Chapter 12

Systems of Equations and Inequalities

12.2 Systems of Linear Equations: Three Equations Containing Three Variables

1. Substituting the values of the variables:

$$\begin{cases} 3x+3y+2z=4 & \longrightarrow & 3(1)+3(-1)+2(2)=3-3+4=4 \\ x-y-z=0 & \longrightarrow & 1-(-1)-2=1+1-2=0 \\ 2y-3z=-8 & \longrightarrow & 2(-1)-3(2)=-2-6=-8 \end{cases}$$

Each equation is satisfied, so $x=1,\ y=-1,\ z=2$ is a solution to the system of equations.

2. Substituting the values of the variables:

$$\begin{cases} 4x-z=7 & \longrightarrow & 4(2)-(1)=8-1=7 \\ 8x+5y-z=0 & \longrightarrow & 8(2)+5(-3)-1=16-15-1=0 \\ -x-y+5z=6 & \longrightarrow & -2-(-3)+5(1)=-2+3+5=6 \end{cases}$$

Each equation is satisfied, so $x=2, y=-3, z=1$ is a solution to the system of equations.

3. Multiply each side of the first equation by –2 and add to the second equation to eliminate x:

$$\begin{cases} x-y=6 & \longrightarrow & -2x+2y = -12 \\ 2x-3z=16 & \longrightarrow & \underline{2x -3z=16} \\ 2y+z=4 & & 2y-3z=4 \end{cases}$$

Multiply each side of the result by –1 and add to the original third equation to eliminate y:

$$\begin{aligned} 2y-3z=4 &\longrightarrow -2y+3z=-4 \\ 2y+z=4 &\longrightarrow \underline{2y+z=4} \\ & \qquad 4z=0 \\ & \qquad z=0 \end{aligned}$$

Substituting and solving for the other variables:

$$\begin{aligned} 2y+0&=4 \\ 2y&=4 \\ y&=2 \end{aligned} \qquad \begin{aligned} 2x-3(0)&=16 \\ 2x&=16 \\ x&=8 \end{aligned}$$

The solution is $x=8,\ y=2,\ z=0$.

4. Multiply each side of the first equation by 2 and add to the second equation to eliminate y:

$$\left\{\begin{array}{rcl} 2x + y = -4 & \xrightarrow{2} & 4x + 2y = -8 \\ -2y + 4z = 0 & \longrightarrow & \underline{-2y + 4z = 0} \\ 3x - 2z = -11 & & 4x + 4z = -8 \end{array}\right.$$

Multiply each side of the result by $\frac{1}{2}$ and add to the original third equation to eliminate z:

$$\begin{array}{rcr} 4x + 4z = -8 & \xrightarrow{1/2} & 2x + 2z = -4 \\ 3x - 2z = -11 & \longrightarrow & \underline{3x - 2z = -11} \\ & & 5x = -15 \\ & & x = -3 \end{array}$$

Substituting and solving for the other variables:

$$\begin{array}{rr} 2(-3) + y = -4 & 3(-3) - 2z = -11 \\ -6 + y = -4 & -9 - 2z = -11 \\ y = 2 & -2z = -2 \\ & z = 1 \end{array}$$

The solution is $x = -3,\ y = 2,\ z = 1$.

5. Multiply each side of the first equation by –2 and add to the second equation to eliminate x; and multiply each side of the first equation by 3 and add to the third equation to eliminate x:

$$\left\{\begin{array}{rcrcr} x - 2y + 3z = 7 & \xrightarrow{-2} & -2x + 4y - 6z = -14 & & \\ 2x + y + z = 4 & \longrightarrow & \underline{2x + y + z = 4} & & \\ -3x + 2y - 2z = -10 & & 5y - 5z = -10 & \xrightarrow{1/5} & y - z = -2 \\ & \xrightarrow{3} & 3x - 6y + 9z = 21 & & \\ & \longrightarrow & \underline{-3x + 2y - 2z = -10} & & \\ & & -4y + 7z = 11 & & \end{array}\right.$$

Multiply each side of the first result by 4 and add to the second result to eliminate y:

$$\begin{array}{rcr} y - z = -2 & \xrightarrow{4} & 4y - 4z = -8 \\ -4y + 7z = 11 & \longrightarrow & \underline{-4y + 7z = 11} \\ & & 3z = 3 \\ & & z = 1 \end{array}$$

Substituting and solving for the other variables:

$$\begin{array}{rr} y - 1 = -2 & x - 2(-1) + 3(1) = 7 \\ y = -1 & x + 2 + 3 = 7 \\ & x = 2 \end{array}$$

The solution is $x = 2,\ y = -1,\ z = 1$.

