Chapter 12

Systems of Equations and Inequalities

12.5 Matrix Algebra

1. $A+B=\begin{bmatrix}0 & 3 & -5\\ 1 & 2 & 6\end{bmatrix}+\begin{bmatrix}4 & 1 & 0\\ -2 & 3 & -2\end{bmatrix}=\begin{bmatrix}0+4 & 3+1 & -5+0\\ 1+(-2) & 2+3 & 6+(-2)\end{bmatrix}=\begin{bmatrix}4 & 4 & -5\\ -1 & 5 & 4\end{bmatrix}$

2. $A-B=\begin{bmatrix}0 & 3 & -5\\ 1 & 2 & 6\end{bmatrix}-\begin{bmatrix}4 & 1 & 0\\ -2 & 3 & -2\end{bmatrix}=\begin{bmatrix}0-4 & 3-1 & -5-0\\ 1-(-2) & 2-3 & 6-(-2)\end{bmatrix}=\begin{bmatrix}-4 & 2 & -5\\ 3 & -1 & 8\end{bmatrix}$

3. $4A=4\begin{bmatrix}0 & 3 & -5\\ 1 & 2 & 6\end{bmatrix}=\begin{bmatrix}4\cdot 0 & 4\cdot 3 & 4(-5)\\ 4\cdot 1 & 4\cdot 2 & 4\cdot 6\end{bmatrix}=\begin{bmatrix}0 & 12 & -20\\ 4 & 8 & 24\end{bmatrix}$

4. $-3B=-3\begin{bmatrix}4 & 1 & 0\\ -2 & 3 & -2\end{bmatrix}=\begin{bmatrix}-3\cdot 4 & -3\cdot 1 & -3\cdot 0\\ -3\cdot(-2) & -3\cdot 3 & -3\cdot(-2)\end{bmatrix}=\begin{bmatrix}-12 & -3 & 0\\ 6 & -9 & 6\end{bmatrix}$

5. $3A-2B=3\begin{bmatrix}0 & 3 & -5\\ 1 & 2 & 6\end{bmatrix}-2\begin{bmatrix}4 & 1 & 0\\ -2 & 3 & -2\end{bmatrix}$

$=\begin{bmatrix}0 & 9 & -15\\ 3 & 6 & 18\end{bmatrix}-\begin{bmatrix}8 & 2 & 0\\ -4 & 6 & -4\end{bmatrix}=\begin{bmatrix}-8 & 7 & -15\\ 7 & 0 & 22\end{bmatrix}$

6. $2A+4B=2\begin{bmatrix}0 & 3 & -5\\ 1 & 2 & 6\end{bmatrix}+4\begin{bmatrix}4 & 1 & 0\\ -2 & 3 & -2\end{bmatrix}$

$=\begin{bmatrix}0 & 6 & -10\\ 2 & 4 & 12\end{bmatrix}+\begin{bmatrix}16 & 4 & 0\\ -8 & 12 & -8\end{bmatrix}=\begin{bmatrix}16 & 10 & -10\\ -6 & 16 & 4\end{bmatrix}$

7. $AC=\begin{bmatrix}0 & 3 & -5\\ 1 & 2 & 6\end{bmatrix}\begin{bmatrix}4 & 1\\ 6 & 2\\ -2 & 3\end{bmatrix}$

$=\begin{bmatrix}0(4)+3(6)+(-5)(-2) & 0(1)+3(2)+(-5)(3)\\ 1(4)+2(6)+6(-2) & 1(1)+2(2)+6(3)\end{bmatrix}=\begin{bmatrix}28 & -9\\ 4 & 23\end{bmatrix}$

8. $BC=\begin{bmatrix}4 & 1 & 0\\ -2 & 3 & -2\end{bmatrix}\begin{bmatrix}4 & 1\\ 6 & 2\\ -2 & 3\end{bmatrix}$

$=\begin{bmatrix}4(4)+1(6)+0(-2) & 4(1)+1(2)+0(3)\\ -2(4)+3(6)+(-2)(-2) & -2(1)+3(2)+(-2)(3)\end{bmatrix}=\begin{bmatrix}22 & 6\\ 14 & -2\end{bmatrix}$

9. $CA = \begin{bmatrix} 4 & 1 \\ 6 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 & -5 \\ 1 & 2 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 4(0)+1(1) & 4(3)+1(2) & 4(-5)+1(6) \\ 6(0)+2(1) & 6(3)+2(2) & 6(-5)+2(6) \\ -2(0)+3(1) & -2(3)+3(2) & -2(-5)+3(6) \end{bmatrix} = \begin{bmatrix} 1 & 14 & -14 \\ 2 & 22 & -18 \\ 3 & 0 & 28 \end{bmatrix}$$

