Chapter 12

Systems of Equations and Inequalities

12.3 Systems of Linear Equations: Matrices

1. Writing the augmented matrix for the system of equations:

$$\begin{cases} x-5y=5 \\ 4x+3y=6 \end{cases} \rightarrow \left[\begin{array}{cc|c} 1 & -5 & 5 \\ 4 & 3 & 6 \end{array}\right]$$

2. Writing the augmented matrix for the system of equations:

$$\begin{cases} 3x+4y=7 \\ 4x-2y=5 \end{cases} \rightarrow \left[\begin{array}{cc|c} 3 & 4 & 7 \\ 4 & -2 & 5 \end{array}\right]$$

3. Writing the augmented matrix for the system of equations:

$$\begin{cases} 2x+3y-6=0 \\ 4x-6y+2=0 \end{cases} \rightarrow \begin{cases} 2x+3y=6 \\ 4x-6y=-2 \end{cases} \rightarrow \left[\begin{array}{cc|c} 2 & 3 & 6 \\ 4 & -6 & -2 \end{array}\right]$$

4. Writing the augmented matrix for the system of equations:

$$\begin{cases} 9x-y=0 \\ 3x-y-4=0 \end{cases} \rightarrow \begin{cases} 9x-y=0 \\ 3x-y=4 \end{cases} \rightarrow \left[\begin{array}{cc|c} 9 & -1 & 0 \\ 3 & -1 & 4 \end{array}\right]$$

5. Writing the augmented matrix for the system of equations:

$$\begin{cases} 0.01x-0.03y=0.06 \\ 0.13x+0.10y=0.20 \end{cases} \rightarrow \left[\begin{array}{cc|c} 0.01 & -0.03 & 0.06 \\ 0.13 & 0.10 & 0.20 \end{array}\right]$$

6. Writing the augmented matrix for the system of equations:

$$\begin{cases} \frac{4}{3}x-\frac{3}{2}y=\frac{3}{4} \\ -\frac{1}{4}x+\frac{1}{3}y=\frac{2}{3} \end{cases} \rightarrow \left[\begin{array}{cc|c} \frac{4}{3} & -\frac{3}{2} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{3} & \frac{2}{3} \end{array}\right]$$

7. Writing the augmented matrix for the system of equations:

$$\begin{cases} x-y+z=10 \\ 3x+3y=5 \\ x+y+2z=2 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 3 & 3 & 0 & 5 \\ 1 & 1 & 2 & 2 \end{array}\right]$$

8. Writing the augmented matrix for the system of equations:

$$\begin{cases} 5x - y - z = 0 \\ x + y = 5 \\ 2x - 3z = 2 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 5 & -1 & -1 & 0 \\ 1 & 1 & 0 & 5 \\ 2 & 0 & -3 & 2 \end{array}\right]$$

9. Writing the augmented matrix for the system of equations:

$$\begin{cases} x + y - z = 2 \\ 3x - 2y = 2 \\ 5x + 3y - z = 1 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 3 & -2 & 0 & 2 \\ 5 & 3 & -1 & 1 \end{array}\right]$$

10. Writing the augmented matrix for the system of equations:

$$\begin{cases} 2x + 3y - 4z = 0 \\ x - 5z + 2 = 0 \\ x + 2y - 3z = -2 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 2 & 3 & -4 & 0 \\ 1 & 0 & -5 & -2 \\ 1 & 2 & -3 & -2 \end{array}\right]$$

11. Writing the augmented matrix for the system of equations:

$$\begin{cases} x - y - z = 10 \\ 2x + y + 2z = -1 \\ -3x + 4y = 5 \\ 4x - 5y + z = 0 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 10 \\ 2 & 1 & 2 & -1 \\ -3 & 4 & 0 & 5 \\ 4 & -5 & 1 & 0 \end{array}\right]$$

12. Writing the augmented matrix for the system of equations:

$$\begin{cases} x - y + 2z - w = 5 \\ x + 3y - 4z + 2w = 2 \\ 3x - y - 5z - w = -1 \end{cases} \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 5 \\ 1 & 3 & -4 & 2 & 2 \\ 3 & -1 & -5 & -1 & -1 \end{array}\right]$$

13. $$\left[\begin{array}{cc|c} 1 & -3 & -2 \\ 2 & -5 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 1 & 9 \end{array}\right]$$

$$R_2 = -2r_1 + r_2$$

14. $$\left[\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & -5 & -4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 1 & 2 \end{array}\right]$$

$$R_2 = -2r_1 + r_2$$

15. $$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 2 & -5 & 6 & 6 \\ -3 & 3 & 4 & 6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ -3 & 3 & 4 & 6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 4 & 3 \\ 0 & 1 & -2 & 0 \\ 0 & -6 & 16 & 15 \end{array}\right]$$

(a) $R_2 = -2r_1 + r_2$ (b) $R_3 = 3r_1 + r_3$

16. $\left[\begin{array}{ccc|c} 1 & -3 & 3 & -5 \\ 2 & -5 & -3 & -5 \\ -3 & -2 & 4 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & 3 & -5 \\ 0 & 1 & -9 & 5 \\ -3 & -2 & 4 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & 3 & -5 \\ 0 & 1 & -9 & 5 \\ 0 & -11 & 13 & -9 \end{array}\right]$

(a) $R_2 = -2r_1 + r_2$ (b) $R_3 = 3r_1 + r_3$

17. $\left[\begin{array}{ccc|c} 1 & -3 & 2 & -6 \\ 2 & -5 & 3 & -4 \\ -3 & -6 & 4 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & 2 & -6 \\ 0 & 1 & -1 & 8 \\ -3 & -6 & 4 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & 2 & -6 \\ 0 & 1 & -1 & 8 \\ 0 & -15 & 10 & -12 \end{array}\right]$

(a) $\mathrm{R}_2 = -2r_1 + r_2$ (b) $R_3 = 3r_1 + r_3$

18. $\left[\begin{array}{ccc|c} 1 & -3 & -4 & -6 \\ 2 & -5 & 6 & -6 \\ -3 & 1 & 4 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & -4 & -6 \\ 0 & 1 & 14 & 6 \\ -3 & 1 & 4 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & -4 & -6 \\ 0 & 1 & 14 & 6 \\ 0 & -8 & -8 & -12 \end{array}\right]$

(a) $R_2 = -2r_1 + r_2$ (b) $R_3 = 3r_1 + r_3$

19. $\left[\begin{array}{ccc|c} 1 & -3 & 1 & -2 \\ 2 & -5 & 6 & -2 \\ -3 & 1 & 4 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & 1 & -2 \\ 0 & 1 & 4 & 2 \\ -3 & 1 & 4 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & 1 & -2 \\ 0 & 1 & 4 & 2 \\ 0 & -8 & 7 & 0 \end{array}\right]$

(a) $\mathrm{R}_2 = -2r_1 + r_2$ (b) $R_3 = 3r_1 + r_3$

20. $\left[\begin{array}{ccc|c} 1 & -3 & -1 & 2 \\ 2 & -5 & 2 & 6 \\ -3 & -6 & 4 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & -1 & 2 \\ 0 & 1 & 4 & 2 \\ -3 & -6 & 4 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & -1 & 2 \\ 0 & 1 & 4 & 2 \\ 0 & -15 & 1 & 12 \end{array}\right]$

(a) $R_2 = -2r_1 + r_2$ (b) $R_3 = 3r_1 + r_3$

21. $\begin{cases} x = 5 \\ y = -1 \end{cases}$ consistent $x = 5$, $y = -1$

22. $\begin{cases} x = -4 \\ y = 0 \end{cases}$ consistent $x = -4$, $y = 0$

23. $\begin{cases} x = 1 \\ y = 2 \\ 0 = 3 \end{cases}$ inconsistent

24. $\begin{cases} x = 0 \\ y = 0 \\ 0 = 2 \end{cases}$ inconsistent

25. $\begin{cases} x + 2z = -1 \\ y - 4z = -2 \\ 0 = 0 \end{cases}$ consistent $x = -1 - 2z,\ y = -2 + 4z,\ z$ is any real number

26. $\begin{cases} x + 4z = 4 \\ y + 3z = 2 \\ 0 = 0 \end{cases}$ consistent $x = 4 - 4z,\ y = 2 - 3z,\ z$ is any real number

27. $\begin{cases} x_1 = 1 \\ x_2 + x_4 = 2 \\ x_3 + 2x_4 = 3 \end{cases}$ consistent $x_1 = 1, x_2 = 2 - x_4, x_3 = 3 - 2x_4, x_4$ is any real number

28. $\begin{cases} x_1 = 1 \\ x_2 + 2x_4 = 2 \\ x_3 + 3x_4 = 0 \end{cases}$ consistent $x_1 = 1, x_2 = 2 - 2x_4, x_3 = -3x_4, x_4$ is any real number

29. $\begin{cases} x_1 + 4x_4 = 2 \\ x_2 + +x_3 + 3x_4 = 3 \\ 0 = 0 \end{cases}$ consistent $x_1 = 2 - 4x_4, x_2 = 3 - x_3 - 3x_4,$ x_3, x_4 are any real numbers

30. $\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 + 2x_4 = 3 \end{cases}$ consistent $x_1 = 1, x_2 = 2, x_3 = 3 - 2x_4,$ x_4 is any real number

31. $\begin{cases} x_1 + x_4 = -2 \\ x_2 + 2x_4 = 2 \\ x_3 - x_4 = 0 \\ 0 = 0 \end{cases}$ consistent $x_1 = -2 - x_4, x_2 = 2 - 2x_4, x_3 = x_4,$ x_4 is any real number

32. $\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \\ x_4 = 0 \end{cases}$ consistent $x_1 = 1_4, x_2 = 2, x_3 = 3, x_4 = 0$

33. $\begin{cases} x+y=8 \\ x-y=4 \end{cases}$ can be written as: $\left[\begin{array}{cc|c} 1 & 1 & 8 \\ 1 & -1 & 4 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & -2 & -4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 2 \end{array}\right]$$

$R_2 = -r_1 + r_2 \quad R_2 = -\frac{1}{2} r_2 \quad R_1 = -r_2 + r_1$

The solution is $x = 6,\ y = 2$.