6. Multiply each side of the first equation by –2 and add to the second equation to eliminate y; and multiply each side of the first equation by 4 and add to the third equation to eliminate y:

$$\left\{\begin{array}{rcr} 2x + y - 3z = 0 & \xrightarrow{-2} & -4x - 2y + 6z = 0 \\ -2x + 2y + z = -7 & \longrightarrow & \underline{-2x + 2y + z = -7} \\ 3x - 4y - 3z = 7 & & -6x + 7z = -7 \end{array}\right.$$

$$\begin{aligned} &\xrightarrow{4} & 8x+4y-12z &= 0 \\ &\longrightarrow & \underline{3x-4y-3z} &= \underline{7} \\ & & 11x \qquad -15z &= 7 \end{aligned}$$

Multiply each side of the first result by 11 and multiply each side of the second result by 6 to eliminate x:

$$\begin{aligned} -6x+7z &= -7 & \xrightarrow{11} & \quad -66x+77z = -77 \\ 11x-15z &= 7 & \xrightarrow{6} & \quad \underline{66x-90z = 42} \\ & & & \quad -13z = -35 \\ & & & \quad z = \frac{35}{13} \end{aligned}$$

Substituting and solving for the other variables:

$$\begin{aligned} -6x+7\left(\frac{35}{13}\right) &= -7 \\ -6x+\frac{245}{13} &= -7 \\ -6x &= -\frac{336}{13} \\ x &= \frac{56}{12} \end{aligned} \qquad \begin{aligned} 2\left(\frac{56}{13}\right)+y-3\left(\frac{35}{13}\right) &= 0 \\ \frac{112}{13}+y-\frac{105}{13} &= 0 \\ y &= -\frac{7}{13} \end{aligned}$$

The solution is $x=\frac{56}{13},\ y=-\frac{7}{13},\ z=\frac{35}{13}$.

7. Add the first and second equations to eliminate z:

$$\begin{cases} x-y-z=1 \\ 2x+3y+z=2 \\ 3x+2y \quad =0 \end{cases} \qquad \begin{aligned} &\longrightarrow & x-y-z &= 1 \\ &\longrightarrow & \underline{2x+3y+z} &= \underline{2} \\ & & 3x+2y \quad &= 3 \end{aligned}$$

Multiply each side of the result by –1 and add to the original third equation to eliminate y:

$$\begin{aligned} 3x+2y=3 & \xrightarrow{-1} & -3x-2y &= -3 \\ 3x+2y=0 & \longrightarrow & \underline{3x+2y} &= \underline{0} \\ & & 0 &= -3 \end{aligned}$$

This result has no solution, so the system is inconsistent.

8. Add the first and second equations to eliminate z; then add the second and third equations to eliminate z:

$$\begin{cases} 2x-3y-z=0 \\ -x+2y+z=5 \\ 3x-4y-z=1 \end{cases} \qquad \begin{aligned} &\longrightarrow & 2x-3y-z &= 0 \\ &\longrightarrow & \underline{-x+2y+z} &= \underline{5} \\ & & x-y \quad &= 5 \end{aligned}$$

$$\begin{aligned} &\longrightarrow & -x+2y+z &= 5 \\ &\longrightarrow & \underline{3x-4y-z} &= \underline{1} \\ & & 2x-2y \quad &= 6 \quad \xrightarrow{1/2} \quad x-y=3 \end{aligned}$$

Multiply each side of the first result by –1 and add to the second result to eliminate y:

$$\begin{aligned} x - y = 5 &\xrightarrow{-1} & -x + y &= -5 \\ x - y = 3 &\longrightarrow & x - y &= 3 \\ \hline & & 0 &= -2 \end{aligned}$$

This result has no solution, so the system is inconsistent.

9. Add the first and second equations to eliminate x; and multiply the first equation by –3 and add to the third equation to eliminate x:

$$\begin{cases} x - y - z = 1 \\ -x + 2y - 3z = -4 \\ 3x - 2y - 7z = 0 \end{cases}$$

$$\begin{aligned} x - y - z &= 1 \\ -x + 2y - 3z &= -4 \\ \hline y - 4z &= -3 \end{aligned}$$

$$\begin{aligned} &\xrightarrow{-3} & -3x + 3y + 3z &= -3 \\ &\longrightarrow & 3x - 2y - 7z &= 0 \\ \hline & & y - 4z &= -3 \end{aligned}$$

Multiply each side of the first result by –1 and add to the second result to eliminate y:

$$\begin{aligned} y - 4z = -3 &\xrightarrow{-1} & -y + 4z &= 3 \\ y - 4z = -3 &\longrightarrow & y - 4z &= -3 \\ \hline & & 0 &= 0 \end{aligned}$$

The system is dependent. If z is any real number, then $y = 4z - 3$.

Solving for x in terms of z in the first equation:

$$\begin{aligned} x - (4z - 3) - z &= 1 \\ x - 4z + 3 - z &= 1 \\ x - 5z + 3 &= 1 \\ x &= 5z - 2 \end{aligned}$$

The solution is $x = 5z - 2$, $y = 4z - 3$, z is any real number.