10. $CB = \begin{bmatrix} 4 & 1 \\ 6 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ -2 & 3 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 4(4)+1(-2) & 4(1)+1(3) & 4(0)+1(-2) \\ 6(4)+2(-2) & 6(1)+2(3) & 6(0)+2(-2) \\ -2(4)+3(-2) & -2(1)+3(3) & -2(0)+3(-2) \end{bmatrix} = \begin{bmatrix} 14 & 7 & -2 \\ 20 & 12 & -4 \\ -14 & 7 & -6 \end{bmatrix}$$

11. $C(A+B) = \begin{bmatrix} 4 & 1 \\ 6 & 2 \\ -2 & 3 \end{bmatrix} \left(\begin{bmatrix} 0 & 3 & -5 \\ 1 & 2 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 0 \\ -2 & 3 & -2 \end{bmatrix} \right)$

$$= \begin{bmatrix} 4 & 1 \\ 6 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 & -5 \\ -1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 15 & 21 & -16 \\ 22 & 34 & -22 \\ -11 & 7 & 22 \end{bmatrix}$$

12. $(A+B)C = \left(\begin{bmatrix} 0 & 3 & -5 \\ 1 & 2 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 0 \\ -2 & 3 & -2 \end{bmatrix} \right) \begin{bmatrix} 4 & 1 \\ 6 & 2 \\ -2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 4 & -5 \\ -1 & 5 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 6 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 50 & -3 \\ 18 & 21 \end{bmatrix}$$

13. $AC - 3I_2 = \begin{bmatrix} 0 & 3 & -5 \\ 1 & 2 & 6 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 6 & 2 \\ -2 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 28 & -9 \\ 4 & 23 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 25 & -9 \\ 4 & 20 \end{bmatrix}$

14. $CA + 5I_3 = \begin{bmatrix} 4 & 1 \\ 6 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 & -5 \\ 1 & 2 & 6 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 14 & -14 \\ 2 & 22 & -18 \\ 3 & 0 & 28 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 14 & -14 \\ 2 & 27 & -18 \\ 3 & 0 & 33 \end{bmatrix}$$

15. $CA - CB = \begin{bmatrix} 4 & 1 \\ 6 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 & -5 \\ 1 & 2 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 6 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ -2 & 3 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 14 & -14 \\ 2 & 22 & -18 \\ 3 & 0 & 28 \end{bmatrix} - \begin{bmatrix} 14 & 7 & -2 \\ 20 & 12 & -4 \\ -14 & 7 & -6 \end{bmatrix} = \begin{bmatrix} -13 & 7 & -12 \\ -18 & 10 & -14 \\ 17 & -7 & 34 \end{bmatrix}$$

16. $AC + BC = \begin{bmatrix} 0 & 3 & -5 \\ 1 & 2 & 6 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 6 & 2 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 0 \\ -2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 6 & 2 \\ -2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 28 & -9 \\ 4 & 23 \end{bmatrix} + \begin{bmatrix} 22 & 6 \\ 14 & -2 \end{bmatrix} = \begin{bmatrix} 50 & -3 \\ 18 & 21 \end{bmatrix}$$

17. $\begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 & 6 \\ 3 & -1 & 3 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2(2)+(-2)(3) & 2(1)+(-2)(-1) & 2(4)+(-2)(3) & 2(6)+(-2)(2) \\ 1(2)+0(3) & 1(1)+0(-1) & 1(4)+0(3) & 1(6)+0(2) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4 & 2 & 8 \\ 2 & 1 & 4 & 6 \end{bmatrix}$$

18. $\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -6 & 6 & 1 & 0 \\ 2 & 5 & 4 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 4(-6)+1(2) & 4(6)+1(5) & 4(1)+1(4) & 4(0)+1(-1) \\ 2(-6)+1(2) & 2(6)+1(5) & 2(1)+1(4) & 2(0)+1(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -22 & 29 & 8 & -1 \\ -10 & 17 & 6 & -1 \end{bmatrix}$$

19. $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 4 & 1 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 6 & 2 \\ 8 & -1 \end{bmatrix} = \begin{bmatrix} 1(1)+0(6)+1(8) & 1(3)+0(2)+1(-1) \\ 2(1)+4(6)+1(8) & 2(3)+4(2)+1(-1) \\ 3(1)+6(6)+1(8) & 3(3)+6(2)+1(-1) \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 34 & 13 \\ 47 & 20 \end{bmatrix}$

20. $\begin{bmatrix} 4 & -2 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4(2)+(-2)(1)+3(0) & 4(6)+(-2)(-1)+3(2) \\ 0(2)+1(1)+2(0) & 0(6)+1(-1)+2(2) \\ -1(2)+0(1)+1(0) & -1(6)+0(-1)+1(2) \end{bmatrix}$

$$= \begin{bmatrix} 6 & 32 \\ 1 & 3 \\ -2 & -4 \end{bmatrix}$$

21. Augment the matrix with the identity and use row operations to find the inverse:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

Interchange r_1 and r_2 $\quad R_2 = -2r_1 + r_2 \quad R_2 = -r_2 \quad R_1 = -r_2 + r_1$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