34. $\begin{cases} x+2y=5 \\ x+\ \ y=3 \end{cases}$ can be written as: $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 1 & 1 & 3 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -1 & -2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

$R_2 = -r_1 + r_2 \quad R_2 = -r_2 \quad R_1 = -2r_2 + r_1$

The solution is $x = 1,\ y = 2$.

35. $\begin{cases} 2x-4y=-2 \\ 3x+2y=\ \ 3 \end{cases}$ can be written as: $\left[\begin{array}{cc|c} 2 & -4 & -2 \\ 3 & 2 & 3 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 3 & 2 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 8 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & \frac{3}{4} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{4} \end{array}\right]$$

$R_1 = \frac{1}{2} r_1 \quad R_2 = -3r_1 + r_2 \quad R_2 = \frac{1}{8} r_2 \quad R_1 = 2r_2 + r_1$

The solution is $x = \dfrac{1}{2},\ y = \dfrac{3}{4}$.

36. $\begin{cases} 3x+3y=3 \\ 4x+2y=\frac{8}{3} \end{cases}$ can be written as: $\left[\begin{array}{cc|c} 3 & 3 & 3 \\ 4 & 2 & \frac{8}{3} \end{array}\right]$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & 2 & \frac{8}{3} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & -\frac{4}{3} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & \frac{2}{3} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \end{array}\right]$$

$R_1 = \frac{1}{3} r_1 \quad R_2 = -4r_1 + r_2 \quad R_2 = -\frac{1}{2} r_2 \quad R_1 = -r_2 + r_1$

The solution is $x = \dfrac{1}{3},\ y = \dfrac{2}{3}$.

37. $\begin{cases} x+2y=4 \\ 2x+4y=8 \end{cases}$ can be written as: $\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 4 & 8 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

$R_2 = -2r_1 + r_2$

This is a dependent system and the solution is $x = -2y + 4$, y is any real number.

38. $\begin{cases} 3x - y = 7 \\ 9x - 3y = 21 \end{cases}$ can be written as: $\left[\begin{array}{cc|c} 3 & -1 & 7 \\ 9 & -3 & 21 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & \frac{7}{3} \\ 9 & -3 & 21 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & \frac{7}{3} \\ 0 & 0 & 0 \end{array}\right]$$

$R_1 = \frac{1}{3}r_1 \qquad R_2 = -9r_1 + r_2$

This is a dependent system and the solution is $x = \frac{1}{3}y + \frac{7}{3}$, y is any real number.

39. $\begin{cases} 2x + 3y = 6 \\ x - y = \dfrac{1}{2} \end{cases}$ can be written as: $\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 1 & -1 & \frac{1}{2} \end{array}\right]$

$$\rightarrow \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 3 \\ 1 & -1 & \frac{1}{2} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 3 \\ 0 & -\frac{5}{2} & -\frac{5}{2} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 3 \\ 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 1 \end{array}\right]$$

$R_1 = \frac{1}{2}r_1 \qquad R_2 = -r_1 + r_2 \qquad R_2 = -\frac{2}{5}r_2 \qquad R_1 = -\frac{3}{2}r_2 + r_1$

The solution is $x = \dfrac{3}{2}, y = 1$.

40. $\begin{cases} \dfrac{1}{2}x + y = -2 \\ x - 2y = 8 \end{cases}$ can be written as: $\left[\begin{array}{cc|c} \frac{1}{2} & 1 & -2 \\ 1 & -2 & 8 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & -4 \\ 1 & -2 & 8 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & -4 \\ 0 & -4 & 12 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & -4 \\ 0 & 1 & -3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right]$$

$R_1 = 2r_1 \qquad R_2 = -r_1 + r_2 \qquad R_2 = -\frac{1}{4}r_2 \qquad R_1 = -2r_2 + r_1$

The solution is $x = 2, y = -3$.

41. $\begin{cases} 3x - 5y = 3 \\ 15x + 5y = 21 \end{cases}$ can be written as: $\left[\begin{array}{cc|c} 3 & -5 & 3 \\ 15 & 5 & 21 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -\frac{5}{3} & 1 \\ 15 & 5 & 21 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -\frac{5}{3} & 1 \\ 0 & 30 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -\frac{5}{3} & 1 \\ 0 & 1 & \frac{1}{5} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{4}{3} \\ 0 & 1 & \frac{1}{5} \end{array}\right]$$

$R_1 = \frac{1}{3}r_1 \qquad R_2 = -15r_1 + r_2 \qquad R_2 = \frac{1}{30}r_2 \qquad R_1 = \frac{5}{3}r_2 + r_1$

The solution is $x = \dfrac{4}{3}, y = \dfrac{1}{5}$.

42. $\begin{cases} 2x - y = -1 \\ x + \dfrac{1}{2}y = \dfrac{3}{2} \end{cases}$ can be written as: $\left[\begin{array}{cc|c} 2 & -1 & -1 \\ 1 & \frac{1}{2} & \frac{3}{2} \end{array}\right]$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & \frac{1}{2} & \frac{3}{2} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 2 \end{array}\right]$$

$R_1 = \frac{1}{2}r_1 \qquad R_2 = -1r_1 + r_2 \qquad R_1 = \frac{1}{2}r_2 + r_1$

The solution is $x = \dfrac{1}{2}, y = 2$.

43. $\begin{cases} x - y = 6 \\ 2x - 3z = 16 \\ 2y + z = 4 \end{cases}$ can be written as: $\left[\begin{array}{rrr|r} 1 & -1 & 0 & 6 \\ 2 & 0 & -3 & 16 \\ 0 & 2 & 1 & 4 \end{array}\right]$

$$\rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 0 & 6 \\ 0 & 2 & -3 & 4 \\ 0 & 2 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 0 & 6 \\ 0 & 1 & -\frac{3}{2} & 2 \\ 0 & 2 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -\frac{3}{2} & 8 \\ 0 & 1 & -\frac{3}{2} & 2 \\ 0 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -\frac{3}{2} & 8 \\ 0 & 1 & -\frac{3}{2} & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_2 = -2r_1 + r_2$ $\quad R_2 = \frac{1}{2} r_2$ $\quad R_1 = r_2 + r_1$, $R_3 = -2r_2 + r_3$ $\quad R_3 = \frac{1}{4} r_3$

$$\rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_1 = \frac{3}{2} r_3 + r_1$

$R_2 = \frac{3}{2} r_3 + r_2$

The solution is $x = 8,\ y = 2,\ z = 0$.

44. $\begin{cases} 2x + y = -4 \\ -2y + 4z = 0 \\ 3x - 2z = -11 \end{cases}$ can be written as: $\left[\begin{array}{rrr|r} 2 & 1 & 0 & -4 \\ 0 & -2 & 4 & 0 \\ 3 & 0 & -2 & -11 \end{array}\right]$

$$\rightarrow \left[\begin{array}{rrr|r} 1 & \frac{1}{2} & 0 & -2 \\ 0 & -2 & 4 & 0 \\ 3 & 0 & -2 & -11 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & \frac{1}{2} & 0 & -2 \\ 0 & -2 & 4 & 0 \\ 0 & -\frac{3}{2} & -2 & -5 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & \frac{1}{2} & 0 & -2 \\ 0 & 1 & -2 & 0 \\ 0 & -\frac{3}{2} & -2 & -5 \end{array}\right]$$

$R_1 = \frac{1}{2} r_1$ $\quad R_3 = -3r_1 + r_3$ $\quad R_2 = -\frac{1}{2} r_2$

$$\rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -5 & -5 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 = -\frac{1}{2} r_2 + r_1$, $R_3 = \frac{3}{2} r_2 + r_3$ $\quad R_3 = -\frac{1}{5} r_3$ $\quad R_1 = -r_3 + r_1$, $R_2 = 2r_3 + r_2$

The solution is $x = -3,\ y = 2,\ z = 1$.

45. $\begin{cases} x-2y+3z=7 \\ 2x+y+z=4 \\ -3x+2y-2z=-10 \end{cases}$ can be written as: $\left[\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ -3 & 2 & -2 & -10 \end{array}\right]$

$$\to \left[\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & 5 & -5 & -10 \\ 0 & -4 & 7 & 11 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & -4 & 7 & 11 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$R_2=-2r_1+r_2$ $\quad R_2=\frac{1}{5}r_2$ $\quad R_1=2r_2+r_1$

$R_3=3r_1+r_3$ $\quad R_3=4r_2+r_3$

$$\to \left[\begin{array}{rrr|r} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_3=\frac{1}{3}r_3$ $\quad R_1=-r_3+r_1$

$R_2=r_3+r_2$

The solution is $x=2$, $y=-1$, $z=1$.