10. Multiply the first equation by 2 and add to the second equation to eliminate z; and multiply the first equation by 3 and add to the third equation to eliminate z:

$$\begin{cases} 2x - 3y - z = 0 \\ 3x + 2y + 2z = 2 \\ x + 5y + 3z = 2 \end{cases}$$

$$\begin{aligned} &\xrightarrow{2} & 4x - 6y - 2z &= 0 \\ &\longrightarrow & 3x + 2y + 2z &= 2 \\ \hline & & 7x - 4y &= 2 \end{aligned}$$

$$\begin{aligned} &\xrightarrow{3} & 6x - 9y - 3z &= 0 \\ &\longrightarrow & x + 5y + 3z &= 2 \\ \hline & & 7x - 4y &= 2 \end{aligned}$$

Multiply each side of the first result by –1 and add to the second result to eliminate y:

$$\begin{aligned} 7x - 4y = 2 &\xrightarrow{-1} & -7x + 4y &= -2 \\ 7x - 4y = 2 &\longrightarrow & 7x - 4y &= 2 \\ \hline & & 0 &= 0 \end{aligned}$$

The system is dependent. If x is any real number, then $y = \frac{7x - 2}{4}$.

Solving for z in terms of x in the first equation:

$$z = 2x - 3y = 2x - 3\left(\frac{7x-2}{4}\right) = \frac{8x - 21x + 6}{4} = \frac{6-13x}{4}$$

The solution is $y = \frac{7x-2}{4},\ z = \frac{6-13x}{4},\ x$ is any real number

11. Multiply the first equation by –2 and add to the second equation to eliminate x; and add the first and third equations to eliminate x:

$$\begin{cases} 2x - 2y + 3z = 6 \\ 4x - 3y + 2z = 0 \\ -2x + 3y - 7z = 1 \end{cases}$$

$$\begin{array}{rl} \xrightarrow{\times(-2)} & -4x + 4y - 6z = -12 \\ \longrightarrow & \underline{4x - 3y + 2z = 0} \\ & y - 4z = -12 \end{array}$$

$$\begin{array}{rl} \longrightarrow & 2x - 2y + 3z = 6 \\ \longrightarrow & \underline{-2x + 3y - 7z = 1} \\ & y - 4z = 7 \end{array}$$

Multiply each side of the first result by –1 and add to the second result to eliminate y:

$$\begin{array}{rll} y - 4z = -12 & \xrightarrow{\times(-1)} & -y + 4z = 12 \\ y - 4z = 7 & \longrightarrow & \underline{y - 4z = 7} \\ & & 0 = 19 \end{array}$$

This result has no solution, so the system is inconsistent.

12. Multiply the first equation by –1 and add to the second equation to eliminate z; and multiply the first equation by –2 and add to the third equation to eliminate z:

$$\begin{cases} 3x - 2y + 2z = 6 \\ 7x - 3y + 2z = -1 \\ 2x - 3y + 4z = 0 \end{cases}$$

$$\begin{array}{rl} \xrightarrow{\times(-1)} & -3x + 2y - 2z = -6 \\ \longrightarrow & \underline{7x - 3y + 2z = -1} \\ & 4x - y = -7 \end{array}$$

$$\begin{array}{rl} \xrightarrow{\times(-2)} & -6x + 4y - 4z = -12 \\ \longrightarrow & \underline{2x - 3y + 4z = 0} \\ & -4x + y = -12 \end{array}$$

Add the first result to the second result to eliminate y:

$$\begin{array}{r} 4x - y = -7 \\ \underline{-4x + y = -12} \\ 0 = -19 \end{array}$$

This result has no solution, so the system is inconsistent.

13. Add the first and second equations to eliminate z; and multiply the second equation by 2 and add to the third equation to eliminate z:

$$\begin{cases} x + y - z = 6 \\ 3x - 2y + z = -5 \\ x + 3y - 2z = 14 \end{cases}$$

$$\begin{array}{rl} \longrightarrow & x + y - z = 6 \\ \longrightarrow & \underline{3x - 2y + z = -5} \\ & 4x - y = 1 \end{array}$$

$$\begin{array}{rl} \xrightarrow{\times 2} & 6x - 4y + 2z = -10 \\ \longrightarrow & \underline{x + 3y - 2z = 14} \\ & 7x - y = 4 \end{array}$$

Multiply each side of the first result by –1 and add to the second result to eliminate y:

$$
\begin{array}{rcl}
4x - y = 1 & \xrightarrow{-1} & -4x + y = -1 \\
7x - y = 4 & \xrightarrow{1} & \underline{7x - y = \ \ 4} \\
 & & 3x \quad = \ 3 \\
 & & x \quad = \ 1
\end{array}
$$

Substituting and solving for the other variables:

$$
\begin{aligned}
4(1) - y &= 1 \\
-y &= -3 \\
y &= 3
\end{aligned}
\qquad
\begin{aligned}
3(1) - 2(3) + z &= -5 \\
3 - 6 + z &= -5 \\
z &= -2
\end{aligned}
$$

The solution is $x = 1$, $y = 3$, $z = -2$.

