the maximum height:

$$-16(1.79)^2 + 100\sin 35^\circ(1.79) + 6 = 57.4 \text{ feet}$$

(d) Find the horizontal displacement:

$$x = (100\cos(35^\circ))(3.69) \approx 302 \text{ feet}$$

(e)