22. Augment the matrix with the identity and use row operations to find the inverse:

$$A = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \rightarrow \left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rr|rr} 1 & 0 & 1 & 1 \\ -2 & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rr|rr} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

$R_1 = r_2 + r_1 \qquad R_2 = 2r_1 + r_2$

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

23. Augment the matrix with the identity and use row operations to find the inverse:

$$A = \begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix} \rightarrow \left[\begin{array}{rr|rr} 6 & 5 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rr|rr} 2 & 2 & 0 & 1 \\ 6 & 5 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|rr} 2 & 2 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{array}\right] \rightarrow \left[\begin{array}{rr|rr} 1 & 1 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{rr|rr} 1 & 0 & 1 & -\frac{5}{2} \\ 0 & 1 & -1 & 3 \end{array}\right]$$

Interchange r_1 and r_2 $\qquad R_2 = -3r_1 + r_2 \qquad R_1 = \frac{1}{2}r_1$, $R_2 = -r_2 \qquad R_1 = -r_2 + r_1$

$$A^{-1} = \begin{bmatrix} 1 & -\frac{5}{2} \\ -1 & 3 \end{bmatrix}$$

24. Augment the matrix with the identity and use row operations to find the inverse:

$$A = \begin{bmatrix} -4 & 1 \\ 6 & -2 \end{bmatrix} \rightarrow \left[\begin{array}{rr|rr} -4 & 1 & 1 & 0 \\ 6 & -2 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rr|rr} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 6 & -2 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rr|rr} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & \frac{3}{2} & 1 \end{array}\right] \rightarrow \left[\begin{array}{rr|rr} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & -3 & -2 \end{array}\right] \rightarrow \left[\begin{array}{rr|rr} 1 & 0 & -1 & -\frac{1}{2} \\ 0 & 1 & -3 & -2 \end{array}\right]$$

$R_1 = -\frac{1}{4}r_1 \qquad R_2 = -6r_1 + r_2 \qquad R_2 = -2r_2 \qquad R_1 = \frac{1}{4}r_2 + r_1$

$$A^{-1} = \begin{bmatrix} -1 & -\frac{1}{2} \\ -3 & -2 \end{bmatrix}$$

25. Augment the matrix with the identity and use row operations to find the inverse:

$$A = \begin{bmatrix} 2 & 1 \\ a & a \end{bmatrix} \rightarrow \left[\begin{array}{rr|rr} 2 & 1 & 1 & 0 \\ a & a & 0 & 1 \end{array}\right] \text{ where } a \neq 0.$$

$$\rightarrow \left[\begin{array}{rr|rr} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ a & a & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rr|rr} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2}a & -\frac{1}{2}a & 1 \end{array}\right] \rightarrow \left[\begin{array}{rr|rr} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & \frac{2}{a} \end{array}\right] \rightarrow \left[\begin{array}{rr|rr} 1 & 0 & 1 & -\frac{1}{a} \\ 0 & 1 & -1 & \frac{2}{a} \end{array}\right]$$

$R_1 = \frac{1}{2}r_1 \qquad R_2 = -a\,r_1 + r_2 \qquad R_2 = \left(\frac{2}{a}\right)r_2 \qquad R_1 = -\frac{1}{2}r_2 + r_1$

$$A^{-1} = \begin{bmatrix} 1 & -\frac{1}{a} \\ -1 & \frac{2}{a} \end{bmatrix}$$

26. Augment the matrix with the identity and use row operations to find the inverse:

$$A = \begin{bmatrix} b & 3 \\ b & 2 \end{bmatrix} \rightarrow \left[\begin{array}{rr|rr} b & 3 & 1 & 0 \\ b & 2 & 0 & 1 \end{array}\right] \text{ where } b \neq 0.$$

$$\rightarrow \left[\begin{array}{rr|rr} b & 3 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rr|rr} 1 & \frac{3}{b} & \frac{1}{b} & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rr|rr} 1 & \frac{3}{b} & \frac{1}{b} & 0 \\ 0 & 1 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{rr|rr} 1 & 0 & -\frac{2}{b} & \frac{3}{b} \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$R_2 = -r_1 + r_2 \qquad R_1 = \frac{1}{b}r_1 \qquad R_2 = -r_2 \qquad R_1 = -\frac{3}{b}r_2 + r_1$

$$A^{-1} = \begin{bmatrix} -\frac{2}{b} & \frac{3}{b} \\ 1 & -1 \end{bmatrix}$$

27. Augment the matrix with the identity and use row operations to find the inverse:

$$A=\begin{bmatrix}1&-1&1\\0&-2&1\\-2&-3&0\end{bmatrix}\rightarrow\left[\begin{array}{ccc|ccc}1&-1&1&1&0&0\\0&-2&1&0&1&0\\-2&-3&0&0&0&1\end{array}\right]$$

$$\rightarrow\left[\begin{array}{ccc|ccc}1&-1&1&1&0&0\\0&-2&1&0&1&0\\0&-5&2&2&0&1\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1&-1&1&1&0&0\\0&1&-\frac{1}{2}&0&-\frac{1}{2}&0\\0&-5&2&2&0&1\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1&0&\frac{1}{2}&1&-\frac{1}{2}&0\\0&1&-\frac{1}{2}&0&-\frac{1}{2}&0\\0&0&-\frac{1}{2}&2&-\frac{5}{2}&1\end{array}\right]$$