14. Multiply the first equation by –3 and add to the second equation to eliminate y; and add the first and third equations to eliminate y:

$$
\begin{cases}
x - y + z = -4 \\
2x - 3y + 4z = -15 \\
5x + y - 2z = 12
\end{cases}
\qquad
\begin{array}{rcl}
 & \xrightarrow{-3} & -3x + 3y - 3z = \ 12 \\
 & \xrightarrow{1} & \underline{2x - 3y + 4z = -15} \\
 & & -x \quad + z = -3 \quad \to \quad z = x - 3 \\
 & \xrightarrow{1} & x - y + z = -4 \\
 & \xrightarrow{1} & \underline{5x + y - 2z = \ 12} \\
 & & 6x \quad - z = \ 8
\end{array}
$$

Substitute and solve:

$$
\begin{aligned}
6x - (x - 3) &= 8 \\
6x - x + 3 &= 8 \\
5x &= 5 \qquad z = 1 - 3 = -2 \\
x &= 1 \qquad y = 12 - 5x + 2z = 12 - 5(1) + 2(-2) = 12 - 5 - 4 = 3
\end{aligned}
$$

The solution is $x = 1$, $y = 3$, $z = -2$.

15. Add the first and second equations to eliminate z; and multiply the second equation by 3 and add to the third equation to eliminate z:

$$
\begin{cases}
x + 2y - z = -3 \\
2x - 4y + z = -7 \\
-2x + 2y - 3z = 4
\end{cases}
\qquad
\begin{array}{rcl}
 & \xrightarrow{1} & x + 2y - z = -3 \\
 & \xrightarrow{1} & \underline{2x - 4y + z = -7} \\
 & & 3x - 2y \quad = -10 \\
 & \xrightarrow{3} & 6x - 12y + 3z = -21 \\
 & \xrightarrow{1} & \underline{-2x + 2y - 3z = \ \ 4} \\
 & & 4x - 10y \quad = -17
\end{array}
$$

Multiply each side of the first result by –5 and add to the second result to eliminate y:

$$
\begin{array}{rcl}
3x - 2y = -10 & \xrightarrow{-5} & -15x + 10y = \ 50 \\
4x - 10y = -17 & \xrightarrow{1} & \underline{4x - 10y = -17} \\
 & & -11x \quad = \ 33 \\
 & & x \quad = -3
\end{array}
$$

Substituting and solving for the other variables:

$$\begin{aligned} 3(-3)-2y &= -10 \\ -9-2y &= -10 \\ -2y &= -1 \\ y &= \frac{1}{2} \end{aligned} \qquad \begin{aligned} -3+2\left(\frac{1}{2}\right)-z &= -3 \\ -3+1-z &= -3 \\ -z &= -1 \\ z &= 1 \end{aligned}$$

The solution is $x=-3,\ y=\frac{1}{2},\ z=1$.

16. Add the first and second equations to eliminate z; and multiply the first equation by 2 and add to the third equation to eliminate z:

$$\begin{cases} x+4y-3z=-8 \\ 3x-y+3z=12 \\ x+y+6z=1 \end{cases} \longrightarrow \begin{aligned} x+4y-3z&=-8 \\ \underline{3x-y+3z}&\underline{=12} \\ 4x+3y&=4 \end{aligned}$$

$$\xrightarrow{2} \begin{aligned} 2x+8y-6z&=-16 \\ \underline{x+y+6z}&\underline{=1} \\ 3x+9y&=-15 \end{aligned} \xrightarrow{1/3} x+3y=-5$$

Multiply each side of the second result by –1 and add to the first result to eliminate y:

$$\begin{aligned} 4x+3y&=4 \\ x+3y&=-5 \end{aligned} \quad \begin{aligned} &\longrightarrow \\ &\xrightarrow{-1} \end{aligned} \quad \begin{aligned} 4x+3y&=4 \\ \underline{-x-3y}&\underline{=5} \\ 3x&=9 \\ x&=3 \end{aligned}$$

Substituting and solving for the other variables:

$$\begin{aligned} 3+3y&=-5 \\ 3y&=-8 \\ y&=-\frac{8}{3} \end{aligned} \qquad \begin{aligned} 3+\left(-\frac{8}{3}\right)+6z&=1 \\ 6z&=\frac{2}{3} \\ z&=\frac{1}{9} \end{aligned}$$

The solution is $x=3,\ y=-\frac{8}{3},\ z=\frac{1}{9}$.

