Chapter 12

Systems of Equations and Inequalities

12.4 Systems of Linear Equations: Determinants

1. Evaluating the determinant:
$$\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 3(2) - 4(1) = 6 - 4 = 2$$

2. Evaluating the determinant:
$$\begin{vmatrix} 6 & 1 \\ 5 & 2 \end{vmatrix} = 6(2) - 5(1) = 12 - 5 = 7$$

3. Evaluating the determinant:
$$\begin{vmatrix} 6 & 4 \\ -1 & 3 \end{vmatrix} = 6(3) - (-1)(4) = 18 + 4 = 22$$

4. Evaluating the determinant:
$$\begin{vmatrix} 8 & -3 \\ 4 & 2 \end{vmatrix} = 8(2) - 4(-3) = 16 + 12 = 28$$

5. Evaluating the determinant:
$$\begin{vmatrix} -3 & -1 \\ 4 & 2 \end{vmatrix} = -3(2) - 4(-1) = -6 + 4 = -2$$

6. Evaluating the determinant:
$$\begin{vmatrix} -4 & 2 \\ -5 & 3 \end{vmatrix} = -4(3) - (-5)(2) = -12 + 10 = -2$$

7. Evaluating the determinant:
$$\begin{vmatrix} 3 & 4 & 2 \\ 1 & -1 & 5 \\ 1 & 2 & -2 \end{vmatrix} = 3\begin{vmatrix} -1 & 5 \\ 2 & -2 \end{vmatrix} - 4\begin{vmatrix} 1 & 5 \\ 1 & -2 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}$$
$$\begin{aligned} &= 3\left[(-1)(-2) - 2(5)\right] - 4\left[1(-2) - 1(5)\right] + 2\left[1(2) - 1(-1)\right] \\ &= 3(2 - 10) - 4(-2 - 5) + 2(2 + 1) \\ &= 3(-8) - 4(-7) + 2(3) \\ &= -24 + 28 + 6 \\ &= 10 \end{aligned}$$

8. Evaluating the determinant:

$$\begin{vmatrix} 1 & 3 & -2 \\ 6 & 1 & -5 \\ 8 & 2 & 3 \end{vmatrix} = 1\begin{vmatrix} 1 & -5 \\ 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 6 & -5 \\ 8 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 6 & 1 \\ 8 & 2 \end{vmatrix}$$

$$\begin{aligned} &= 1[1(3) - 2(-5)] - 3[6(3) - 8(-5)] - 2[6(2) - 8(1)] \\ &= 1(3+10) - 3(18 + 40) - 2(12 - 8) \\ &= 1(13) - 3(58) - 2(4) \\ &= 13 - 174 - 8 \\ &= -169 \end{aligned}$$

9. Evaluating the determinant:

$$\begin{vmatrix} 4 & -1 & 2 \\ 6 & -1 & 0 \\ 1 & -3 & 4 \end{vmatrix} = 4\begin{vmatrix} -1 & 0 \\ -3 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 6 & 0 \\ 1 & 4 \end{vmatrix} + 2\begin{vmatrix} 6 & -1 \\ 1 & -3 \end{vmatrix}$$

$$\begin{aligned} &= 4[-1(4) - 0(-3)] + 1[6(4) - 1(0)] + 2[6(-3) - 1(-1)] \\ &= 4(-4) + 1(24) + 2(-17) \\ &= -16 + 24 - 34 \\ &= -26 \end{aligned}$$

10. Evaluating the determinant:

$$\begin{vmatrix} 3 & -9 & 4 \\ 1 & 4 & 0 \\ 8 & -3 & 1 \end{vmatrix} = 3\begin{vmatrix} 4 & 0 \\ -3 & 1 \end{vmatrix} - (-9)\begin{vmatrix} 1 & 0 \\ 8 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & 4 \\ 8 & -3 \end{vmatrix}$$

$$\begin{aligned} &= 3[4(1) - (-3)(0)] + 9[1(1) - 8(0)] + 4[1(-3) - 8(4)] \\ &= 3(4) + 9(1) + 4(-35) \\ &= 12 + 9 - 140 \\ &= -119 \end{aligned}$$

11. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} x + y = 8 \\ x - y = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 1(-1) - 1(1) = -1 - 1 = -2$$

$$D_x = \begin{vmatrix} 8 & 1 \\ 4 & -1 \end{vmatrix} = 8(-1) - 4(1) = -8 - 4 = -12$$

$$D_y = \begin{vmatrix} 1 & 8 \\ 1 & 4 \end{vmatrix} = 1(4) - 1(8) = 4 - 8 = -4$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{-12}{-2} = 6 \qquad y = \frac{D_y}{D} = \frac{-4}{-2} = 2$$

12. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} x + 2y = 5 \\ x - y = 3 \end{cases}$$

$$D = \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = 1(-1) - 1(2) = -1 - 2 = -3$$

$$D_x = \begin{vmatrix} 5 & 2 \\ 3 & -1 \end{vmatrix} = 5(-1) - 3(2) = -5 - 6 = -11$$

$$D_y = \begin{vmatrix} 1 & 5 \\ 1 & 3 \end{vmatrix} = 1(3) - 1(5) = 3 - 5 = -2$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{-11}{-3} = \frac{11}{3} \qquad y = \frac{D_y}{D} = \frac{-2}{-3} = \frac{2}{3}$$

13. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} 5x - y = 13 \\ 2x + 3y = 12 \end{cases}$$

$$D = \begin{vmatrix} 5 & -1 \\ 2 & 3 \end{vmatrix} = 5(3) - 2(-1) = 15 + 2 = 17$$

$$D_x = \begin{vmatrix} 13 & -1 \\ 12 & 3 \end{vmatrix} = 13(3) - 12(-1) = 39 + 12 = 51$$

$$D_y = \begin{vmatrix} 5 & 13 \\ 2 & 12 \end{vmatrix} = 5(12) - 2(13) = 60 - 26 = 34$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{51}{17} = 3 \qquad y = \frac{D_y}{D} = \frac{34}{17} = 2$$

14. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} x + 3y = 5 \\ 2x - 3y = -8 \end{cases}$$

$$D = \begin{vmatrix} 1 & 3 \\ 2 & -3 \end{vmatrix} = 1(-3) - 2(3) = -3 - 6 = -9$$

$$D_x = \begin{vmatrix} 5 & 3 \\ -8 & -3 \end{vmatrix} = 5(-3) - (-8)(3) = -15 + 24 = 9$$

$$D_y = \begin{vmatrix} 1 & 5 \\ 2 & -8 \end{vmatrix} = 1(-8) - 2(5) = -8 - 10 = -18$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{9}{-9} = -1 \qquad y = \frac{D_y}{D} = \frac{-18}{-9} = 2$$

15. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} 3x = 24 \\ x + 2y = 0 \end{cases}$$

$$D = \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} = 6 - 0 = 6$$

$$D_x = \begin{vmatrix} 24 & 0 \\ 0 & 2 \end{vmatrix} = 48 - 0 = 48$$

$$D_y = \begin{vmatrix} 3 & 24 \\ 1 & 0 \end{vmatrix} = 0 - 24 = -24$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{48}{6} = 8 \qquad y = \frac{D_y}{D} = \frac{-24}{6} = -4$$

16. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} 4x + 5y = -3 \\ -2y = -4 \end{cases}$$

$$D = \begin{vmatrix} 4 & 5 \\ 0 & -2 \end{vmatrix} = -8 - 0 = -8$$

$$D_x = \begin{vmatrix} -3 & 5 \\ -4 & -2 \end{vmatrix} = 6 + 20 = 26$$

$$D_y = \begin{vmatrix} 4 & -3 \\ 0 & -4 \end{vmatrix} = -16 - 0 = -16$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{26}{-8} = -\frac{13}{4} \qquad y = \frac{D_y}{D} = \frac{-16}{-8} = 2$$

17. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} 3x - 6y = 24 \\ 5x + 4y = 12 \end{cases}$$

$$D = \begin{vmatrix} 3 & -6 \\ 5 & 4 \end{vmatrix} = 12 - (-30) = 42$$

$$D_x = \begin{vmatrix} 24 & -6 \\ 12 & 4 \end{vmatrix} = 96 - (-72) = 168$$

$$D_y = \begin{vmatrix} 3 & 24 \\ 5 & 12 \end{vmatrix} = 36 - 120 = -84$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{168}{42} = 4 \qquad y = \frac{D_y}{D} = \frac{-84}{42} = -2$$

18. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} 2x + 4y = 16 \\ 3x - 5y = -9 \end{cases}$$

$$D = \begin{vmatrix} 2 & 4 \\ 3 & -5 \end{vmatrix} = -10 - 12 = -22$$

$$D_x = \begin{vmatrix} 16 & 4 \\ -9 & -5 \end{vmatrix} = -80 + 36 = -44$$

$$D_y = \begin{vmatrix} 2 & 16 \\ 3 & -9 \end{vmatrix} = -18 - 48 = -66$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{-44}{-22} = 2 \qquad y = \frac{D_y}{D} = \frac{-66}{-22} = 3$$

19. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} 3x - 2y = 4 \\ 6x - 4y = 0 \end{cases}$$

$$D = \begin{vmatrix} 3 & -2 \\ 6 & -4 \end{vmatrix} = -12 - (-12) = 0$$

Since $D = 0$, Cramer's Rule does not apply.

20. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} -x + 2y = 5 \\ 4x - 8y = 6 \end{cases}$$

$$D = \begin{vmatrix} -1 & 2 \\ 4 & -8 \end{vmatrix} = 8 - 8 = 0$$

Since $D = 0$, Cramer's Rule does not apply.

21. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} 2x - 4y = -2 \\ 3x + 2y = 3 \end{cases}$$

$$D = \begin{vmatrix} 2 & -4 \\ 3 & 2 \end{vmatrix} = 4 - (-12) = 16$$

$$D_x = \begin{vmatrix} -2 & -4 \\ 3 & 2 \end{vmatrix} = -4 - (-12) = 8$$

$$D_y = \begin{vmatrix} 2 & -2 \\ 3 & 3 \end{vmatrix} = 6 - (-6) = 12$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{8}{16} = \frac{1}{2} \qquad y = \frac{D_y}{D} = \frac{12}{16} = \frac{3}{4}$$

22. (a) Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} 3x + 3y = 3 \\ 4x + 2y = \dfrac{8}{3} \end{cases}$$

$$D = \begin{vmatrix} 3 & 3 \\ 4 & 2 \end{vmatrix} = 6 - 12 = -6$$

$$D_x = \begin{vmatrix} 3 & 3 \\ \frac{8}{3} & 2 \end{vmatrix} = 6 - 8 = -2$$

$$D_y = \begin{vmatrix} 3 & 3 \\ 4 & \frac{8}{3} \end{vmatrix} = 8 - 12 = -4$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{-2}{-6} = \frac{1}{3} \qquad y = \frac{D_y}{D} = \frac{-4}{-6} = \frac{2}{3}$$

23. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} 2x - 3y = -1 \\ 10x + 10y = 5 \end{cases}$$

$$D = \begin{vmatrix} 2 & -3 \\ 10 & 10 \end{vmatrix} = 20 - (-30) = 50$$

$$D_x = \begin{vmatrix} -1 & -3 \\ 5 & 10 \end{vmatrix} = -10 - (-15) = 5$$

$$D_y = \begin{vmatrix} 2 & -1 \\ 10 & 5 \end{vmatrix} = 10 - (-10) = 20$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{5}{50} = \frac{1}{10} \qquad y = \frac{D_y}{D} = \frac{20}{50} = \frac{2}{5}$$

24. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} 3x - 2y = 0 \\ 5x + 10y = 4 \end{cases}$$

$$D = \begin{vmatrix} 3 & -2 \\ 5 & 10 \end{vmatrix} = 30 - (-10) = 40$$

$$D_x = \begin{vmatrix} 0 & -2 \\ 4 & 10 \end{vmatrix} = 0 - (-8) = 8$$

$$D_y = \begin{vmatrix} 3 & 0 \\ 5 & 4 \end{vmatrix} = 12 - 0 = 12$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{8}{40} = \frac{1}{5} \qquad y = \frac{D_y}{D} = \frac{12}{40} = \frac{3}{10}$$

25. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} 2x+3y=6 \\ x-\ y=\dfrac{1}{2} \end{cases}$$

$$D=\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}=-2-3=-5$$

$$D_x=\begin{vmatrix} 6 & 3 \\ \frac{1}{2} & -1 \end{vmatrix}=-6-\frac{3}{2}=-\frac{15}{2}$$

$$D_y=\begin{vmatrix} 2 & 6 \\ 1 & \frac{1}{2} \end{vmatrix}=1-6=-5$$

Find the solutions by Cramer's Rule:

$$x=\frac{D_x}{D}=\frac{\left(-\frac{15}{2}\right)}{-5}=\frac{3}{2} \qquad y=\frac{D_y}{D}=\frac{-5}{-5}=1$$

26. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} \dfrac{1}{2}x+\ y=-2 \\ x-2y=\ \ 8 \end{cases}$$

$$D=\begin{vmatrix} \frac{1}{2} & 1 \\ 1 & -2 \end{vmatrix}=-1-1=-2$$

$$D_x=\begin{vmatrix} -2 & 1 \\ 8 & -2 \end{vmatrix}=4-8=-4$$

$$D_y=\begin{vmatrix} \frac{1}{2} & -2 \\ 1 & 8 \end{vmatrix}=4-(-2)=6$$

Find the solutions by Cramer's Rule:

$$x=\frac{D_x}{D}=\frac{-4}{-2}=2 \qquad y=\frac{D_y}{D}=\frac{6}{-2}=-3$$

27. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} 3x-5y=3 \\ 15x+5y=21 \end{cases}$$

$$D=\begin{vmatrix} 3 & -5 \\ 15 & 5 \end{vmatrix}=15-(-75)=90$$

$$D_x=\begin{vmatrix} 3 & -5 \\ 21 & 5 \end{vmatrix}=15-(-105)=120$$

$$D_y=\begin{vmatrix} 3 & 3 \\ 15 & 21 \end{vmatrix}=63-45=18$$

Find the solutions by Cramer's Rule:

$$x=\frac{D_x}{D}=\frac{120}{90}=\frac{4}{3} \qquad y=\frac{D_y}{D}=\frac{18}{90}=\frac{1}{5}$$

28. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} 2x - y = -1 \\ x + \frac{1}{2}y = \frac{3}{2} \end{cases}$$

$$D = \begin{vmatrix} 2 & -1 \\ 1 & \frac{1}{2} \end{vmatrix} = 1 - (-1) = 2$$

$$D_x = \begin{vmatrix} -1 & -1 \\ \frac{3}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2} - \left(-\frac{3}{2}\right) = 1$$

$$D_y = \begin{vmatrix} 2 & -1 \\ 1 & \frac{3}{2} \end{vmatrix} = 3 - (-1) = 4$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{1}{2} \qquad y = \frac{D_y}{D} = \frac{4}{2} = 2$$

29. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} x + y - z = 6 \\ 3x - 2y + z = -5 \\ x + 3y - 2z = 14 \end{cases}$$

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 3 & -2 & 1 \\ 1 & 3 & -2 \end{vmatrix} = 1\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix}$$
$$= 1(4-3) - 1(-6-1) - 1(9+2) = 1 + 7 - 11 = -3$$

$$D_x = \begin{vmatrix} 6 & 1 & -1 \\ -5 & -2 & 1 \\ 14 & 3 & -2 \end{vmatrix} = 6\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} - 1\begin{vmatrix} -5 & 1 \\ 14 & -2 \end{vmatrix} + (-1)\begin{vmatrix} -5 & -2 \\ 14 & 3 \end{vmatrix}$$
$$= 6(4-3) - 1(10 - 14) - 1(-15+28) = 6 + 4 - 13 = -3$$

$$D_y = \begin{vmatrix} 1 & 6 & -1 \\ 3 & -5 & 1 \\ 1 & 14 & -2 \end{vmatrix} = 1\begin{vmatrix} -5 & 1 \\ 14 & -2 \end{vmatrix} - 6\begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -5 \\ 1 & 14 \end{vmatrix}$$
$$= 1(10 - 14) - 6(-6-1) - 1(42+5) = -4 + 42 - 47 = -9$$

$$D_z = \begin{vmatrix} 1 & 1 & 6 \\ 3 & -2 & -5 \\ 1 & 3 & 14 \end{vmatrix} = 1\begin{vmatrix} -2 & -5 \\ 3 & 14 \end{vmatrix} - 1\begin{vmatrix} 3 & -5 \\ 1 & 14 \end{vmatrix} + 6\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix}$$
$$= 1(-28+15) - 1(42+5) + 6(9+2) = -13 - 47 + 66 = 6$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{-3}{-3} = 1 \qquad y = \frac{D_y}{D} = \frac{-9}{-3} = 3 \qquad z = \frac{D_z}{D} = \frac{6}{-3} = -2$$

30. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} x - y + z = -4 \\ 2x - 3y + 4z = -15 \\ 5x + y - 2z = 12 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 2 & -3 & 4 \\ 5 & 1 & -2 \end{vmatrix} = 1\begin{vmatrix} -3 & 4 \\ 1 & -2 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 4 \\ 5 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix}$$

$$= 1(6-4) + 1(-4-20) + 1(2+15) = 2 - 24 + 17 = -5$$

$$D_x = \begin{vmatrix} -4 & -1 & 1 \\ -15 & -3 & 4 \\ 12 & 1 & -2 \end{vmatrix} = -4\begin{vmatrix} -3 & 4 \\ 1 & -2 \end{vmatrix} - (-1)\begin{vmatrix} -15 & 4 \\ 12 & -2 \end{vmatrix} + 1\begin{vmatrix} -15 & -3 \\ 12 & 1 \end{vmatrix}$$

$$= -4(6-4) + 1(30-48) + 1(-15+36) = -8 - 18 + 21 = -5$$

$$D_y = \begin{vmatrix} 1 & -4 & 1 \\ 2 & -15 & 4 \\ 5 & 12 & -2 \end{vmatrix} = 1\begin{vmatrix} -15 & 4 \\ 12 & -2 \end{vmatrix} - (-4)\begin{vmatrix} 2 & 4 \\ 5 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -15 \\ 5 & 12 \end{vmatrix}$$

$$= 1(30-48) + 4(-4-20) + 1(24+75) = -18 - 96 + 99 = -15$$

$$D_z = \begin{vmatrix} 1 & -1 & -4 \\ 2 & -3 & -15 \\ 5 & 1 & 12 \end{vmatrix} = 1\begin{vmatrix} -3 & -15 \\ 1 & 12 \end{vmatrix} - (-1)\begin{vmatrix} 2 & -15 \\ 5 & 12 \end{vmatrix} + (-4)\begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix}$$

$$= 1(-36+15) + 1(24+75) - 4(2+15) = -21 + 99 - 68 = 10$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{-5}{-5} = 1 \qquad y = \frac{D_y}{D} = \frac{-15}{-5} = 3 \qquad z = \frac{D_z}{D} = \frac{10}{-5} = -2$$

31. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} x + 2y - z = -3 \\ 2x - 4y + z = -7 \\ -2x + 2y - 3z = 4 \end{cases}$$

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -4 & 1 \\ -2 & 2 & -3 \end{vmatrix} = 1\begin{vmatrix} -4 & 1 \\ 2 & -3 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ -2 & -3 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -4 \\ -2 & 2 \end{vmatrix}$$

$$= 1(12-2) - 2(-6+2) - 1(4-8) = 10 + 8 + 4 = 22$$

$$D_x = \begin{vmatrix} -3 & 2 & -1 \\ -7 & -4 & 1 \\ 4 & 2 & -3 \end{vmatrix} = -3\begin{vmatrix} -4 & 1 \\ 2 & -3 \end{vmatrix} - 2\begin{vmatrix} -7 & 1 \\ 4 & -3 \end{vmatrix} + (-1)\begin{vmatrix} -7 & -4 \\ 4 & 2 \end{vmatrix}$$

$$= -3(12-2) - 2(21-4) - 1(-14+16) = -30 - 34 - 2 = -66$$

$$D_y = \begin{vmatrix} 1 & -3 & -1 \\ 2 & -7 & 1 \\ -2 & 4 & -3 \end{vmatrix} = 1\begin{vmatrix} -7 & 1 \\ 4 & -3 \end{vmatrix} - (-3)\begin{vmatrix} 2 & 1 \\ -2 & -3 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -7 \\ -2 & 4 \end{vmatrix}$$

$$= 1(21-4) + 3(-6+2) - 1(8-14) = 17 - 12 + 6 = 11$$

$$D_z = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -4 & -7 \\ -2 & 2 & 4 \end{vmatrix} = 1\begin{vmatrix} -4 & -7 \\ 2 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & -7 \\ -2 & 4 \end{vmatrix} + (-3)\begin{vmatrix} 2 & -4 \\ -2 & 2 \end{vmatrix}$$

$$= 1(-16+14) - 2(8-14) - 3(4-8) = -2+12+12 = 22$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{-66}{22} = -3 \qquad y = \frac{D_y}{D} = \frac{11}{22} = \frac{1}{2} \qquad z = \frac{D_z}{D} = \frac{22}{22} = 1$$

32. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} x + 4y - 3z = -8 \\ 3x - y + 3z = 12 \\ x + y + 6z = 1 \end{cases}$$

$$D = \begin{vmatrix} 1 & 4 & -3 \\ 3 & -1 & 3 \\ 1 & 1 & 6 \end{vmatrix} = 1\begin{vmatrix} -1 & 3 \\ 1 & 6 \end{vmatrix} - 4\begin{vmatrix} 3 & 3 \\ 1 & 6 \end{vmatrix} + (-3)\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 1(-6-3) - 4(18-3) - 3(3+1) = -9 - 60 - 12 = -81$$

$$D_x = \begin{vmatrix} -8 & 4 & -3 \\ 12 & -1 & 3 \\ 1 & 1 & 6 \end{vmatrix} = -8\begin{vmatrix} -1 & 3 \\ 1 & 6 \end{vmatrix} - 4\begin{vmatrix} 12 & 3 \\ 1 & 6 \end{vmatrix} + (-3)\begin{vmatrix} 12 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= -8(-6-3) - 4(72-3) - 3(12+1) = 72 - 276 - 39 = -243$$