46. $\begin{cases} 2x+y-3z=0 \\ -2x+2y+z=-7 \\ 3x-4y-3z=7 \end{cases}$ can be written as: $\left[\begin{array}{rrr|r} 2 & 1 & -3 & 0 \\ -2 & 2 & 1 & -7 \\ 3 & -4 & -3 & 7 \end{array}\right]$

$$\to \left[\begin{array}{rrr|r} 1 & \frac{1}{2} & -\frac{3}{2} & 0 \\ -2 & 2 & 1 & -7 \\ 3 & -4 & -3 & 7 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & \frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 3 & -2 & -7 \\ 0 & -\frac{11}{2} & \frac{3}{2} & 7 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & \frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{7}{3} \\ 0 & -\frac{11}{2} & \frac{3}{2} & 7 \end{array}\right]$$

$R_1=\frac{1}{2}r_1$ $\quad R_2=2r_1+r_2$ $\quad R_2=\frac{1}{3}r_2$

$R_3=-3r_1+r_3$

$$\to \left[\begin{array}{rrr|r} 1 & 0 & -\frac{7}{6} & \frac{7}{6} \\ 0 & 1 & -\frac{2}{3} & -\frac{7}{3} \\ 0 & 0 & -\frac{13}{6} & -\frac{35}{6} \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -\frac{7}{6} & \frac{7}{6} \\ 0 & 1 & -\frac{2}{3} & -\frac{7}{3} \\ 0 & 0 & 1 & \frac{35}{13} \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & \frac{56}{13} \\ 0 & 1 & 0 & -\frac{7}{13} \\ 0 & 0 & 1 & \frac{35}{13} \end{array}\right]$$

$R_1=-\frac{1}{2}r_2+r_1$ $\quad R_3=-\frac{6}{13}r_3$ $\quad R_1=\frac{7}{6}r_3+r_1$

$R_3=\frac{11}{2}r_2+r_3$ $\quad R_2=\frac{2}{3}r_3+r_2$

The solution is $x=\frac{56}{13}$, $y=-\frac{7}{13}$, $z=\frac{35}{13}$.

47. $\begin{cases} 2x-2y-2z=2 \\ 2x+3y+z=2 \\ 3x+2y=0 \end{cases}$ can be written as: $\left[\begin{array}{ccc|c} 2 & -2 & -2 & 2 \\ 2 & 3 & 1 & 2 \\ 3 & 2 & 0 & 0 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 2 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 5 & 3 & 0 \\ 0 & 5 & 3 & -3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

$R_1 = \frac{1}{2} r_1$ $\quad R_2 = -2r_1 + r_2$ $\quad R_3 = -r_2 + r_3$

$R_3 = -3r_1 + r_3$

There is no solution. The system is inconsistent.

48. $\begin{cases} 2x-3y-z=0 \\ -x+2y+z=5 \\ 3x-4y-z=1 \end{cases}$ can be written as: $\left[\begin{array}{ccc|c} 2 & -3 & -1 & 0 \\ -1 & 2 & 1 & 5 \\ 3 & -4 & -1 & 1 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & -5 \\ 2 & -3 & -1 & 0 \\ 3 & -4 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & -5 \\ 0 & 1 & 1 & 10 \\ 0 & 2 & 2 & 16 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -1 & -5 \\ 0 & 1 & 1 & 10 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

Interchange r_1 and $-r_2$ $\quad R_2 = -2r_1 + r_2$ $\quad R_3 = -2r_2 + r_3$

$R_3 = -3r_1 + r_3$

There is no solution. The system is inconsistent.

49. $\begin{cases} -x+y+z=-1 \\ -x+2y-3z=-4 \\ 3x-2y-7z=0 \end{cases}$ can be written as: $\left[\begin{array}{ccc|c} -1 & 1 & 1 & -1 \\ -1 & 2 & -3 & -4 \\ 3 & -2 & -7 & 0 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ -1 & 2 & -3 & -4 \\ 3 & -2 & -7 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & -4 & -3 \\ 0 & 1 & -4 & -3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x-5z=-2 \\ y-4z=-3 \end{array}$$

$R_1 = -r_1$ $\quad R_2 = r_1 + r_2$ $\quad R_1 = r_2 + r_1$

$R_3 = -3r_1 + r_3$ $\quad R_3 = -r_2 + r_3$

The solution is $x = 5z-2,\ y = 4z-3,\ z$ is any real number.

50. $\begin{cases} 2x-3y-z=0 \\ 3x+2y+2z=2 \\ x+5y+3z=2 \end{cases}$ can be written as: $\left[\begin{array}{ccc|c} 2 & -3 & -1 & 0 \\ 3 & 2 & 2 & 2 \\ 1 & 5 & 3 & 2 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 3 & 2 \\ 3 & 2 & 2 & 2 \\ 2 & -3 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 3 & 2 \\ 0 & -13 & -7 & -4 \\ 0 & -13 & -7 & -4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 3 & 2 \\ 0 & 1 & \frac{7}{13} & \frac{4}{13} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Interchange r_1 and r_3 $\quad R_2 = -3r_1 + r_2$ $\quad R_3 = -r_2 + r_3$

$R_3 = -2r_1 + r_3$ $\quad R_2 = -\frac{1}{13} r_2$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{13} & \frac{6}{13} \\ 0 & 1 & \frac{7}{13} & \frac{4}{13} \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x + \frac{4}{13}z = \frac{6}{13} \\ y + \frac{7}{13}z = \frac{4}{13} \end{array}$$

$R_1 = -5r_2 + r_1$

The solution is $x = \dfrac{6}{13} - \dfrac{4}{13}z,\ y = \dfrac{4}{13} - \dfrac{7}{13}z,\ z$ is any real number.

51. $\begin{cases} 2x - 2y + 3z = 6 \\ 4x - 3y + 2z = 0 \\ -2x + 3y - 7z = 1 \end{cases}$ can be written as: $\left[\begin{array}{ccc|c} 2 & -2 & 3 & 6 \\ 4 & -3 & 2 & 0 \\ -2 & 3 & -7 & 1 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & \frac{3}{2} & 3 \\ 4 & -3 & 2 & 0 \\ -2 & 3 & -7 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & \frac{3}{2} & 3 \\ 0 & 1 & -4 & -12 \\ 0 & 1 & -4 & 7 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{2} & -9 \\ 0 & 1 & -4 & -12 \\ 0 & 0 & 0 & 19 \end{array}\right]$$

$R_1 = \frac{1}{2}r_1$ $\quad$ $R_2 = -4r_1 + r_2$, $R_3 = 2r_1 + r_3$ $\quad$ $R_1 = r_2 + r_1$, $R_3 = -r_2 + r_3$

There is no solution. The system is inconsistent.

52. $\begin{cases} 3x - 2y + 2z = 6 \\ 7x - 3y + 2z = -1 \\ 2x - 3y + 4z = 0 \end{cases}$ can be written as: $\left[\begin{array}{ccc|c} 3 & -2 & 2 & 6 \\ 7 & -3 & 2 & -1 \\ 2 & -3 & 4 & 0 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{2}{3} & 2 \\ 7 & -3 & 2 & -1 \\ 2 & -3 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{2}{3} & 2 \\ 0 & \frac{5}{3} & -\frac{8}{3} & -15 \\ 0 & -\frac{5}{3} & \frac{8}{3} & -4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{2}{3} & 2 \\ 0 & \frac{5}{3} & -\frac{8}{3} & -15 \\ 0 & 0 & 0 & -19 \end{array}\right]$$

$R_1 = \frac{1}{3}r_1$ $\quad$ $R_2 = -7r_1 + r_2$, $R_3 = -2r_1 + r_3$ $\quad$ $R_3 = r_2 + r_3$

There is no solution. The system is inconsistent.

53. $\begin{cases} x + y - z = 6 \\ 3x - 2y + z = -5 \\ x + 3y - 2z = 14 \end{cases}$ can be written as: $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 3 & -2 & 1 & -5 \\ 1 & 3 & -2 & 14 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & -5 & 4 & -23 \\ 0 & 2 & -1 & 8 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & 1 & -\frac{4}{5} & \frac{23}{5} \\ 0 & 2 & -1 & 8 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{5} & \frac{7}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{23}{5} \\ 0 & 0 & \frac{3}{5} & -\frac{6}{5} \end{array}\right]$$

$R_2 = -3r_1 + r_2$, $R_3 = -r_1 + r_3$ $\quad$ $R_2 = -\frac{1}{5}r_2$ $\quad$ $R_1 = -r_2 + r_1$, $R_3 = -2r_2 + r_3$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{5} & \frac{7}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{23}{5} \\ 0 & 0 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$R_3 = \tfrac{5}{3} r_3 \qquad R_1 = \tfrac{1}{5} r_3 + r_1$$
$$R_2 = \tfrac{4}{5} r_3 + r_2$$

The solution is $x = 1,\ y = 3,\ z = -2$.

54. $\begin{cases} x - y + z = -4 \\ 2x - 3y + 4z = -15 \\ 5x + y - 2z = 12 \end{cases}$ can be written as: $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -4 \\ 2 & -3 & 4 & -15 \\ 5 & 1 & -2 & 12 \end{array}\right]$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 1 & -4 \\ 0 & -1 & 2 & -7 \\ 0 & 6 & -7 & 32 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & -4 \\ 0 & 1 & -2 & 7 \\ 0 & 6 & -7 & 32 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

$$R_2 = -2r_1 + r_2 \qquad R_2 = -r_2 \qquad R_1 = r_2 + r_1$$
$$R_3 = -5r_1 + r_3 \qquad R_3 = -6r_2 + r_3$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$R_3 = \tfrac{1}{5} r_3 \qquad R_1 = r_3 + r_1$$
$$R_2 = 2r_3 + r_2$$

The solution is $x = 1,\ y = 3,\ z = -2$.