17. $y=ax^2+bx+c$

At (–1, 4) the equation becomes:

$$\begin{aligned} 4 &= a(-1)^2+b(-1)+c \\ 4 &= a-b+c \\ a-b+c &= 4 \end{aligned}$$

At (2, 3) the equation becomes:

$$\begin{aligned} 3 &= a(2)^2+b(2)+c \\ 3 &= 4a+2b+c \\ 4a+2b+c &= 3 \end{aligned}$$

At (0, 1) the equation becomes:

$$1 = a(0)^2 + b(0) + c$$
$$c = 1$$

The system of equations is:

$$\begin{cases} a - b + c = 4 \\ 4a + 2b + c = 3 \\ c = 1 \end{cases}$$

Substitute $c = 1$ into the first and second equations and simplify:

$$\begin{cases} a - b + 1 = 4 \rightarrow a - b = 3 \rightarrow a = b + 3 \\ 4a + 2b + 1 = 3 \rightarrow 4a + 2b = 2 \end{cases}$$

Solve the first equation for a, substitute into the second equation and solve:

$$4(b+3) + 2b = 2$$
$$4b + 12 + 2b = 2$$
$$6b = -10 \rightarrow b = -\frac{5}{3}$$
$$a = -\frac{5}{3} + 3 = \frac{4}{3}$$

The solution is $a = \frac{4}{3}$, $b = -\frac{5}{3}$, $c = 1$. So the equation is $y = \frac{4}{3}x^2 - \frac{5}{3}x + 1$.

18. $y = ax^2 + bx + c$

At (–1, –2) the equation becomes:

$$-2 = a(-1)^2 + b(-1) + c$$
$$-2 = a - b + c$$
$$a - b + c = -2$$

At (1, –4) the equation becomes:

$$-4 = a(1)^2 + b(1) + c$$
$$a + b + c = -4$$

At (2, 4) the equation becomes:

$$4 = a(2)^2 + b(2) + c$$
$$4a + 2b + c = 4$$

The system of equations is:

$$\begin{cases} a - b + c = -2 \\ a + b + c = -4 \\ 4a + 2b + c = 4 \end{cases}$$

Multiply the first equation by –1 and add to the second equation; and multiply the first equation by –1 and add to the third equation to eliminate c:

$$\begin{cases} a - b + c = -2 \xrightarrow{-1} -a + b - c = 2 \\ a + b + c = -4 \longrightarrow \underline{a + b + c = -4} \\ 4a + 2b + c = 4 \qquad 2b = -2 \xrightarrow{1/2} b = -1 \end{cases}$$

$$\xrightarrow{-1} -a + b - c = 2$$
$$\longrightarrow \underline{4a + 2b + c = 4}$$
$$3a + 3b = 6 \xrightarrow{1/3} a + b = 2$$

Substitute and solve:

$$a+(-1)=2 \quad\to\quad a=3$$
$$c=-a-b-4=-3-(-1)-4=-6$$

The solution is $a=3,\ b=-1,\ c=-6$. The equation is $y=3x^2-x-6$

19. Substitute the expression for I_2 into the second and third equations and simplify:

$$\begin{cases} I_2 = I_1 + I_3 \\ 5-3I_1-5I_2=0 \quad\to\quad 5-3I_1-5(I_1+I_3)=0 \quad\to\quad -8I_1-5I_3=-5 \\ 10-5I_2-7I_3=0 \quad\to\quad 10-5(I_1+I_3)-7I_3=0 \quad\to\quad -5I_1-12I_3=-10 \end{cases}$$

Multiply both sides of the second equation by 5 and multiply both sides of the third equation by –8 to eliminate I_1:

$$\begin{array}{rcr} -8I_1-5I_3=-5 & \xrightarrow{5} & -40I_1-25I_3=-25 \\ -5I_1-12I_3=-10 & \xrightarrow{-8} & 40I_1+96I_3=80 \\ \hline & & 71I_3=55 \\ & & I_3=\dfrac{55}{71} \end{array}$$

Substituting and solving for the other variables:

$$-8I_1-5\left(\frac{55}{71}\right)=-5 \qquad I_2=\frac{10}{71}+\frac{55}{71}$$
$$-8I_1-\frac{275}{71}=-5 \qquad I_2=\frac{65}{71}$$
$$-8I_1=-\frac{80}{71}$$
$$I_1=\frac{10}{71}$$

The solution is $I_1=\dfrac{10}{71},\ I_2=\dfrac{65}{71},\ I_3=\dfrac{55}{71}$.

20. Substitute the expression for I_3 into the second equation and simplify:

$$\begin{cases} I_3=I_1+I_2 \\ 8=4I_3+6I_2 \quad\to\quad 8=4(I_1+I_2)+6I_2 \quad\to\quad 8=4I_1+10I_2 \\ 8I_1=4+6I_2 \quad\to\quad 8I_1-6I_2=4 \end{cases}$$

Multiply both sides of the second equation by –2 and add to the third equation to eliminate I_1:

$$\begin{array}{rcr} 4I_1+10I_2=8 & \xrightarrow{-2} & -8I_1-20I_2=-16 \\ 8I_1-6I_2=4 & \xrightarrow{1} & 8I_1-6I_2=4 \\ \hline & & -26I_2=-12 \\ & & I_2=\dfrac{6}{13} \end{array}$$

Substituting and solving for the other variables:

$$4I_1+10\left(\frac{6}{13}\right)=8 \quad\to\quad 4I_1+\frac{60}{13}=8 \quad\to\quad 4I_1=\frac{44}{13} \quad\to\quad I_1=\frac{11}{13}$$

$$I_3=I_1+I_2=\frac{11}{13}+\frac{6}{13}=\frac{17}{13}$$

The solution is $I_1=\dfrac{11}{13},\ I_2=\dfrac{6}{13},\ I_3=\dfrac{17}{13}$.