$$D_y = \begin{vmatrix} 1 & -8 & -3 \\ 3 & 12 & 3 \\ 1 & 1 & 6 \end{vmatrix} = 1\begin{vmatrix} 12 & 3 \\ 1 & 6 \end{vmatrix} - (-8)\begin{vmatrix} 3 & 3 \\ 1 & 6 \end{vmatrix} + (-3)\begin{vmatrix} 3 & 12 \\ 1 & 1 \end{vmatrix}$$

$$= 1(72-3) + 8(18-3) - 3(3-12) = 69 + 120 + 27 = 216$$

$$D_z = \begin{vmatrix} 1 & 4 & -8 \\ 3 & -1 & 12 \\ 1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 12 \\ 1 & 1 \end{vmatrix} - 4\begin{vmatrix} 3 & 12 \\ 1 & 1 \end{vmatrix} + (-8)\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 1(-1-12) - 4(3-12) - 8(3+1) = -13 + 36 - 32 = -9$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{-243}{-81} = 3 \qquad y = \frac{D_y}{D} = \frac{216}{-81} = -\frac{8}{3} \qquad z = \frac{D_z}{D} = \frac{-9}{-81} = \frac{1}{9}$$

33. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} x - 2y + 3z = 1 \\ 3x + y - 2z = 0 \\ 2x - 4y + 6z = 2 \end{cases}$$

$$D = \begin{vmatrix} 1 & -2 & 3 \\ 3 & 1 & -2 \\ 2 & -4 & 6 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 \\ -4 & 6 \end{vmatrix} - (-2)\begin{vmatrix} 3 & -2 \\ 2 & 6 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 2 & -4 \end{vmatrix}$$

$$= 1(6-8) + 2(18+4) + 3(-12-2) = -2 + 44 - 42 = 0$$

Since $D = 0$, Cramer's Rule does not apply.

34. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} x - y + 2z = 5 \\ 3x + 2y = 4 \\ -2x + 2y - 4z = -10 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -2 & 2 & -4 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 2 & -4 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 0 \\ -2 & -4 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ -2 & 2 \end{vmatrix}$$

$$= 1(-8 - 0) + 1(-12 - 0) + 2(6 + 4) = -8 - 12 + 20 = 0$$

Since $D = 0$, Cramer's Rule does not apply.

35. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} x + 2y - z = 0 \\ 2x - 4y + z = 0 \\ -2x + 2y - 3z = 0 \end{cases}$$

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -4 & 1 \\ -2 & 2 & -3 \end{vmatrix} = 1\begin{vmatrix} -4 & 1 \\ 2 & -3 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ -2 & -3 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -4 \\ -2 & 2 \end{vmatrix}$$

$$= 1(12 - 2) - 2(-6 + 2) - 1(4 - 8) = 10 + 8 + 4 = 22$$

$$D_x = \begin{vmatrix} 0 & 2 & -1 \\ 0 & -4 & 1 \\ 0 & 2 & -3 \end{vmatrix} = 0 \quad \text{(By Theorem 12)}$$

$$D_y = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ -2 & 0 & -3 \end{vmatrix} = 0 \quad \text{(By Theorem 12)}$$

$$D_z = \begin{vmatrix} 1 & 2 & 0 \\ 2 & -4 & 0 \\ -2 & 2 & 0 \end{vmatrix} = 0 \quad \text{(By Theorem 12)}$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{0}{22} = 0 \qquad y = \frac{D_y}{D} = \frac{0}{22} = 0 \qquad z = \frac{D_z}{D} = \frac{0}{22} = 0$$

36. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} x + 4y - 3z = 0 \\ 3x - y + 3z = 0 \\ x + y + 6z = 0 \end{cases}$$

$$D = \begin{vmatrix} 1 & 4 & -3 \\ 3 & -1 & 3 \\ 1 & 1 & 6 \end{vmatrix} = 1\begin{vmatrix} -1 & 3 \\ 1 & 6 \end{vmatrix} - 4\begin{vmatrix} 3 & 3 \\ 1 & 6 \end{vmatrix} + (-3)\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 1(-6 - 3) - 4(18 - 3) - 3(3 + 1) = -9 - 60 - 12 = -81$$

$$D_x = \begin{vmatrix} 0 & 4 & -3 \\ 0 & -1 & 3 \\ 0 & 1 & 6 \end{vmatrix} = 0 \quad \text{(By Theorem 12)}$$

$$D_y = \begin{vmatrix} 1 & 0 & -3 \\ 3 & 0 & 3 \\ 1 & 0 & 6 \end{vmatrix} = 0 \quad \text{(By Theorem 12)}$$

$$D_z = \begin{vmatrix} 1 & 4 & 0 \\ 3 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0 \quad \text{(By Theorem 12)}$$

Find the solutions by Cramer's Rule:

$$x = \frac{D_x}{D} = \frac{0}{-81} = 0 \qquad y = \frac{D_y}{D} = \frac{0}{-81} = 0 \qquad z = \frac{D_z}{D} = \frac{0}{-81} = 0$$

37. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} x - 2y + 3z = 0 \\ 3x + y - 2z = 0 \\ 2x - 4y + 6z = 0 \end{cases}$$

$$D = \begin{vmatrix} 1 & -2 & 3 \\ 3 & 1 & -2 \\ 2 & -4 & 6 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 \\ -4 & 6 \end{vmatrix} - (-2)\begin{vmatrix} 3 & -2 \\ 2 & 6 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 2 & -4 \end{vmatrix}$$

$$= 1(6-8) + 2(18+4) + 3(-12-2) = -2 + 44 - 42 = 0$$

Since $D = 0$, Cramer's Rule does not apply.