55. $\begin{cases} x + 2y - z = -3 \\ 2x - 4y + z = -7 \\ -2x + 2y - 3z = 4 \end{cases}$ can be written as: $\left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 2 & -4 & 1 & -7 \\ -2 & 2 & -3 & 4 \end{array}\right]$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & -8 & 3 & -1 \\ 0 & 6 & -5 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 1 & -\frac{3}{8} & \frac{1}{8} \\ 0 & 6 & -5 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{4} & -\frac{13}{4} \\ 0 & 1 & -\frac{3}{8} & \frac{1}{8} \\ 0 & 0 & -\frac{11}{4} & -\frac{11}{4} \end{array}\right]$$

$$R_2 = -2r_1 + r_2 \qquad R_2 = -\tfrac{1}{8} r_2 \qquad R_1 = -2r_2 + r_1$$
$$R_3 = 2r_1 + r_3 \qquad R_3 = -6r_2 + r_3$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{4} & -\frac{13}{4} \\ 0 & 1 & -\frac{3}{8} & \frac{1}{8} \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$R_3 = -\tfrac{4}{11} r_3 \qquad R_1 = \tfrac{1}{4} r_3 + r_1$$
$$R_2 = \tfrac{3}{8} r_3 + r_2$$

The solution is $x = -3, y = \dfrac{1}{2}, z = 1$.

56. $\begin{cases} x+4y-3z=-8 \\ 3x-y+3z=12 \\ x+y+6z=1 \end{cases}$ can be written as: $\left[\begin{array}{rrr|r} 1 & 4 & -3 & -8 \\ 3 & -1 & 3 & 12 \\ 1 & 1 & 6 & 1 \end{array}\right]$

$$\rightarrow \left[\begin{array}{rrr|r} 1 & 4 & -3 & -8 \\ 0 & -13 & 12 & 36 \\ 0 & -3 & 9 & 9 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 4 & -3 & -8 \\ 0 & 1 & -\frac{12}{13} & -\frac{36}{13} \\ 0 & -3 & 9 & 9 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & \frac{9}{13} & \frac{40}{13} \\ 0 & 1 & -\frac{12}{13} & -\frac{36}{13} \\ 0 & 0 & \frac{81}{13} & \frac{9}{13} \end{array}\right]$$

$R_2=-3r_1+r_2$ $\quad R_2=-\frac{1}{13}r_2$ $\quad R_1=-4r_2+r_1$

$R_3=-r_1+r_3$ $\quad R_3=3r_2+r_3$

$$\rightarrow \left[\begin{array}{rrr|r} 1 & 0 & \frac{9}{13} & \frac{40}{13} \\ 0 & 1 & -\frac{12}{13} & -\frac{36}{13} \\ 0 & 0 & 1 & \frac{1}{9} \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -\frac{8}{3} \\ 0 & 0 & 1 & \frac{1}{9} \end{array}\right]$$

$R_3=\frac{13}{81}r_3$ $\quad R_1=-\frac{9}{13}r_3+r_1$

$R_2=\frac{12}{13}r_3+r_2$

The solution is $x=3, y=-\dfrac{8}{3}, z=\dfrac{1}{9}$.

57. $\begin{cases} 3x+y-z=\dfrac{2}{3} \\ 2x-y+z=1 \\ 4x+2y=\dfrac{8}{3} \end{cases}$ can be written as: $\left[\begin{array}{rrr|r} 3 & 1 & -1 & \frac{2}{3} \\ 2 & -1 & 1 & 1 \\ 4 & 2 & 0 & \frac{8}{3} \end{array}\right]$

$$\rightarrow \left[\begin{array}{rrr|r} 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{9} \\ 2 & -1 & 1 & 1 \\ 4 & 2 & 0 & \frac{8}{3} \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{9} \\ 0 & -\frac{5}{3} & \frac{5}{3} & \frac{5}{9} \\ 0 & \frac{2}{3} & \frac{4}{3} & \frac{16}{9} \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{9} \\ 0 & 1 & -1 & -\frac{1}{3} \\ 0 & \frac{2}{3} & \frac{4}{3} & \frac{16}{9} \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & -1 & -\frac{1}{3} \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$R_1=\frac{1}{3}r_1$ $\quad R_2=-2r_1+r_2$ $\quad R_2=-\frac{3}{5}r_2$ $\quad R_1=-\frac{1}{3}r_2+r_1$

$R_3=-4r_1+r_3$ $\quad R_3=-\frac{2}{3}r_2+r_3$

$$\rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & -1 & -\frac{1}{3} \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_3=\frac{1}{2}r_3$ $\quad R_2=r_3+r_2$

The solution is $x=\dfrac{1}{3}, y=\dfrac{2}{3}, z=1$.

58. $\begin{cases} x + y = 1 \\ 2x - y + z = 1 \\ x + 2y + z = \frac{8}{3} \end{cases}$ can be written as: $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & -1 & 1 & 1 \\ 1 & 2 & 1 & \frac{8}{3} \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & 1 & 1 & \frac{5}{3} \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & \frac{5}{3} \\ 0 & -3 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -\frac{2}{3} \\ 0 & 1 & 1 & \frac{5}{3} \\ 0 & 0 & 4 & 4 \end{array}\right]$$

$R_2 = -2r_1 + r_2$ Interchange r_2 and r_3 $R_1 = -r_2 + r_1$

$R_3 = -r_1 + r_3$ $R_3 = 3r_2 + r_3$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -\frac{2}{3} \\ 0 & 1 & 1 & \frac{5}{3} \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_3 = \frac{1}{4}r_3$ $R_1 = r_3 + r_1$

$R_2 = -r_3 + r_2$

The solution is $x = \dfrac{1}{3}, y = \dfrac{2}{3}, z = 1$.

59. $\begin{cases} x + y + z + w = 4 \\ 2x - y + z = 0 \\ 3x + 2y + z - w = 6 \\ x - 2y - 2z + 2w = -1 \end{cases}$ can be written as: $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 2 & -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & -1 & 6 \\ 1 & -2 & -2 & 2 & -1 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & -3 & -1 & -2 & -8 \\ 0 & -1 & -2 & -4 & -6 \\ 0 & -3 & -3 & 1 & -5 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & -1 & -2 & -4 & -6 \\ 0 & -3 & -1 & -2 & -8 \\ 0 & -3 & -3 & 1 & -5 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & -3 & -1 & -2 & -8 \\ 0 & -3 & -3 & 1 & -5 \end{array}\right]$$

$R_2 = -2r_1 + r_2$ Interchange r_2 and r_3 $R_2 = -r_2$

$R_3 = -3r_1 + r_3$

$R_4 = -r_1 + r_4$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & -2 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 5 & 10 & 10 \\ 0 & 0 & 3 & 13 & 13 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & -2 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 13 & 13 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 7 & 7 \end{array}\right]$$

$R_1 = -r_2 + r_1$ $R_3 = \frac{1}{5}r_3$ $R_1 = r_3 + r_1$

$R_3 = 3r_2 + r_3$ $R_2 = -2r_3 + r_2$

$R_4 = 3r_2 + r_4$ $R_4 = -3r_3 + r_4$

$$\rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$R_4 = \frac{1}{7} r_4$ $\qquad$ $R_1 = r_4 + r_1$

$R_3 = -2r_4 + r_3$

The solution is $x = 1$, $y = 2$, $z = 0$, $w = 1$.

60. $\begin{cases} x + y + z + w = 4 \\ -x + 2y + z = 0 \\ 2x + 3y + z - w = 6 \\ -2x + y - 2z + 2w = -1 \end{cases}$ can be written as: $\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 4 \\ -1 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & -1 & 6 \\ -2 & 1 & -2 & 2 & -1 \end{array}\right]$

$$\rightarrow \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 4 \\ 0 & 3 & 2 & 1 & 4 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 3 & 0 & 4 & 7 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 3 & 2 & 1 & 4 \\ 0 & 3 & 0 & 4 & 7 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 4 & 6 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 5 & 10 & 10 \\ 0 & 0 & 3 & 13 & 13 \end{array}\right]$$

$R_2 = r_1 + r_2$ $\qquad$ Interchange r_2 and r_3 $\qquad$ $R_1 = -r_2 + r_1$

$R_3 = -2r_1 + r_3$ $\qquad$ $R_3 = -3r_2 + r_3$

$R_4 = 2r_1 + r_4$ $\qquad$ $R_4 = -3r_2 + r_4$

$$\rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 4 & 6 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 5 & 10 & 10 \\ 0 & 0 & 0 & -35 & -35 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 4 & 6 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$R_4 = 3r_3 - 5r_4$ $\qquad$ $R_3 = \frac{1}{5} r_3$

$R_4 = -\frac{1}{35} r_4$

So w =1 . Now substitute and solve.

$$w = 1$$
$$z + 2(1) = 2 \rightarrow z = 0$$
$$y - (0) - 3(1) = -2 \rightarrow y = 1$$
$$x + 0(1) + 2(0) + 4(1) = 6 \rightarrow x = 2$$

the solution is $x = 2, y = 1, z = 0, w = 1$.