$R_3=2r_1+r_3$ $\quad R_2=-\frac{1}{2}r_2$ $\quad R_1=r_2+r_1$

$R_3=5r_2+r_3$

$$\rightarrow\left[\begin{array}{ccc|ccc}1&0&\frac{1}{2}&1&-\frac{1}{2}&0\\0&1&-\frac{1}{2}&0&-\frac{1}{2}&0\\0&0&1&-4&5&-2\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1&0&0&3&-3&1\\0&1&0&-2&2&-1\\0&0&1&-4&5&-2\end{array}\right]$$

$R_3=-2r_3$ $\quad R_1=-\frac{1}{2}r_3+r_1$

$R_2=\frac{1}{2}r_3+r_2$

$$A^{-1}=\begin{bmatrix}3&-3&1\\-2&2&-1\\-4&5&-2\end{bmatrix}$$

28. Augment the matrix with the identity and use row operations to find the inverse:

$$A=\begin{bmatrix}1&0&2\\-1&2&3\\1&-1&0\end{bmatrix}\rightarrow\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\-1&2&3&0&1&0\\1&-1&0&0&0&1\end{array}\right]$$

$$\rightarrow\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&2&5&1&1&0\\0&-1&-2&-1&0&1\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&\frac{5}{2}&\frac{1}{2}&\frac{1}{2}&0\\0&-1&-2&-1&0&1\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&\frac{5}{2}&\frac{1}{2}&\frac{1}{2}&0\\0&0&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&1\end{array}\right]$$

$R_2=r_1+r_2$ $\quad R_2=\frac{1}{2}r_2$ $\quad R_3=r_2+r_3$

$R_3=-r_1+r_3$

$$\rightarrow\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&\frac{5}{2}&\frac{1}{2}&\frac{1}{2}&0\\0&0&1&-1&1&2\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1&0&0&3&-2&-4\\0&1&0&3&-2&-5\\0&0&1&-1&1&2\end{array}\right]$$

$R_3=2r_3$ $\quad R_1=-2r_3+r_1$

$R_2=-\frac{5}{2}r_3+r_2$

$$A^{-1}=\begin{bmatrix}3&-2&-4\\3&-2&-5\\-1&1&2\end{bmatrix}$$

29. Augment the matrix with the identity and use row operations to find the inverse:

$$A=\begin{bmatrix}1&1&1\\3&2&-1\\3&1&2\end{bmatrix}\rightarrow\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\3&2&-1&0&1&0\\3&1&2&0&0&1\end{array}\right]$$

$$\rightarrow\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&-1&-4&-3&1&0\\0&-2&-1&-3&0&1\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&4&3&-1&0\\0&-2&-1&-3&0&1\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1&0&-3&-2&1&0\\0&1&4&3&-1&0\\0&0&7&3&-2&1\end{array}\right]$$

$R_2=-3r_1+r_2$ $\quad R_2=-r_2$ $\quad R_1=-r_2+r_1$

$R_3=-3r_1+r_3$ $\quad R_3=2r_2+r_3$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 1 & 4 & 3 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ 0 & 1 & 0 & \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ 0 & 0 & 1 & \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{array}\right]$$

$R_3 = \frac{1}{7} r_3$ $\qquad$ $R_1 = 3r_3 + r_1$

$\qquad$ $R_2 = -4r_3 + r_2$

$$A^{-1} = \begin{bmatrix} -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{bmatrix}$$

30. Augment the matrix with the identity and use row operations to find the inverse:

$$A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|ccc} 3 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 3 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -3 & -2 & 1 & -3 & 0 \\ 0 & -5 & -1 & 0 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & -5 & -1 & 0 & -2 & 1 \end{array}\right]$$

Interchange r_1 and r_2 $\qquad$ $R_2 = -3r_1 + r_2$, $R_3 = -2r_1 + r_3$ $\qquad$ $R_2 = -\frac{1}{3} r_2$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{7}{3} & -\frac{5}{3} & 3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ 0 & 0 & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{array}\right]$$

$R_1 = -2r_2 + r_1$ $\qquad$ $R_3 = \frac{3}{7} r_3$ $\qquad$ $R_1 = \frac{1}{3} r_3 + r_1$

$R_3 = 5r_2 + r_3$ $\qquad$ $R_2 = -\frac{2}{3} r_3 + r_2$

$$A^{-1} = \begin{bmatrix} \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{bmatrix}$$

31. Rewrite the system of equations in matrix form:

$$\begin{cases} 2x + y = 8 \\ x + y = 5 \end{cases} \qquad A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

Find the inverse of A and solve $X = A^{-1}B$:

From Problem 21, $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ and $X = A^{-1}B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

The solution is $x = 3$, $y = 2$.