38. Set up and evaluate the determinants to use Cramer's Rule:

$$\begin{cases} x - y + 2z = 0 \\ 3x + 2y = 0 \\ -2x + 2y - 4z = 0 \end{cases}$$

$$D = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -2 & 2 & -4 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 2 & -4 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 0 \\ -2 & -4 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ -2 & 2 \end{vmatrix}$$

$$= 1(-8-0) + 1(-12-0) + 2(6+4) = -8 - 12 + 20 = 0$$

Since $D = 0$, Cramer's Rule does not apply.

39. Rewrite the system letting $u = \dfrac{1}{x}$ and $v = \dfrac{1}{y}$:

$$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 8 \\ \dfrac{3}{x} - \dfrac{5}{y} = 0 \end{cases} \quad \rightarrow \quad \begin{cases} u + v = 8 \\ 3u - 5v = 0 \end{cases}$$

Set up and evaluate the determinants to use Cramer's Rule:

$$D = \begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix} = -5 - 3 = -8$$

$$D_u = \begin{vmatrix} 8 & 1 \\ 0 & -5 \end{vmatrix} = -40 - 0 = -40$$

$$D_v = \begin{vmatrix} 1 & 8 \\ 3 & 0 \end{vmatrix} = 0 - 24 = -24$$

Find the solutions by Cramer's Rule:

$$u = \frac{D_u}{D} = \frac{-40}{-8} = 5 \qquad v = \frac{D_v}{D} = \frac{-24}{-8} = 3$$

The solutions are $x = \frac{1}{5},\ y = \frac{1}{3}$

40. Rewrite the system letting $u = \frac{1}{x}$ and $v = \frac{1}{y}$:

$$\begin{cases} \frac{4}{x} - \frac{3}{y} = 0 \\ \frac{6}{x} + \frac{3}{2y} = 2 \end{cases} \rightarrow \begin{cases} 4u - 3v = 0 \\ 6u + \frac{3}{2}v = 2 \end{cases}$$

Set up and evaluate the determinants to use Cramer's Rule:

$$D = \begin{vmatrix} 4 & -3 \\ 6 & \frac{3}{2} \end{vmatrix} = 6 + 18 = 24$$

$$D_u = \begin{vmatrix} 0 & -3 \\ 2 & \frac{3}{2} \end{vmatrix} = 0 + 6 = 6$$

$$D_v = \begin{vmatrix} 4 & 0 \\ 6 & 2 \end{vmatrix} = 8 - 0 = 8$$

Find the solutions by Cramer's Rule:

$$u = \frac{D_u}{D} = \frac{6}{24} = \frac{1}{4} \qquad v = \frac{D_v}{D} = \frac{8}{24} = \frac{1}{3}$$

The solutions are $x = 4,\ y = 3$

41. Solve for x:

$$\begin{vmatrix} x & x \\ 4 & 3 \end{vmatrix} = 3x - 4x = -x$$

$$-x = 5 \rightarrow x = -5$$

42. Solve for x:

$$\begin{vmatrix} x & 1 \\ 3 & x \end{vmatrix} = x^2 - 3 = -2 \rightarrow x^2 = 1 \rightarrow x = \pm 1$$

43. Solve for x:

$$\begin{vmatrix} x & 1 & 1 \\ 4 & 3 & 2 \\ -1 & 2 & 5 \end{vmatrix} = x\begin{vmatrix} 3 & 2 \\ 2 & 5 \end{vmatrix} - 1\begin{vmatrix} 4 & 2 \\ -1 & 5 \end{vmatrix} + 1\begin{vmatrix} 4 & 3 \\ -1 & 2 \end{vmatrix}$$

$$= x(15 - 4) - (20 + 2) + (8 + 3) = 11x - 22 + 11 = 11x - 11$$

So, $11x - 11 = 2 \rightarrow 11x = 13 \rightarrow x = \frac{13}{11}$

44. Solve for x:

$$\begin{vmatrix} 3 & 2 & 4 \\ 1 & x & 5 \\ 0 & 1 & -2 \end{vmatrix} = 3\begin{vmatrix} x & 5 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} + 4\begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$

$$= 3(-2x-5) - 2(-2-0) + 4(1-0) = -6x - 15 + 4 + 4 = -6x - 7$$

$$-6x - 7 = 0 \rightarrow -6x = 7 \rightarrow x = -\frac{7}{6}$$

45. Solve for x:

$$\begin{vmatrix} x & 2 & 3 \\ 1 & x & 0 \\ 6 & 1 & -2 \end{vmatrix} = x\begin{vmatrix} x & 0 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 6 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & x \\ 6 & 1 \end{vmatrix}$$

$$= x(-2x-0) - 2(-2-0) + 3(1-6x)$$

$$= -2x^2 + 4 + 3 - 18x = -2x^2 - 18x + 7$$

So, $-2x^2 - 18x + 7 = 7$

$$-2x^2 - 18x = 0 \rightarrow -2x(x+9) = 0 \rightarrow x = 0 \text{ or } x = -9$$

46. Solve for x:

$$\begin{vmatrix} x & 1 & 2 \\ 1 & x & 3 \\ 0 & 1 & 2 \end{vmatrix} = x\begin{vmatrix} x & 3 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$

$$= x(2x-3) - 1(2-0) + 2(1-0) = 2x^2 - 3x - 2 + 2 = 2x^2 - 3x$$

So, $2x^2 - 3x = -4x \rightarrow 2x^2 + x = 0 \rightarrow x(2x+1) = 0$

$$x = 0 \text{ or } x = -\frac{1}{2}$$

47. Let $\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix} = 4$

Then $\begin{vmatrix} 1 & 2 & 3 \\ u & v & w \\ x & y & z \end{vmatrix} = -4$ by Theorem 11 --

The value of the determinant changes sign when two rows are interchanged.