61. $\begin{cases} x + 2y + z = 1 \\ 2x - y + 2z = 2 \\ 3x + y + 3z = 3 \end{cases}$ can be written as: $\left[\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 3 & 3 \end{array}\right]$

$$\rightarrow \left[\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & -5 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x + 2y + z = 1 \\ -5y = 0 \end{array}$$

$R_2 = -2r_1 + r_2$ $\qquad$ $R_3 = -r_2 + r_3$

$R_3 = -3r_1 + r_3$

Substitute and solve:

$$\begin{aligned} y &= 0 \\ x + 2(0) + z &= 1 \\ x + z &= 1 \\ x &= 1 - z \end{aligned}$$

The solution is $y = 0,\ x = 1 - z,\ z$ is any real number.

62. $\begin{cases} x + 2y - z = 3 \\ 2x - y + 2z = 6 \\ x - 3y + 3z = 4 \end{cases}$ can be written as: $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -1 & 2 & 6 \\ 1 & -3 & 3 & 4 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -5 & 4 & 0 \\ 0 & -5 & 4 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -5 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = -2r_1 + r_2$ $\quad R_3 = -r_2 + r_3$

$R_3 = -r_1 + r_3$

There is no solution. The system is inconsistent.

63. $\begin{cases} x - y + z = 5 \\ 3x + 2y - 2z = 0 \end{cases}$ can be written as: $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 3 & 2 & -2 & 0 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 5 & -5 & -15 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -1 & -3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -3 \end{array}\right]$$

$R_2 = -3r_1 + r_2$ $\quad R_2 = \frac{1}{5} r_2$ $\quad R_1 = r_2 + r_1$

The matrix in the third step represents the system $\begin{cases} x = 2 \\ y - z = -3 \end{cases}$

Therefore the solution is $x = 2;\ y = -3 + z;\ z$ is any real number

or

$x = 2;\ z = y + 3;\ y$ is any real number

64. $\begin{cases} 2x + y - z = 4 \\ -x + y + 3z = 1 \end{cases}$ can be written as: $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ -1 & 1 & 3 & 1 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -3 & -1 \\ 2 & 1 & -1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -3 & -1 \\ 0 & 3 & 5 & 6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & \frac{5}{3} & 2 \end{array}\right]$$

interchange r_1 and $-r_2$ $\quad R_2 = -2r_1 + r_2$ $\quad R_2 = \frac{1}{3} r_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 1 & \frac{5}{3} & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 1 \\ 0 & 1 & \frac{5}{3} & 2 \end{array}\right]$$

$R_1 = r_2 + r_1$

The matrix in the last step represents the system $\begin{cases} x + \frac{4}{3}z = 1 \\ y - \frac{5}{3}z = 2 \end{cases}$

Therefore the solution is:

$$x = 1 - \frac{4}{3}z;\; y = 2 - \frac{5}{3}z;\; z \text{ is any real number}$$

65.

$\begin{cases} 2x + 3y - z = 3 \\ x - y - z = 0 \\ -x + y + z = 0 \\ x + y + 3z = 5 \end{cases}$ can be written as: $\left[\begin{array}{ccc|c} 2 & 3 & -1 & 3 \\ 1 & -1 & -1 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 3 & 5 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 2 & 3 & -1 & 3 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 3 & 5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 5 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 5 & 1 & 3 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

interchange r_1 and r_2 $\quad$ $R_2 = -2r_1 + r_2$, $R_3 = r_1 + r_3$, $R_4 = -r_1 + r_4$ $\quad$ interchange r_3 and r_4

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -7 & -7 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -8 & -7 \\ 0 & 1 & -7 & -7 \\ 0 & 1 & 18 & 19 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -8 & -7 \\ 0 & 1 & -7 & -7 \\ 0 & 0 & 1 & \frac{19}{18} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 = -2r_3 + r_2$ $\quad$ $R_1 = r_2 + r_1$, $R_3 = -2r_2 + r_3$ $\quad$ $R_3 = \frac{1}{18}r_3$

The matrix in the last step represents the system $\begin{cases} x - 8z = -7 \\ y - 7z = -7 \\ z = \frac{19}{18} \end{cases}$

Therefore the solution is

$z = \frac{19}{18}$

$x = -7 + 8z = -7 + 8\left(\frac{19}{18}\right) = \frac{13}{9};\quad y = -7 + 7z = -7 + 7\left(\frac{19}{18}\right) = \frac{7}{18}$

66.
$$\begin{cases} x-3y+z=1 \\ 2x-y-4z=0 \\ x-3y+2z=1 \\ x-2y=5 \end{cases} \text{ can be written as: } \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 1 & -1 & -4 & 0 \\ 1 & -3 & 2 & 1 \\ 1 & -2 & 0 & 5 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 4 & 5 & 1 \\ 0 & 6 & -1 & 0 \\ 0 & 5 & -4 & -5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 4 & 5 & 1 \\ 0 & 0 & -34 & -6 \\ 0 & 0 & 41 & 25 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 4 & 5 & 1 \\ 0 & 0 & 1 & \frac{2}{17} \\ 0 & 0 & 1 & \frac{25}{41} \end{array}\right]$$

$R_2 = r_1 - r_2$ $\quad R_3 = 4r_3 - 6r_2$ $\quad R_3 = -\frac{1}{34} r_3$

$R_3 = r_1 - r_3$ $\quad R_4 = 5r_2 - 4r_4$ $\quad R_4 = \frac{1}{41} r_4$

$R_4 = r_1 - r_4$

The matrix in the last step represents an inconsistent system since

R_3 $\quad z = \dfrac{2}{17}$ and R_4 $\quad z = \dfrac{25}{41}$.

67. $\begin{cases} 4x+y+z-w=4 \\ x-y+2z+3w=3 \end{cases}$ can be written as: $\left[\begin{array}{cccc|c} 4 & 1 & 1 & -1 & 4 \\ 1 & -1 & 2 & 3 & 3 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 3 \\ 4 & 1 & 1 & -1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 3 \\ 0 & 5 & -7 & -13 & -8 \end{array}\right]$$

interchange r_1 and r_2 $\quad R_2 = -4r_1 + r_2$

The matrix in the last step represents the system $\begin{cases} x-y+2z+3w=3 \\ 5y-7z-13w=-8 \end{cases}$

The second equation yields

$$5y-7z-13w=-8 \rightarrow 5y=-8+7z+13w \rightarrow y=-\frac{8}{5}+\frac{7}{5}z+\frac{13}{5}w$$

The first equation yields

$$x-y+2z+3w=3 \rightarrow x=3+y-2z-3w$$

substituting for y

$$x=3+\left(-\frac{8}{5}+\frac{7}{5}z+\frac{13}{5}w\right)-2z-3w$$

$$x=-\frac{3}{5}z-\frac{2}{5}w+\frac{7}{5}$$

Therefore the solution is

$$x = -\frac{3}{5}z - \frac{2}{5}w + \frac{7}{5};\ \ y = -\frac{8}{5} + \frac{7}{5}z + \frac{13}{5}w$$

z and w are any real numbers

68. $\begin{cases} -4x + y = 5 \\ 2x - y + z - w = 5 \\ z + w = 4 \end{cases}$ can be written as: $\left[\begin{array}{cccc|c} -4 & 1 & 0 & 0 & 5 \\ 2 & -1 & 1 & -1 & 5 \\ 0 & 0 & 1 & 1 & 4 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & -\frac{1}{4} & 0 & 0 & -\frac{5}{4} \\ 2 & -1 & 1 & -1 & 5 \\ 0 & 0 & 1 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -\frac{1}{4} & 0 & 0 & -\frac{5}{4} \\ 0 & -\frac{1}{2} & 1 & -1 & \frac{15}{2} \\ 0 & 0 & 1 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -\frac{1}{4} & 0 & 0 & -\frac{5}{4} \\ 0 & 1 & -2 & 2 & -15 \\ 0 & 0 & 1 & 1 & 4 \end{array}\right]$$

$R_1 = -\frac{1}{4}r_1$ $\quad$ $R_2 = -2r_1 + r_2$ $\quad$ $R_2 = -2r_2$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & -5 \\ 0 & 1 & -2 & 2 & -15 \\ 0 & 0 & 1 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 4 & -7 \\ 0 & 0 & 1 & 1 & 4 \end{array}\right]$$

$R_1 = \frac{1}{4}r_2 + r_1$ $\quad$ $R_1 = \frac{1}{2}r_3 + r_1$

$R_2 = 2r_3 + r_2$

The matrix in the last step represents the system $\begin{cases} x + w = -3 \\ y + 4w = -7 \\ z + w = 4 \end{cases}$

Therefore the solution is

$x = -3 - w$

$y = -7 - 4w$

$z = 4 - w$

w is any real number

69. Each of the points must satisfy the equation $y = ax^2 + bx + c$.

$(1,2)$: $2 = a + b + c$

$(-2,-7)$: $-7 = 4a - 2b + c$

$(2,-3)$: $-3 = 4a + 2b + c$

Set up a matrix and solve:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 4 & -2 & 1 & -7 \\ 4 & 2 & 1 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -6 & -3 & -15 \\ 0 & -2 & -3 & -11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & -2 & -3 & -11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & -2 & -6 \end{array}\right]$$

$R_2 = -4r_1 + r_2$ $\quad R_2 = -\frac{1}{6}r_2$ $\quad R_1 = -r_2 + r_1$

$R_3 = -4r_1 + r_3$ $\quad R_3 = 2r_2 + r_3$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$R_3 = -\frac{1}{2}r_3$ $\quad R_1 = -\frac{1}{2}r_3 + r_1$

$R_2 = -\frac{1}{2}r_3 + r_2$

The solution is $a = -2,\ b = 1,\ c = 3$; so the equation is $y = -2x^2 + x + 3$.