32. Rewrite the system of equations in matrix form:

$$\begin{cases} 3x - y = 8 \\ -2x + y = 4 \end{cases} \qquad A = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

Find the inverse of A and solve $X = A^{-1}B$:

From Problem 22, $A^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $X = A^{-1}B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 28 \end{bmatrix}$.

The solution is $x = 12$, $y = 28$.

33. Rewrite the system of equations in matrix form:
$$\begin{cases} 2x + y = 0 \\ x + y = 5 \end{cases} \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$
Find the inverse of A and solve $X = A^{-1}B$:
From Problem 21, $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ and $X = A^{-1}B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$.
The solution is $x = -5,\ y = 10$.

34. Rewrite the system of equations in matrix form:
$$\begin{cases} 3x - y = 4 \\ -2x + y = 5 \end{cases} \quad A = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$
Find the inverse of A and solve $X = A^{-1}B$:
From Problem 22, $A^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $X = A^{-1}B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 23 \end{bmatrix}$.
The solution is $x = 9,\ y = 23$.

35. Rewrite the system of equations in matrix form:
$$\begin{cases} 6x + 5y = 7 \\ 2x + 2y = 2 \end{cases} \quad A = \begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ B = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$
Find the inverse of A and solve $X = A^{-1}B$:
From Problem 23, $A^{-1} = \begin{bmatrix} 1 & -\frac{5}{2} \\ -1 & 3 \end{bmatrix}$ and $X = A^{-1}B = \begin{bmatrix} 1 & -\frac{5}{2} \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.
The solution is $x = 2,\ y = -1$.

36. Rewrite the system of equations in matrix form:
$$\begin{cases} -4x + y = 0 \\ 6x - 2y = 14 \end{cases} \quad A = \begin{bmatrix} -4 & 1 \\ 6 & -2 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 14 \end{bmatrix}$$
Find the inverse of A and solve $X = A^{-1}B$:
From Problem 24, $A^{-1} = \begin{bmatrix} -1 & -\frac{1}{2} \\ -3 & -2 \end{bmatrix}$ and $X = A^{-1}B = \begin{bmatrix} -1 & -\frac{1}{2} \\ -3 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 14 \end{bmatrix} = \begin{bmatrix} -7 \\ -28 \end{bmatrix}$.
The solution is $x = -7,\ y = -28$.

37. Rewrite the system of equations in matrix form:
$$\begin{cases} 6x + 5y = 13 \\ 2x + 2y = 5 \end{cases} \quad A = \begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ B = \begin{bmatrix} 13 \\ 5 \end{bmatrix}$$
Find the inverse of A and solve $X = A^{-1}B$:
From Problem 23, $A^{-1} = \begin{bmatrix} 1 & -\frac{5}{2} \\ -1 & 3 \end{bmatrix}$ and $X = A^{-1}B = \begin{bmatrix} 1 & -\frac{5}{2} \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 13 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}$.
The solution is $x = \frac{1}{2},\ y = 2$.

38. Rewrite the system of equations in matrix form:
$$\begin{cases} -4x + y = 5 \\ 6x - 2y = -9 \end{cases} \quad A = \begin{bmatrix} -4 & 1 \\ 6 & -2 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ B = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$
Find the inverse of A and solve $X = A^{-1}B$:

From Problem 24, $A^{-1}=\begin{bmatrix}-1 & -\frac{1}{2}\\ -3 & -2\end{bmatrix}$ and $X=A^{-1}B=\begin{bmatrix}-1 & -\frac{1}{2}\\ -3 & -2\end{bmatrix}\begin{bmatrix}5\\ -9\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}\\ 3\end{bmatrix}$.

The solution is $x=-\dfrac{1}{2},\ y=3$.

39. Rewrite the system of equations in matrix form:

$$\begin{cases}2x+\ \ y=-3\\ ax+ay=-a\end{cases}\quad a\neq 0\qquad A=\begin{bmatrix}2 & 1\\ a & a\end{bmatrix},\quad X=\begin{bmatrix}x\\ y\end{bmatrix},\quad B=\begin{bmatrix}-3\\ -a\end{bmatrix}$$

Find the inverse of A and solve $X=A^{-1}B$:

From Problem 25, $A^{-1}=\begin{bmatrix}1 & -\frac{1}{a}\\ -1 & \frac{2}{a}\end{bmatrix}$ and $X=A^{-1}B=\begin{bmatrix}1 & -\frac{1}{a}\\ -1 & \frac{2}{a}\end{bmatrix}\begin{bmatrix}-3\\ -a\end{bmatrix}=\begin{bmatrix}-2\\ 1\end{bmatrix}$.

The solution is $x=-2,\ y=1$.