21. Let x = the number of orchestra seats.
Let y = the number of main seats.
Let z = the number of balcony seats.
Since the total number of seats is 500, $x + y + z = 500$.
Since the total revenue is \$17,100 if all seats are sold, $50x + 35y + 25z = 17{,}100$.
If only half of the orchestra seats are sold, the revenue is \$14,600. So,

$$50\left(\tfrac{1}{2}x\right) + 35y + 25z = 14{,}600$$

Multiply each side of the first equation by –25 and add to the second equation to eliminate z; and multiply each side of the third equation by –1 and add to the second equation to eliminate z:

$$\begin{cases} x + y + z = 500 \\ 50x + 35y + 25z = 17100 \\ 25x + 35y + 25z = 14600 \end{cases}$$

$$\begin{array}{l} \xrightarrow{-25} \\ \longrightarrow \\ \\ \end{array} \begin{array}{r} -25x - 25y - 25z = -12500 \\ 50x + 35y + 25z = 17100 \\ \hline 25x + 10y \qquad = 4600 \end{array}$$

$$\begin{array}{l} \longrightarrow \\ \xrightarrow{-1} \\ \\ \\ \end{array} \begin{array}{r} 50x + 35y + 25z = 17100 \\ -25x - 35y - 25z = -14600 \\ \hline 25x \qquad = 2500 \\ x = 100 \end{array}$$

Substituting and solving for the other variables:

$$\begin{aligned} 25(100) + 10y &= 4600 \\ 2500 + 10y &= 4600 \\ 10y &= 2100 \\ y &= 210 \end{aligned} \qquad \begin{aligned} 100 + 210 + z &= 500 \\ 310 + z &= 500 \\ z &= 190 \end{aligned}$$

There are 100 orchestra seats, 210 main seats, and 190 balcony seats.

22. Let x = the number of adult tickets.
Let y = the number of child tickets.
Let z = the number of senior citizen tickets.
Since the total number of tickets is 405, $x + y + z = 405$.
Since the total revenue is \$2320, $8x + 4.50y + 6z = 2320$.
Twice as many children's tickets as adult tickets are sold. So, $y = 2x$.
Substitute for y in the first two equations and simplify:

$$\begin{cases} x + y + z = 405 & \to \quad x + 2x + z = 405 \quad \to \quad 3x + z = 405 \\ 8x + 4.50y + 6z = 2320 & \to \quad 8x + 4.50(2x) + 6z = 2320 \quad \to \quad 17x + 6z = 2320 \\ y = 2x & \end{cases}$$

Multiply the first equation by –6 and add to the second equation to eliminate z:

$$\begin{cases} 3x + z = 405 \\ 17x + 6z = 2320 \end{cases} \begin{array}{l} \xrightarrow{-6} \\ \longrightarrow \\ \\ \end{array} \begin{array}{r} -18x - 6z = -2430 \\ 17x + 6z = 2320 \\ \hline -x \qquad = -110 \end{array} \quad \to \quad x = 110$$

$y = 2x = 2(110) = 220$

$3x + z = 405 \quad \to \quad z = 405 - 3(110) = 405 - 330 = 75$

There were 110 adults, 220 children, and 75 senior citizens that bought tickets.

23. Let x = the number of servings of chicken.
Let y = the number of servings of corn.
Let z = the number of servings of 2% milk.
Protein equation: $30x+3y+9z=66$
Carbohydrate equation: $35x+16y+13z=94.5$
Calcium equation: $200x+10y+300z=910$
Multiply each side of the first equation by –16 and multiply each side of the second equation by 3 and add them to eliminate y; and multiply each side of the second equation by –5 and multiply each side of the third equation by 8 and add to eliminate y:

$$\begin{cases} 30x+3y+9z=66 & \xrightarrow{-16} & -480x-48y-144z=-1056 \\ 35x+16y+13z=94.5 & \xrightarrow{3} & 105x+48y+39z=283.5 \\ 200x+10y+300z=910 & & \overline{-375x\qquad -105z=-772.5} \end{cases}$$

$$\begin{aligned} &\xrightarrow{-5} & -175x-80y-65z&=-472.5 \\ &\xrightarrow{8} & 1600x+80y+2400z&=7280 \\ & & \overline{1425x\qquad +2335z}&=6807.5 \end{aligned}$$

Multiply each side of the first result by 19 and multiply each side of the second result by 5 to eliminate x:

$$\begin{aligned} -375x-105z&=-772.5 & \xrightarrow{19} & & -7125x-1995z&=-14677.5 \\ 1425x+2335z&=6807.5 & \xrightarrow{5} & & 7125x+11675z&=34037.5 \\ & & & & \overline{9680z}&=19360 \\ & & & & z&=2 \end{aligned}$$

Substituting and solving for the other variables:

$$\begin{aligned} -375x-105(2)&=-772.5 & \qquad 30(1.5)+3y+9(2)&=66 \\ -375x-210&=-772.5 & 45+3y+18&=66 \\ -375x&=-562.5 & 3y&=3 \\ x&=1.5 & y&=1 \end{aligned}$$

The dietitian should serve 1.5 servings of chicken, 1 serving of corn, and 2 servings of 2% milk.