70. Each of the points must satisfy the equation $y = ax^2 + bx + c$.

$(1,-1)$: $-1 = a + b + c$

$(3,-1)$: $-1 = 9a + 3b + c$

$(-2,14)$: $14 = 4a - 2b + c$

Set up a matrix and solve:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 9 & 3 & 1 & -1 \\ 4 & -2 & 1 & 14 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -6 & -8 & 8 \\ 0 & -6 & -3 & 18 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & \frac{4}{3} & -\frac{4}{3} \\ 0 & 0 & 5 & 10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{4}{3} & -\frac{4}{3} \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$R_2 = -9r_1 + r_2$ $\quad R_2 = -\frac{1}{6}r_2$ $\quad R_1 = -r_2 + r_1$

$R_3 = -4r_1 + r_3$ $\quad R_3 = -r_2 + r_3$ $\quad R_3 = \frac{1}{5}r_3$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$R_1 = \frac{1}{3}r_3 + r_1$

$R_2 = -\frac{4}{3}r_3 + r_2$

The solution is $a = 1,\ b = -4,\ c = 2$; so the equation is $y = x^2 - 4x + 2$.

71. Each of the points must satisfy the equation $f(x) = ax^3 + bx^2 + cx + d$.

$f(-3) = -112$: $-27a + 9b - 3c + d = -112$

$f(-1) = -2$: $-a + b - c + d = -2$

$f(1) = 4$: $a + b + c + d = 4$

$f(2) = 13$: $8a + 4b + 2c + d = 13$

Set up a matrix and solve:

$$\left[\begin{array}{cccc|c} -27 & 9 & -3 & 1 & -112 \\ -1 & 1 & -1 & 1 & -2 \\ 1 & 1 & 1 & 1 & 4 \\ 8 & 4 & 2 & 1 & 13 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ -1 & 1 & -1 & 1 & -2 \\ -27 & 9 & -3 & 1 & -112 \\ 8 & 4 & 2 & 1 & 13 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 36 & 24 & 28 & -4 \\ 0 & -4 & -6 & -7 & -19 \end{array}\right]$$

Interchange r_3 and r_1 $\quad R_2 = r_1 + r_2$

$R_3 = 27 r_1 + r_3$

$R_4 = -8 r_1 + r_4$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 36 & 24 & 28 & -4 \\ 0 & -4 & -6 & -7 & -19 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 24 & -8 & -40 \\ 0 & 0 & -6 & -3 & -15 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{5}{3} \\ 0 & 0 & -6 & -3 & -15 \end{array}\right]$$

$R_2 = \frac{1}{2} r_2$ $\quad R_1 = -r_2 + r_1$ $\quad R_3 = \frac{1}{24} r_3$

$R_3 = -36 r_2 + r_3$

$R_4 = 4 r_2 + r_4$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{3} & \frac{14}{3} \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{5}{3} \\ 0 & 0 & 0 & -5 & -25 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{3} & \frac{14}{3} \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{5}{3} \\ 0 & 0 & 0 & 1 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

$R_1 = -r_3 + r_1$ $\quad R_4 = -\frac{1}{5} r_4$ $\quad R_1 = -\frac{1}{3} r_4 + r_1$

$R_4 = 6 r_3 + r_4$ $\quad R_2 = -r_4 + r_2$

$R_3 = \frac{1}{3} r_4 + r_3$

The solution is $a = 3,\ b = -4,\ c = 0,\ d = 5$; so the equation is $f(x) = 3x^3 - 4x^2 + 5$.

72. Each of the points must satisfy the equation $f(x) = ax^3 + bx^2 + cx + d$.

$f(-2) = -10$: $\quad -8a + 4b - 2c + d = -10$

$f(-1) = 3$: $\quad -a + b - c + d = 3$

$f(1) = 5$: $\quad a + b + c + d = 5$

$f(3) = 15$: $\quad 27a + 9b + 3c + d = 15$

Set up a matrix and solve:

$$\left[\begin{array}{cccc|c} -8 & 4 & -2 & 1 & -10 \\ -1 & 1 & -1 & 1 & 3 \\ 1 & 1 & 1 & 1 & 5 \\ 27 & 9 & 3 & 1 & 15 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ -1 & 1 & -1 & 1 & 3 \\ -8 & 4 & -2 & 1 & -10 \\ 27 & 9 & 3 & 1 & 15 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 2 & 0 & 2 & 8 \\ 0 & 12 & 6 & 9 & 30 \\ 0 & -18 & -24 & -26 & -120 \end{array}\right]$$

Interchange r_3 and r_1 $\quad R_2 = r_1 + r_2$

$R_3 = 8 r_1 + r_3$

$R_4 = -27 r_1 + r_4$

$$\to\begin{bmatrix}1&1&1&1&5\\0&1&0&1&4\\0&12&6&9&30\\0&-18&-24&-26&-120\end{bmatrix}\to\begin{bmatrix}1&0&1&0&1\\0&1&0&1&4\\0&0&6&-3&-18\\0&0&-24&-8&-48\end{bmatrix}\to\begin{bmatrix}1&0&1&0&1\\0&1&0&1&4\\0&0&1&-\frac{1}{2}&-3\\0&0&-24&-8&-48\end{bmatrix}$$

$R_2=\frac{1}{2}r_2$ $\quad R_1=-r_2+r_1$, $R_3=-12r_2+r_3$, $R_4=18r_2+r_4$ $\quad R_3=\frac{1}{6}r_3$

$$\to\begin{bmatrix}1&0&0&\frac{1}{2}&4\\0&1&0&1&4\\0&0&1&-\frac{1}{2}&-3\\0&0&0&-20&-120\end{bmatrix}\to\begin{bmatrix}1&0&0&\frac{1}{2}&4\\0&1&0&1&4\\0&0&1&-\frac{1}{2}&-3\\0&0&0&1&6\end{bmatrix}\to\begin{bmatrix}1&0&0&0&1\\0&1&0&0&-2\\0&0&1&0&0\\0&0&0&1&6\end{bmatrix}$$

$R_1=-r_3+r_1$, $R_4=24r_3+r_4$ $\quad R_4=-\frac{1}{20}r_4$ $\quad R_1=-\frac{1}{2}r_4+r_1$, $R_2=-r_4+r_2$, $R_3=\frac{1}{2}r_4+r_3$

The solution is $a=1,\ b=-2,\ c=0,\ d=6$; so the equation is $f(x)=x^3-2x^2+6$.

73. Let x = the number of servings of salmon steak.
Let y = the number of servings of baked eggs.
Let z = the number of servings of acorn squash.
Protein equation: $30x+15y+3z=78$
Carbohydrate equation: $20x+2y+25z=59$
Vitamin A equation: $2x+20y+32z=75$
Set up a matrix and solve:

$$\begin{bmatrix}30&15&3&78\\20&2&25&59\\2&20&32&75\end{bmatrix}\to\begin{bmatrix}2&20&32&75\\20&2&25&59\\30&15&3&78\end{bmatrix}\to\begin{bmatrix}1&10&16&37.5\\20&2&25&59\\30&15&3&78\end{bmatrix}$$

Interchange r_3 and r_1 $\quad R_1=\frac{1}{2}r_1$

$$\to\begin{bmatrix}1&10&16&37.5\\0&-198&-295&-691\\0&-285&-477&-1047\end{bmatrix}\to\begin{bmatrix}1&10&16&37.5\\0&-198&-295&-691\\0&0&-\frac{3457}{66}&-\frac{3457}{66}\end{bmatrix}$$

$R_2=-20r_1+r_2$, $R_3=-30r_1+r_3$ $\quad R_3=-\frac{95}{66}r_2+r_3$

$$\to\begin{bmatrix}1&10&16&37.5\\0&-198&-295&-691\\0&0&1&1\end{bmatrix}$$

$R_3=-\frac{66}{3457}r_3$

Substitute $z=1$ and solve:

$$\begin{aligned}-198y-295(1)&=-691\\-198y&=-396\\y&=2\end{aligned}\qquad\begin{aligned}x+10(2)+16(1)&=37.5\\x+36&=37.5\\x&=1.5\end{aligned}$$

The dietitian should serve 1.5 servings of salmon steak, 2 servings of baked eggs, and 1 serving of acorn squash.

74. Let x = the number of servings of pork chops.
Let y = the number of servings of corn on the cob.
Let z = the number of servings of 2% milk.
Protein equation: $23x + 3y + 9z = 47$
Carbohydrate equation: $16y + 13z = 58$
Calcium equation: $10x + 10y + 300z = 630$

Set up a matrix and solve:

$$\left[\begin{array}{ccc|c} 23 & 3 & 9 & 47 \\ 0 & 16 & 13 & 58 \\ 10 & 10 & 300 & 630 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 30 & 63 \\ 0 & 16 & 13 & 58 \\ 23 & 3 & 9 & 47 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 30 & 63 \\ 0 & 1 & \frac{13}{16} & \frac{29}{8} \\ 0 & -20 & -681 & -1402 \end{array}\right]$$

Interchange $\frac{1}{10}r_3$ and r_1 $\quad R_3 = -23r_1 + r_3$
$R_2 = \frac{1}{16}r_2$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & \frac{467}{16} & \frac{475}{8} \\ 0 & 1 & \frac{13}{16} & \frac{29}{8} \\ 0 & 0 & -\frac{2659}{4} & -\frac{2659}{2} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{467}{16} & \frac{475}{8} \\ 0 & 1 & \frac{13}{16} & \frac{29}{8} \\ 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$R_1 = -r_2 + r_1$ $\quad R_3 = -\frac{4}{2659}r_3$ $\quad R_1 = -\frac{467}{16}r_3 + r_1$
$R_3 = 20r_2 + r_3$ $\quad R_2 = -\frac{13}{16}r_3 + r_2$

The dietitian should provide 1 serving of pork chops, 2 servings of corn on the cob, and 2 servings of 2% milk.