40. Rewrite the system of equations in matrix form:

$$\begin{cases}bx+3y=2b+3\\ bx+2y=2b+2\end{cases}\quad b\neq 0\qquad A=\begin{bmatrix}b & 3\\ b & 2\end{bmatrix},\quad X=\begin{bmatrix}x\\ y\end{bmatrix},\quad B=\begin{bmatrix}2b+3\\ 2b+2\end{bmatrix}$$

Find the inverse of A and solve $X=A^{-1}B$:

From Problem 26, $A^{-1}=\begin{bmatrix}-\frac{2}{b} & \frac{3}{b}\\ 1 & -1\end{bmatrix}$ and $X=A^{-1}B=\begin{bmatrix}-\frac{2}{b} & \frac{3}{b}\\ 1 & -1\end{bmatrix}\begin{bmatrix}2b+3\\ 2b+2\end{bmatrix}=\begin{bmatrix}2\\ 1\end{bmatrix}$.

The solution is $x=2,\ y=1$.

41. Rewrite the system of equations in matrix form:

$$\begin{cases}2x+\ \ y=\dfrac{7}{a}\\ ax+ay=5\end{cases}\quad a\neq 0\qquad A=\begin{bmatrix}2 & 1\\ a & a\end{bmatrix},\quad X=\begin{bmatrix}x\\ y\end{bmatrix},\quad B=\begin{bmatrix}\frac{7}{a}\\ 5\end{bmatrix}$$

Find the inverse of A and solve $X=A^{-1}B$:

From Problem 25, $A^{-1}=\begin{bmatrix}1 & -\frac{1}{a}\\ -1 & \frac{2}{a}\end{bmatrix}$ and $X=A^{-1}B=\begin{bmatrix}1 & -\frac{1}{a}\\ -1 & \frac{2}{a}\end{bmatrix}\begin{bmatrix}\frac{7}{a}\\ 5\end{bmatrix}=\begin{bmatrix}\frac{2}{a}\\ \frac{3}{a}\end{bmatrix}$.

The solution is $x=\dfrac{2}{a},\ y=\dfrac{3}{a}$.

42. Rewrite the system of equations in matrix form:

$$\begin{cases}bx+3y=14\\ bx+2y=10\end{cases}\quad b\neq 0\qquad A=\begin{bmatrix}b & 3\\ b & 2\end{bmatrix},\quad X=\begin{bmatrix}x\\ y\end{bmatrix},\quad B=\begin{bmatrix}14\\ 10\end{bmatrix}$$

Find the inverse of A and solve $X=A^{-1}B$:

From Problem 26, $A^{-1}=\begin{bmatrix}-\frac{2}{b} & \frac{3}{b}\\ 1 & -1\end{bmatrix}$ and $X=A^{-1}B=\begin{bmatrix}-\frac{2}{b} & \frac{3}{b}\\ 1 & -1\end{bmatrix}\begin{bmatrix}14\\ 10\end{bmatrix}=\begin{bmatrix}\frac{2}{b}\\ 4\end{bmatrix}$.

The solution is $x=\dfrac{2}{b},\ y=4$.

43. Rewrite the system of equations in matrix form:

$$\begin{cases}x-\ \ y+z=\ \ 0\\ \quad\ -2y+z=-1\\ -2x-3y\quad\ \ =-5\end{cases}\qquad A=\begin{bmatrix}1 & -1 & 1\\ 0 & -2 & 1\\ -2 & -3 & 0\end{bmatrix},\quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix},\quad B=\begin{bmatrix}0\\ -1\\ -5\end{bmatrix}$$

Find the inverse of A and solve $X=A^{-1}B$:

From Problem 27,

$$A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{bmatrix} \text{ and } X = A^{-1}B = \begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ -5 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}.$$

The solution is $x = -2,\ y = 3,\ z = 5$.

44. Rewrite the system of equations in matrix form:

$$\begin{cases} x \quad\quad + 2z = 6 \\ -x + 2y + 3z = -5 \\ x - y \quad\quad = 6 \end{cases} \qquad A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix}$$

Find the inverse of A and solve $X = A^{-1}B$:

From Problem 28,

$$A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \text{ and } X = A^{-1}B = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}.$$

The solution is $x = 4,\ y = -2,\ z = 1$.

45. Rewrite the system of equations in matrix form:

$$\begin{cases} x - y + z = 2 \\ \quad -2y + z = 2 \\ -2x - 3y \quad = \dfrac{1}{2} \end{cases} \qquad A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 2 \\ 2 \\ \frac{1}{2} \end{bmatrix}$$

Find the inverse of A and solve $X = A^{-1}B$:

From Problem 27,

$$A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{bmatrix} \text{ and } X = A^{-1}B = \begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

The solution is $x = \dfrac{1}{2},\ y = -\dfrac{1}{2},\ z = 1$.