48. Let $\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix} = 4$

Then $\begin{vmatrix} x & y & z \\ u & v & w \\ 2 & 4 & 6 \end{vmatrix} = 2\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix} = 2(4) = 8$ by Theorem 14 --

The value of the determinant is multiplied by k when the elements of a row are multiplied by k.

Problems 49 – 54 use the Laws for Determinants in reverse order.

49. Let $\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix} = 4$

$$\begin{vmatrix} x & y & z \\ -3 & -6 & -9 \\ u & v & w \end{vmatrix} = \underbrace{-3\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ u & v & w \end{vmatrix}}_{\text{Theorem 14}} = \underbrace{-3(-1)\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix}}_{\text{Theorem 11}} = 3(4) = 12$$

50. Let $\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix} = 4$

$$\begin{vmatrix} 1 & 2 & 3 \\ x-u & y-v & z-w \\ u & v & w \end{vmatrix} = \underbrace{\begin{vmatrix} 1 & 2 & 3 \\ x & y & z \\ u & v & w \end{vmatrix}}_{\text{Theorem 15}} = \underbrace{(-1)\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ u & v & w \end{vmatrix}}_{\text{Theorem 11}} = \underbrace{(-1)(-1)\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix}}_{\text{Theorem 11}}$$

$$= \begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix} = 4$$

51. Let $\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix} = 4$

$$\begin{vmatrix} 1 & 2 & 3 \\ x-3 & y-6 & z-9 \\ 2u & 2v & 2w \end{vmatrix} = \underbrace{2\begin{vmatrix} 1 & 2 & 3 \\ x-3 & y-6 & z-9 \\ u & v & w \end{vmatrix}}_{\text{Theorem 14}} = \underbrace{2(-1)\begin{vmatrix} x-3 & y-6 & z-9 \\ 1 & 2 & 3 \\ u & v & w \end{vmatrix}}_{\text{Theorem 11}}$$

$$= \underbrace{2(-1)(-1)\begin{vmatrix} x-3 & y-6 & z-9 \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix}}_{\text{Theorem 11}} = \underbrace{2(-1)(-1)\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix}}_{\text{Theorem 15 } (R_1 = -3r_3 + r_1)} = 2(-1)(-1)(4) = 8$$

52. Let $\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix} = 4$

$$= \begin{vmatrix} x & y & z-x \\ u & v & w-u \\ 1 & 2 & 2 \end{vmatrix} = \underbrace{\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix}}_{\text{Theorem 15 } (C_3 = -c_1 + c_3)} = 4$$

53. Let $\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix} = 4$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2x & 2y & 2z \\ u-1 & v-2 & w-3 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ x & y & z \\ u-1 & v-2 & w-3 \end{vmatrix} = 2(-1)\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ u-1 & v-2 & w-3 \end{vmatrix}$$

Theorem 14 Theorem 11

$$= 2(-1)(-1)\begin{vmatrix} x & y & z \\ u-1 & v-2 & w-3 \\ 1 & 2 & 3 \end{vmatrix} = 2(-1)(-1)\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix} = 2(-1)(-1)(4) = 8$$

Theorem 11 Theorem 15 $(R_2 = -r_3 + r_2)$

54. Let $\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix} = 4$

$$\begin{vmatrix} x+3 & y+6 & z+9 \\ 3u-1 & 3v-2 & 3w-3 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} x & y & z \\ 3u-1 & 3v-2 & 3w-3 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} x & y & z \\ 3u & 3v & 3w \\ 1 & 2 & 3 \end{vmatrix}$$

Theorem 15 $(R_1 = 3r_3 + r_1)$ Theorem 15 $(R_2 = -r_3 + r_2)$

$$= 3\begin{vmatrix} x & y & z \\ u & v & w \\ 1 & 2 & 3 \end{vmatrix} = 3(4) = 12$$

Theorem 14

55. Expanding the determinant:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = x\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix} - y\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} + 1\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

$$= x(y_1 - y_2) - y(x_1 - x_2) + (x_1y_2 - x_2y_1) = 0$$

$$x(y_1 - y_2) + y(x_2 - x_1) = x_2y_1 - x_1y_2$$

$$y(x_2 - x_1) = x_2y_1 - x_1y_2 + x(y_2 - y_1)$$

$$y(x_2 - x_1) - y_1(x_2 - x_1) = x_2y_1 - x_1y_2 + x(y_2 - y_1) - y_1(x_2 - x_1)$$

$$(x_2 - x_1)(y - y_1) = x(y_2 - y_1) + x_2y_1 - x_1y_2 - y_1x_2 + y_1x_1$$

$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)x - (y_2 - y_1)x_1$$

$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1)$$

$$(y - y_1) = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$$

56. Any point (x, y) on the line containing (x_2, y_2) and (x_3, y_3) satisfies:

$$\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

If the point (x_1, y_1) is on the line containing (x_2, y_2) and (x_3, y_3) (the points are collinear), then:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Conversely, if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$, then (x_1, y_1) is on the line containing (x_2, y_2) and (x_3, y_3) and the points are collinear.