75. Let x = the amount invested in Treasury bills.
Let y = the amount invested in Treasury bonds.
Let z = the amount invested in corporate bonds.
Total investment equation: $x + y + z = 10000$
Annual income equation: $0.06x + 0.07y + 0.08z = 680$
Condition on investment equation: $z = \frac{1}{2}x$
Set up a matrix and solve:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10000 \\ 0.06 & 0.07 & 0.08 & 680 \\ 1 & 0 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10000 \\ 0 & 0.01 & 0.02 & 80 \\ 0 & -1 & -3 & -10000 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10000 \\ 0 & 1 & 2 & 8000 \\ 0 & -1 & -3 & -10000 \end{array}\right]$$

$R_2 = -0.06r_1 + r_2$ $\quad R_2 = 100r_2$
$R_3 = -r_1 + r_3$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2000 \\ 0 & 1 & 2 & 8000 \\ 0 & 0 & -1 & -2000 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2000 \\ 0 & 1 & 2 & 8000 \\ 0 & 0 & 1 & 2000 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4000 \\ 0 & 1 & 0 & 4000 \\ 0 & 0 & 1 & 2000 \end{array}\right]$$

$R_1 = -r_2 + r_1$ $\quad R_3 = -r_3$ $\quad R_1 = r_3 + r_1$
$R_3 = r_2 + r_3$ $\quad R_2 = -2r_3 + r_2$

Carletta should invest \$4000 in Treasury bills, \$4000 in Treasury bonds, and \$2000 in corporate bonds.

76. Let x = the amount invested in Treasury bills.
Let y = the amount invested in Treasury bonds.
Let z = the amount invested in corporate bonds.
Total investment equation: $x+y+z=20000$
Annual income equation: $0.05x+0.07y+0.09z=1280$
Condition on investment equation: $x=2z$

Set up a matrix and solve:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 20000 \\ 0.05 & 0.07 & 0.09 & 1280 \\ 1 & 0 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 20000 \\ 0 & 0.02 & 0.04 & 280 \\ 0 & -1 & -3 & -20000 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 20000 \\ 0 & 1 & 2 & 14000 \\ 0 & -1 & -3 & -20000 \end{array}\right]$$

$R_2=-0.05r_1+r_2$ $\quad$ $R_2=50r_2$
$R_3=-r_1+r_3$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 6000 \\ 0 & 1 & 2 & 14000 \\ 0 & 0 & -1 & -6000 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 6000 \\ 0 & 1 & 2 & 14000 \\ 0 & 0 & 1 & 6000 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 12000 \\ 0 & 1 & 0 & 2000 \\ 0 & 0 & 1 & 6000 \end{array}\right]$$

$R_1=-r_2+r_1$ $\quad$ $R_3=-r_3$ $\quad$ $R_1=r_3+r_1$
$R_3=r_2+r_3$ $\quad$ $R_2=-2r_3+r_2$

John should invest \$12,000 in Treasury bills, \$2,000 in Treasury bonds, and \$6,000 in corporate bonds.

77. Let x = the number of Deltas produced.
Let y = the number of Betas produced.
Let z = the number of Sigmas produced.
Painting equation: $10x+16y+8z=240$
Drying equation: $3x+5y+2z=69$
Polishing equation: $2x+3y+z=41$
Set up a matrix and solve:

$$\left[\begin{array}{ccc|c} 10 & 16 & 8 & 240 \\ 3 & 5 & 2 & 69 \\ 2 & 3 & 1 & 41 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 33 \\ 3 & 5 & 2 & 69 \\ 2 & 3 & 1 & 41 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 33 \\ 0 & 2 & -4 & -30 \\ 0 & 1 & -3 & -25 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 33 \\ 0 & 1 & -2 & -15 \\ 0 & 1 & -3 & -25 \end{array}\right]$$

$R_1=-3r_2+r_1$ $\quad$ $R_2=-3r_1+r_2$ $\quad$ $R_2=\frac{1}{2}r_2$
$R_3=-2r_1+r_3$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 48 \\ 0 & 1 & -2 & -15 \\ 0 & 0 & -1 & -10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 4 & 48 \\ 0 & 1 & -2 & -15 \\ 0 & 0 & 1 & 10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 10 \end{array}\right]$$

$R_1=-r_2+r_1$ $\quad$ $R_3=-r_3$ $\quad$ $R_1=-4r_3+r_1$
$R_3=-r_2+r_3$ $\quad$ $R_2=2r_3+r_2$

The company should produce 8 Deltas, 5 Betas, and 10 Sigmas.

78. Let x = the number of cases of orange juice produced.
Let y = the number of cases of grapefruit juice produced.
Let z = the number of cases of tomato juice produced.
Sterilizing equation: $9x + 10y + 12z = 398$
Filling equation: $6x + 4y + 4z = 164$
Labeling equation: $x + 2y + z = 58$

Set up a matrix and solve:

$$\left[\begin{array}{ccc|c} 9 & 10 & 12 & 398 \\ 6 & 4 & 4 & 164 \\ 1 & 2 & 1 & 58 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 58 \\ 6 & 4 & 4 & 164 \\ 9 & 10 & 12 & 398 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 58 \\ 0 & -8 & -2 & -184 \\ 0 & -8 & 3 & -124 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 58 \\ 0 & 1 & \frac{1}{4} & 23 \\ 0 & -8 & 3 & -124 \end{array}\right]$$

Interchange r_1 and r_3 $\quad R_2 = -6r_1 + r_2$, $R_3 = -9r_1 + r_3$ $\quad R_2 = -\frac{1}{8}r_2$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 12 \\ 0 & 1 & \frac{1}{4} & 23 \\ 0 & 0 & 5 & 60 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 12 \\ 0 & 1 & \frac{1}{4} & 23 \\ 0 & 0 & 1 & 12 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 12 \end{array}\right]$$

$R_1 = -2r_2 + r_1$, $R_3 = 8r_2 + r_3$ $\quad R_3 = \frac{1}{5}r_3$ $\quad R_1 = -\frac{1}{2}r_3 + r_1$, $R_2 = -\frac{1}{4}r_3 + r_2$

The company should prepare 6 cases of orange juice, 20 cases of grapefruit juice, and 12 cases of tomato juice.

79. Rewrite the system as set up and solve the matrix:

$$\begin{cases} -4 + 8 - 2I_2 = 0 \\ 8 = 5I_4 + I_1 \\ 4 = 3I_3 + I_1 \\ I_3 + I_4 = I_1 \end{cases} \to \begin{cases} 2I_2 = 4 \\ I_1 + 5I_4 = 8 \\ I_1 + 3I_3 = 4 \\ I_1 - I_3 - I_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 0 & 2 & 0 & 0 & 4 \\ 1 & 0 & 0 & 5 & 8 \\ 1 & 0 & 3 & 0 & 4 \\ 1 & 0 & -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 8 \\ 0 & 2 & 0 & 0 & 4 \\ 1 & 0 & 3 & 0 & 4 \\ 1 & 0 & -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 8 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 3 & -5 & -4 \\ 0 & 0 & -1 & -6 & -8 \end{array}\right]$$

Interchange r_2 and r_1 $\quad R_2 = \frac{1}{2}r_2$
$R_3 = -r_1 + r_3$
$R_4 = -r_1 + r_4$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 8 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & -6 & -8 \\ 0 & 0 & 3 & -5 & -4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 8 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 6 & 8 \\ 0 & 0 & 0 & -23 & -28 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 8 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 6 & 8 \\ 0 & 0 & 0 & 1 & \frac{28}{23} \end{array}\right]$$

Interchange r_3 and r_4 $\quad R_3 = -r_3 \quad R_4 = -\frac{1}{23}r_4$

$R_4 = -3r_3 + r_4$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{44}{23} \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & \frac{16}{23} \\ 0 & 0 & 0 & 1 & \frac{28}{23} \end{array}\right]$$

$R_1 = -5r_4 + r_1$

$R_3 = -6r_4 + r_3$

The solution is $I_1 = \dfrac{44}{23},\ I_2 = 2,\ I_3 = \dfrac{16}{23},\ I_4 = \dfrac{28}{23}$.

80. Rewrite the system as set up and solve the matrix:

$$\begin{cases} I_1 = I_3 + I_2 \\ 24 - 6I_1 - 3I_3 = 0 \\ 12 + 24 - 6I_1 - 6I_2 = 0 \end{cases} \rightarrow \begin{cases} I_1 - I_2 - I_3 = 0 \\ -6I_1 - 3I_3 = -24 \\ -6I_1 - 6I_2 = -36 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ -6 & 0 & -3 & -24 \\ -6 & -6 & 0 & -36 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -6 & -9 & -24 \\ 0 & -12 & -6 & -36 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & \frac{3}{2} & 4 \\ 0 & -12 & -6 & -36 \end{array}\right]$$

$R_2 = 6r_1 + r_2 \quad R_2 = -\frac{1}{6}r_2$

$R_3 = 6r_1 + r_3$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 4 \\ 0 & 1 & \frac{3}{2} & 4 \\ 0 & 0 & 12 & 12 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 4 \\ 0 & 1 & \frac{3}{2} & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3.5 \\ 0 & 1 & 0 & 2.5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 = r_2 + r_1 \quad R_3 = \frac{1}{12}r_3 \quad R_1 = -\frac{1}{2}r_3 + r_1$

$R_3 = 12r_2 + r_3 \quad R_2 = -\frac{3}{2}r_3 + r_2$

The solution is $I_1 = 3.5,\ I_2 = 2.5,\ I_3 = 1$.