46. Rewrite the system of equations in matrix form:

$$\begin{cases} x \quad\quad + 2z = 2 \\ -x + 2y + 3z = -\dfrac{3}{2} \\ x - y \quad\quad = 2 \end{cases} \qquad A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 2 \\ -\frac{3}{2} \\ 2 \end{bmatrix}$$

Find the inverse of A and solve $X = A^{-1}B$:

From Problem 28,

$$A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \text{ and } X = A^{-1}B = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -\frac{3}{2} \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ \frac{1}{2} \end{bmatrix}.$$

The solution is $x = 1,\ y = -1,\ z = \dfrac{1}{2}$.

47. Rewrite the system of equations in matrix form:

$$\begin{cases} x + y + z = 9 \\ 3x + 2y - z = 8 \\ 3x + y + 2z = 1 \end{cases} \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ 8 \\ 1 \end{bmatrix}$$

Find the inverse of A and solve $X = A^{-1}B$:

From Problem 29,

$$A^{-1} = \begin{bmatrix} -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{bmatrix} \text{ and } X = A^{-1}B = \begin{bmatrix} -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 9 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{34}{7} \\ \frac{85}{7} \\ \frac{12}{7} \end{bmatrix}.$$

The solution is $x = -\dfrac{34}{7}, y = \dfrac{85}{7}, z = \dfrac{12}{7}$.

48. Rewrite the system of equations in matrix form:

$$\begin{cases} 3x + 3y + z = 8 \\ x + 2y + z = 5 \\ 2x - y + z = 4 \end{cases} \qquad A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix}$$

Find the inverse of A and solve $X = A^{-1}B$:

From Problem 30,

$$A^{-1} = \begin{bmatrix} \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{bmatrix} \text{ and } X = A^{-1}B = \begin{bmatrix} \frac{3}{7} & -\frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} \\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7} \end{bmatrix}\begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{8}{7} \\ \frac{5}{7} \\ \frac{17}{7} \end{bmatrix}.$$

The solution is $x = \dfrac{8}{7}, y = \dfrac{5}{7}, z = \dfrac{17}{7}$.

49. Rewrite the system of equations in matrix form:

$$\begin{cases} x + y + z = 2 \\ 3x + 2y - z = \dfrac{7}{3} \\ 3x + y + 2z = \dfrac{10}{3} \end{cases} \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \\ 3 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ \frac{7}{3} \\ \frac{10}{3} \end{bmatrix}$$

Find the inverse of A and solve $X = A^{-1}B$:

From Problem 29,

$$A^{-1} = \begin{bmatrix} -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{bmatrix} \text{ and } X = A^{-1}B = \begin{bmatrix} -\frac{5}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{9}{7} & \frac{1}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{2}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 2 \\ \frac{7}{3} \\ \frac{10}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \\ \frac{2}{3} \end{bmatrix}.$$

The solution is $x = \dfrac{1}{3}, y = 1, z = \dfrac{2}{3}$.

50. Rewrite the system of equations in matrix form:

$$\begin{cases} 3x + 3y + z = 1 \\ x + 2y + z = 0 \\ 2x - y + z = 4 \end{cases} \qquad A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

Find the inverse of A and solve $X = A^{-1}B$:

From Problem 30,

$$A^{-1}=\begin{bmatrix}\frac{3}{7} & -\frac{4}{7} & \frac{1}{7}\\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7}\\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7}\end{bmatrix} \text{ and } X=A^{-1}B=\begin{bmatrix}\frac{3}{7} & -\frac{4}{7} & \frac{1}{7}\\ \frac{1}{7} & \frac{1}{7} & -\frac{2}{7}\\ -\frac{5}{7} & \frac{9}{7} & \frac{3}{7}\end{bmatrix}\begin{bmatrix}1\\0\\4\end{bmatrix}=\begin{bmatrix}1\\-1\\1\end{bmatrix}.$$

The solution is $x=1,\ y=-1,\ z=1$.

51. Augment the matrix with the identity and use row operations to find the inverse:

$$A=\begin{bmatrix}4 & 2\\ 2 & 1\end{bmatrix} \rightarrow \left[\begin{array}{cc|cc}4 & 2 & 1 & 0\\ 2 & 1 & 0 & 1\end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc}4 & 2 & 1 & 0\\ 0 & 0 & -\frac{1}{2} & 1\end{array}\right] \rightarrow \left[\begin{array}{cc|cc}1 & \frac{1}{2} & \frac{1}{4} & 0\\ 0 & 0 & -\frac{1}{2} & 1\end{array}\right]$$

$$R_2=-\tfrac{1}{2}r_1+r_2 \qquad R_1=\tfrac{1}{4}r_1$$

There is no way to obtain the identity matrix on the left; thus, there is no inverse.