57. Expanding the determinant:

$$\begin{aligned} \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} &= x^2 \begin{vmatrix} y & 1 \\ z & 1 \end{vmatrix} - x \begin{vmatrix} y^2 & 1 \\ z^2 & 1 \end{vmatrix} + 1 \begin{vmatrix} y^2 & y \\ z^2 & z \end{vmatrix} \\ &= x^2(y - z) - x(y^2 - z^2) + 1(y^2 z - z^2 y) \\ &= x^2(y - z) - x(y - z)(y + z) + yz(y - z) \\ &= (y - z)\left[x^2 - xy - xz + yz\right] \\ &= (y - z)\left[x(x - y) - z(x - y)\right] \\ &= (y - z)(x - y)(x - z) \end{aligned}$$

58. Cramer's Rule for two equations containing two variables asserts that the solution to the system of equations

$$\begin{aligned} ax + by &= s \\ cs + dy &= t \end{aligned}$$

is given by $x = \dfrac{\begin{vmatrix} s & b \\ t & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ $y = \dfrac{\begin{vmatrix} a & s \\ c & t \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ provided that $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$.

Case 1: If $a = 0$, then $b \neq 0$ and $c \neq 0$, so $D = -bc \neq 0$.

Then, $x = \dfrac{sd - bt}{-bc}$, $y = \dfrac{-sc}{-bc}$ is a solution.

Case 2: If $b = 0$, then $a \neq 0$ and $d \neq 0$, so $D = ad \neq 0$.

Then, $x = \dfrac{sd}{ad}$, $y = \dfrac{at - sc}{ad}$ is a solution.

Case 3: If $c = 0$, then $a \neq 0$ and $d \neq 0$, so $D = ad \neq 0$.

Then, $x = \dfrac{sd - bt}{ad}$, $y = \dfrac{at}{ad}$ is a solution.

Case 4: If $d = 0$, then $b \neq 0$ and $c \neq 0$, so $D = -bc \neq 0$.

Then, $x = \dfrac{-bt}{-bc}$, $y = \dfrac{at - sc}{-bc}$ is a solution.

59. Evaluating the determinant to show the relationship:

$$\begin{vmatrix} a_{13} & a_{12} & a_{11} \\ a_{23} & a_{22} & a_{21} \\ a_{33} & a_{32} & a_{31} \end{vmatrix} = a_{13}\begin{vmatrix} a_{22} & a_{21} \\ a_{32} & a_{31} \end{vmatrix} - a_{12}\begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} + a_{11}\begin{vmatrix} a_{23} & a_{22} \\ a_{33} & a_{32} \end{vmatrix}$$

$$= a_{13}(a_{22}a_{31} - a_{21}a_{32}) - a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{11}(a_{23}a_{32} - a_{22}a_{33})$$

$$= a_{13}a_{22}a_{31} - a_{13}a_{21}a_{32} - a_{12}a_{23}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32} - a_{11}a_{22}a_{33}$$

$$= -a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31}$$

$$= -a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{21}a_{33} - a_{23}a_{31}) - a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= -a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -\left[a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right]$$

$$= -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

60. Evaluating the determinant to show the relationship:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} ka_{22} & ka_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} ka_{21} & ka_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} ka_{21} & ka_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(ka_{22}a_{33} - ka_{23}a_{32}) - a_{12}(ka_{21}a_{33} - ka_{23}a_{31}) + a_{13}(ka_{21}a_{32} - ka_{22}a_{31})$$

$$= ka_{11}(a_{22}a_{33} - a_{23}a_{32}) - ka_{12}(a_{21}a_{33} - a_{23}a_{31}) + ka_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= k\big(a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})\big)$$

$$= k\left(a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right)$$

$$= k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

61. Set up a 3 by 3 determinant in which the first column and third column are the same and evaluate:

$$\begin{vmatrix} a & b & a \\ c & d & c \\ e & f & e \end{vmatrix} = -b\begin{vmatrix} c & c \\ e & e \end{vmatrix} + d\begin{vmatrix} a & a \\ e & e \end{vmatrix} - f\begin{vmatrix} a & a \\ c & c \end{vmatrix}$$

$$= -b(ce - ce) + d(ae - ae) - f(ac - ac) = -b(0) + d(0) - f(0) = 0$$

62. Evaluating the determinant to show the relationship:

$$\begin{aligned}
&\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= (a_{11}+ka_{21})\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - (a_{12}+ka_{22})\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (a_{13}+ka_{23})\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= (a_{11}+ka_{21})(a_{22}a_{33}-a_{23}a_{32}) - (a_{12}+ka_{22})(a_{21}a_{33}-a_{23}a_{31}) \\
&\qquad + (a_{13}+ka_{23})(a_{21}a_{32}-a_{22}a_{31}) \\
&= a_{11}(a_{22}a_{33}-a_{23}a_{32}) + ka_{21}(a_{22}a_{33}-a_{23}a_{32}) - a_{12}(a_{21}a_{33}-a_{23}a_{31}) \\
&\qquad - ka_{22}(a_{21}a_{33}-a_{23}a_{31}) + a_{13}(a_{21}a_{32}-a_{22}a_{31}) + ka_{23}(a_{21}a_{32}-a_{22}a_{31}) \\
&= a_{11}(a_{22}a_{33}-a_{23}a_{32}) + ka_{21}a_{22}a_{33} - ka_{21}a_{23}a_{32} - a_{12}(a_{21}a_{33}-a_{23}a_{31}) \\
&\qquad - ka_{22}a_{21}a_{33} + ka_{22}a_{23}a_{31} + a_{13}(a_{21}a_{32}-a_{22}a_{31}) + ka_{23}a_{21}a_{32} - ka_{23}a_{22}a_{31} \\
&= a_{11}(a_{22}a_{33}-a_{23}a_{32}) - a_{12}(a_{21}a_{33}-a_{23}a_{31}) + a_{13}(a_{21}a_{32}-a_{22}a_{31}) \\
&= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$