81. Let x = the amount invested in Treasury bills.
Let y = the amount invested in Treasury bonds.
Let z = the amount invested in corporate bonds.

(a) Total investment equation: $x + y + z = 20000$
Annual income equation: $0.07x + 0.09y + 0.11z = 2000$

Set up a matrix and solve:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 20000 \\ .07 & .09 & .11 & 2000 \end{array}\right] \underset{R_2 = 100r_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 20000 \\ 7 & 9 & 11 & 200000 \end{array}\right] \underset{R_2 = r_2 - 7r_1}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 20000 \\ 0 & 2 & 4 & 60000 \end{array}\right]$$

$$\underset{R_2 = \frac{1}{2}r_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 20000 \\ 0 & 1 & 2 & 30000 \end{array}\right] \underset{R_1 = r_1 - r_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -10000 \\ 0 & 1 & 2 & 30000 \end{array}\right]$$

The matrix in the last step represents the system $\begin{cases} x - z = -10000 \\ y + 2z = 30000 \end{cases}$

Therefore the solution is

$x = -10000 + z;\ \ y = 30000 - 2z;\ \ z$ is any real number

Possible investment strategies:

Amount invested at

7%	9%	11%
0	10000	10000
1000	8000	11000
2000	6000	12000
3000	4000	13000
4000	2000	14000
5000	0	15000

(b) Total investment equation: $x + y + z = 25000$
Annual income equation: $0.07x + 0.09y + 0.11z = 2000$

Set up a matrix and solve:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25000 \\ .07 & .09 & .11 & 2000 \end{array}\right] \underset{R_2 = 100r_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25000 \\ 7 & 9 & 11 & 200000 \end{array}\right] \underset{R_2 = r_2 - 7r_1}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25000 \\ 0 & 2 & 4 & 25000 \end{array}\right]$$

$$\underset{R_2 = \frac{1}{2}r_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25000 \\ 0 & 1 & 2 & 12500 \end{array}\right] \underset{R_1 = r_1 - r_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 12500 \\ 0 & 1 & 2 & 12500 \end{array}\right]$$

The matrix in the last step represents the system $\begin{cases} x - z = 12500 \\ y + 2z = 12500 \end{cases}$

Therefore the solution is

$$x = 12500 + z;\ \ y = 12500 - 2z;\ \ z \text{ is any real number}$$

Possible investment strategies:

Amount invested at

7%	9%	11%
12500	12500	0
14500	8500	2000
16500	4500	4000
18750	0	6250

(c) Total investment equation: $x + y + z = 30000$
Annual income equation: $0.07x + 0.09y + 0.11z = 2000$
Set up a matrix and solve:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 30000 \\ .07 & .09 & .11 & 2000 \end{array}\right] \underset{R_2 = 100r_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 30000 \\ 7 & 9 & 11 & 200000 \end{array}\right] \underset{R_1 = r_2 - 7r_1}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 30000 \\ 0 & 2 & 4 & -10000 \end{array}\right]$$

$$\underset{R_2 = \frac{1}{2}r_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 30000 \\ 0 & 1 & 2 & -5000 \end{array}\right] \underset{R_1 = r_1 - r_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 35000 \\ 0 & 1 & 2 & -5000 \end{array}\right]$$

The matrix in the last step represents the system $\begin{cases} x - z = 35000 \\ y + 2z = -5000 \end{cases}$

Therefore the solution is

$$x = 35000 + z;\ \ y = -5000 - 2z;\ \ z \text{ is any real number}$$

One possible investment strategy

Amount invested at

7%	9%	11%
30000	0	0

This will yield (\$30000)(.07) = \$2100, which is more than the required income.

82. Let x = the amount invested in Treasury bills.
Let y = the amount invested in Treasury bonds.
Let z = the amount invested in corporate bonds.
Let I = income

Total investment equation: $x + y + z = 20000$
Annual income equation: $0.07x + 0.09y + 0.11z = I$

Set up a matrix and solve:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 25000 \\ .07 & .09 & .11 & I \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25000 \\ 7 & 9 & 11 & 100I \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 20000 \\ 0 & 2 & 4 & 100I - 175000 \end{array}\right]$$

$$R_2 = 100 r_2 \qquad\qquad R_1 = r_2 - 7r_1$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 25000 \\ 0 & 1 & 2 & 50I - 87500 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 112500 - 50I \\ 0 & 1 & 2 & 50I - 87500 \end{array}\right]$$

$$R_2 = \tfrac{1}{2} r_2 \qquad\qquad R_1 = r_1 - r_2$$

The matrix in the last step represents the system $\begin{cases} x - z = 112500 - 50I \\ y + 2z = 50I - 87500 \end{cases}$

Therefore the solution is

$$x = 112500 - 50I + z$$
$$y = 50I - 87500 - 2z$$
z is any real number

(a) $I = 1500$

$x = 112500 - 50(1500) + z = 37500 + z$

$y = 50I - 87500 - 2z = 50(1500) - 87500 - 2z = -12500 - 2z$

z is any real number

Investing all of the money at 7% yields more than \$1500.

(b) $I = 2000$

$x = 112500 - 50(2000) + z = 12500 + z$

$y = 50I - 87500 - 2z = 50(2000) - 87500 - 2z = 12500 - 2z$

z is any real number

Possible investment strategies:

Amount invested at

7%	9%	11%
12500	12500	0
15500	6500	3000
18750	0	6250

(c) $I = 2500$

$x = 112500 - 50(2500) + z = -12500 + z$

$y = 50I - 87500 - 2z = 50(2500) - 87500 - 2z = 37500 - 2z$

z is any real number

Possible investment strategies:

Amount invested at

7%	9%	11%
0	12500	12500
1000	10500	13500
6250	0	18750

83. Let x = the amount of liquid 1.
Let y = the amount of liquid 2.
Let z = the amount of liquid 3.

$$.20x + .40y + .30z = 40 \text{ Vitamin C}$$
$$.30x + .20y + .50z = 30 \text{ Vitamin D}$$

multiplying each equation by 10 yields

$$2x + 4y + 3z = 400$$
$$3x + 2y + 5z = 300$$

Set up a matrix and solve: $\left[\begin{array}{ccc|c} 2 & 4 & 3 & 400 \\ 3 & 2 & 5 & 300 \end{array}\right] \underset{R_1 = \frac{1}{2}r_1}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 2 & \frac{3}{2} & 200 \\ 3 & 2 & 5 & 300 \end{array}\right] \underset{R_2 = r_2 - 3r_1}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 2 & \frac{3}{2} & 200 \\ 0 & -4 & \frac{1}{2} & -300 \end{array}\right]$

$$\underset{R_2 = -\frac{1}{4}r_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 2 & \frac{3}{2} & 200 \\ 0 & 1 & -\frac{1}{8} & 75 \end{array}\right] \underset{R_1 = r_1 - 2r_2}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{4} & 50 \\ 0 & 1 & -\frac{1}{8} & 75 \end{array}\right]$$

The matrix in the last step represents the system $\begin{cases} x + \frac{7}{4}z = 50 \\ y - \frac{1}{8}z = 75 \end{cases}$

Therefore the solution is

$$x = 50 - \tfrac{7}{4}z; \quad y = 75 + \tfrac{1}{8}z; \quad z \text{ is any real number}$$

Possible combinations:

Liquid 1	Liquid 2	Liquid 3
50mg	75mg	0mg
36mg	76mg	8mg
22mg	77mg	16mg
8mg	78mg	24mg

84. Let x = the amount of powder 1.
Let y = the amount of powder 2.
Let z = the amount of powder 3.

$$.20x + .40y + .30z = 12 \text{ Vitamin } B_{12}$$
$$.30x + .20y + .40z = 12 \text{ Vitamin E}$$

multiplying each equation by 10 yields

$$2x + 4y + 3z = 120$$
$$3x + 2y + 4z = 120$$

Set up a matrix and solve:

$$\left[\begin{array}{ccc|c} 2 & 4 & 3 & 120 \\ 3 & 2 & 4 & 120 \end{array}\right] \underset{R_2 = r_2 - \frac{3}{2}r_1}{\rightarrow} \left[\begin{array}{ccc|c} 2 & 4 & 3 & 120 \\ 0 & -4 & -.5 & -60 \end{array}\right] \underset{R_1 = r_1 + r_2}{\rightarrow} \left[\begin{array}{ccc|c} 2 & 0 & 2.5 & 60 \\ 0 & -4 & -.5 & -60 \end{array}\right]$$

The matrix in the last step represents the system $\begin{cases} 2x + 2.5z = 60 \\ -4y - .5z = -60 \end{cases}$

Therefore the solution is

$x = 30 - 1.25z$

$y = 15 - .125z$

z is any real number

Possible combinations:

Powder 1	Powder 2	Powder 3
30mg	15mg	0mg
20mg	14mg	8mg
10mg	13mg	16mg
0mg	12mg	24mg

85 – 87. Answers will vary.