24. Let x = the amount in Treasury bills.
Let y = the amount in Treasury bonds.
Let z = the amount in corporate bonds.
Since the total investment is \$20,000, $x+y+z=20,000$
Since the total income is to be \$1390, $0.05x+0.07y+0.10z=1390$
The investment in Treasury bills is to be \$3000 more than the investment in corporate bonds. So, $x=3000+z$
Substitute for x in the first two equations and simplify:

$$\begin{cases} x+y+z=20000 & \rightarrow & 3000+z+y+z=20000 & \rightarrow & y+2z=17000 \\ 0.05x+0.07y+0.10z=1390 & \rightarrow & 5(3000+z)+7y+10z=139000 & & \\ x=3000+z & & & \rightarrow & 7y+15z=124000 \end{cases}$$

Multiply each side of the first result by –7 and to the second result to eliminate y:

$$\begin{cases} y + 2z = 17000 \\ 7y + 15z = 124000 \end{cases} \begin{array}{l} \longrightarrow \\ \longrightarrow \end{array} \begin{array}{r} -7y - 14z = -119000 \\ 7y + 15z = 124000 \\ \hline z = 5000 \end{array}$$

$x = 3000 + z = 3000 + 5000 = 8000$

$y + 2z = 17000 \quad \to \quad y = 17000 - 2(5000) = 7000$

Kelly should invest \$8000 in Treasury bills, \$7000 in Treasury bonds, and \$5000 in corporate bonds.

25. Let x = the price of 1 hamburger.
Let y = the price of 1 order of fries.
Let z = the price of 1 drink.

We can construct the system

$$\begin{cases} 8x + 6y + 6z = 26.10 \\ 10x + 6y + 8z = 31.60 \end{cases}$$

A system involving only 2 equations that contain 3 or more unknowns cannot be solved uniquely. In other words, we can create as many solutions as we want by choosing a specific value for one of the variables and then solving the resulting
2 x 2 system.
For example, suppose we know that

$\$1.75 < \text{hamburger price} < \$2.25$

$\$0.75 < \text{fries price} < \$1.00$

$\$0.60 < \text{fries price} < \$0.90$

Pick a specific value for x, y or z	2x2 system	Solution
$x = \$2.00$	$\begin{cases} 8(2) + 6y + 6z = 26.10 \\ 10(2) + 6y + 8z = 31.60 \end{cases}$	$x = \$2.00$
		$y = \$0.93$
	$\begin{cases} 16 + 6y + 6z = 26.10 \\ 20 + 6y + 8z = 31.60 \end{cases}$	$z = \$0.75$
	$\begin{cases} 6y + 6z = 10.10 \\ 6y + 8z = 11.60 \end{cases}$	

$y = \$0.90$

$$\begin{cases} 8x + 6(0.9) + 6z = 26.10 \\ 10x + 6(0.9) + 8z = 31.60 \end{cases}$$

$$\begin{cases} 8x + 5.4 + 6z = 26.10 \\ 10x + 5.4 + 8z = 31.60 \end{cases}$$

$$\begin{cases} 8x + 6z = 20.7 \\ 10x + 8z = 26.2 \end{cases}$$

$x = \$2.00$

$y = \$0.90$

$z = \$0.65$

$z = \$0.80$

$$\begin{cases} 8x + 6y + 6(.8) = 26.10 \\ 10x + 6y + 8(.8) = 31.60 \end{cases}$$

$$\begin{cases} 8x + 6y + 4.8 = 26.10 \\ 10x + 6y + 6.4 = 31.60 \end{cases}$$

$$\begin{cases} 8x + 6y = 21.3 \\ 10x + 6y = 25.2 \end{cases}$$

$x = \$1.95$

$y = \$0.95$

$z = \$0.80$

26. Let x = the price of 1 hamburger.
Let y = the price of 1 order of fries.
Let z = the price of 1 drink

We can construct the system

$$\begin{cases} 8x + 6y + 6z = 26.10 \\ 10x + 6y + 8z = 31.60 \\ 3x + 2y + 4z = 10.95 \end{cases}$$

subtracting the second equation from the first equation yields

$$\begin{array}{r} 8x + 6y + 6z = 26.10 \\ \underline{10x + 6y + 8z = 31.60} \\ -2x - 2z = -5.5 \end{array}$$

Now consider the second and third equations

$$\begin{cases} 10x + 6y + 8z = 31.60 \xrightarrow{\quad} 10x + 6y + 8z = 31.60 \\ 3x + 2y + 4z = 10.95 \xrightarrow{3} -9x - 6y - 12z = -32.85 \end{cases}$$

adding these 2 equations yields

$$\begin{array}{r} 10x + 6y + 8z = 31.60 \\ \underline{-9x - 6y - 12z = -32.85} \\ x - 4z = -1.25 \end{array}$$

Now consider the system

$$\begin{cases} -2x - 2z = -5.5 \xrightarrow{\quad} -2x - 2z = -5.5 \\ x - 4z = -1.25 \xrightarrow{2} 2x - 8z = -2.5 \end{cases}$$

adding these 2 equations yields

$$\begin{array}{r} -2x-2z=-5.5 \\ \underline{2x-8z=-2.5} \\ -10z=-8 \end{array}$$

$\rightarrow z=0.8 \longrightarrow 2x-8(.8)=-2.5 \rightarrow x=1.95$

plugging into the first equation in the original system

$8(1.95)+6y+6(.8)=26.10$

$15.6+6y+4.8=26.10 \rightarrow 6y=5.7 \rightarrow y=0.95$

Therefore, one hamburger costs \$1.95, one order of fries costs \$0.95 and one drink costs \$0.80.

27. Let x = Beth's time working alone.
Let y = Bill's time working alone.
Let z = Edie's time working alone.

We can use the following tables to organize our work:

	Beth	Bill	Edie	Together
Hours to do job	x	y	z	10
Part of job done in 1 hour	$\frac{1}{x}$	$\frac{1}{y}$	$\frac{1}{z}$	$\frac{1}{10}$

Equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{10}$

	Bill	Edie	Together
Hours to do job	y	z	15
Part of job done in 1 hour	$\frac{1}{y}$	$\frac{1}{z}$	$\frac{1}{15}$

Equation $\frac{1}{y}+\frac{1}{z}=\frac{1}{15}$

	Beth	Bill	Edie	All three	Beth and Bill
Hours to do job	x	y	z	4	8
Part of job done in 1 hour	$\frac{1}{x}$	$\frac{1}{y}$	$\frac{1}{z}$	$\frac{1}{4}$	$\frac{1}{8}$

Equation

$$4\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+8\left(\frac{1}{x}+\frac{1}{y}\right)=1 \rightarrow \frac{12}{x}+\frac{12}{y}+\frac{4}{z}=1$$

We can construct the system

$$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{10} \\ \dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{15} \\ \dfrac{12}{x}+\dfrac{12}{y}+\dfrac{4}{z}=1 \end{cases}$$

subtracting the second equation from the first equation yield

$$\begin{array}{l} \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{10} \\ \quad\ \dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{15} \\ \hline \dfrac{1}{x}=\dfrac{1}{10}-\dfrac{1}{15} \end{array} \rightarrow \dfrac{1}{x}=\dfrac{1}{30} \rightarrow x=30$$

Plugging x = 30 into the original system yields

$$\begin{cases} \dfrac{1}{30}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{10} \longrightarrow \dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{10}-\dfrac{1}{30} \longrightarrow \dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{15} \\ \dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{15} \\ \dfrac{12}{30}+\dfrac{12}{y}+\dfrac{4}{z}=1 \longrightarrow \dfrac{12}{y}+\dfrac{4}{z}=1-\dfrac{12}{30} \longrightarrow \dfrac{12}{y}+\dfrac{4}{z}=\dfrac{3}{5} \end{cases}$$

Now consider the system

$$\begin{cases} \dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{15} \xrightarrow{-12} \dfrac{-12}{y}+\dfrac{-12}{z}=\dfrac{-12}{15} \\ \dfrac{12}{y}+\dfrac{4}{z}=\dfrac{3}{5} \longrightarrow \quad \dfrac{12}{y}+\dfrac{4}{z}=\dfrac{3}{5} \end{cases}$$

adding these 2 equations yields

$$\begin{array}{l} \dfrac{-12}{y}+\dfrac{-12}{z}=\dfrac{-12}{15} \\ \quad \dfrac{12}{y}+\dfrac{4}{z}\quad=\dfrac{3}{5} \\ \hline \dfrac{-12}{z}+\dfrac{4}{z}=\dfrac{-12}{15}+\dfrac{3}{5} \end{array} \rightarrow \dfrac{-8}{z}=\dfrac{-3}{15} \rightarrow \dfrac{8}{z}=\dfrac{1}{5} \rightarrow z=40$$

plugging z = 40 into the equation

$$\dfrac{12}{y}+\dfrac{4}{z}=\dfrac{3}{5} \rightarrow \dfrac{12}{y}+\dfrac{4}{40}=\dfrac{3}{5}$$

$$\dfrac{12}{y}+\dfrac{1}{10}=\dfrac{3}{5} \rightarrow \dfrac{12}{y}=\dfrac{3}{5}-\dfrac{1}{10} \rightarrow \dfrac{12}{y}=\dfrac{1}{2} \rightarrow y=24$$

So, working alone, it would take Beth 30 hours, Bill 24 hours and Edie 40 hours to finish the job.