52. Augment the matrix with the identity and use row operations to find the inverse:

$$A=\begin{bmatrix}-3 & \frac{1}{2}\\ 6 & -1\end{bmatrix} \rightarrow \left[\begin{array}{cc|cc}-3 & \frac{1}{2} & 1 & 0\\ 6 & -1 & 0 & 1\end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc}-3 & \frac{1}{2} & 1 & 0\\ 0 & 0 & 2 & 1\end{array}\right] \rightarrow \left[\begin{array}{cc|cc}1 & -\frac{1}{6} & -\frac{1}{3} & 0\\ 0 & 0 & 2 & 1\end{array}\right]$$

$$R_2=2r_1+r_2 \qquad R_1=-\tfrac{1}{3}r_1$$

There is no way to obtain the identity matrix on the left; thus, there is no inverse.

53. Augment the matrix with the identity and use row operations to find the inverse:

$$A=\begin{bmatrix}15 & 3\\ 10 & 2\end{bmatrix} \rightarrow \left[\begin{array}{cc|cc}15 & 3 & 1 & 0\\ 10 & 2 & 0 & 1\end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc}15 & 3 & 1 & 0\\ 0 & 0 & -\frac{2}{3} & 1\end{array}\right] \rightarrow \left[\begin{array}{cc|cc}1 & \frac{1}{5} & \frac{1}{15} & 0\\ 0 & 0 & -\frac{2}{3} & 1\end{array}\right]$$

$$R_2=-\tfrac{2}{3}r_1+r_2 \qquad R_1=\tfrac{1}{15}r_1$$

There is no way to obtain the identity matrix on the left; thus, there is no inverse.

54. Augment the matrix with the identity and use row operations to find the inverse:

$$A=\begin{bmatrix}-3 & 0\\ 4 & 0\end{bmatrix} \rightarrow \left[\begin{array}{cc|cc}-3 & 0 & 1 & 0\\ 4 & 0 & 0 & 1\end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc}-3 & 0 & 1 & 0\\ 0 & 0 & \frac{4}{3} & 1\end{array}\right] \rightarrow \left[\begin{array}{cc|cc}1 & 0 & -\frac{1}{3} & 0\\ 0 & 0 & \frac{4}{3} & 1\end{array}\right]$$

$$R_2=\tfrac{4}{3}r_1+r_2 \qquad R_1=-\tfrac{1}{3}r_1$$

There is no way to obtain the identity matrix on the left; thus, there is no inverse.

55. Augment the matrix with the identity and use row operations to find the inverse:

$$A=\begin{bmatrix}-3 & 1 & -1\\ 1 & -4 & -7\\ 1 & 2 & 5\end{bmatrix} \rightarrow \left[\begin{array}{ccc|ccc}-3 & 1 & -1 & 1 & 0 & 0\\ 1 & -4 & -7 & 0 & 1 & 0\\ 1 & 2 & 5 & 0 & 0 & 1\end{array}\right]$$

$$\to \left[\begin{array}{rrr|rrr} 1 & 2 & 5 & 0 & 0 & 1 \\ 1 & -4 & -7 & 0 & 1 & 0 \\ -3 & 1 & -1 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 2 & 5 & 0 & 0 & 1 \\ 0 & -6 & -12 & 0 & 1 & -1 \\ 0 & 7 & 14 & 1 & 0 & 3 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 2 & 5 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & -\frac{1}{6} & \frac{1}{6} \\ 0 & 7 & 14 & 1 & 0 & 3 \end{array}\right]$$

Interchange r_1 and r_3 $\quad$ $R_2 = -r_1 + r_2$, $R_3 = 3r_1 + r_3$ $\quad$ $R_2 = -\frac{1}{6} r_2$

$$\to \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 2 & 0 & -\frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 1 & \frac{7}{6} & \frac{11}{6} \end{array}\right]$$

$R_1 = -2r_2 + r_1$
$R_3 = -7r_2 + r_3$

There is no way to obtain the identity matrix on the left; thus, there is no inverse.

56. Augment the matrix with the identity and use row operations to find the inverse:

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & -4 & 1 \\ -5 & 7 & 1 \end{bmatrix} \to \left[\begin{array}{rrr|rrr} 1 & 1 & -3 & 1 & 0 & 0 \\ 2 & -4 & 1 & 0 & 1 & 0 \\ -5 & 7 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|rrr} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & -6 & 7 & -2 & 1 & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 12 & -14 & 5 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & -\frac{11}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & 1 & -\frac{7}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array}\right]$$

$R_2 = -2r_1 + r_2$, $R_3 = 5r_1 + r_3$ $\quad$ $R_2 = -\frac{1}{6} r_2$ $\quad$ $R_1 = -r_2 + r_1$, $R_3 = -12r_2 + r_3$

There is no way to obtain the identity matrix on the left; thus, there is no inverse.

57. $\begin{bmatrix} 0.01 & 0.05 & -0.01 \\ 0.01 & -0.02 & 0.01 \\ -0.02 & 0.01 & 0.03 \end{bmatrix}$

58. $\begin{bmatrix} 0.26 & -0.29 & -0.20 \\ -1.21 & 1.63 & 1.20 \\ -1.84 & 2.53 & 1.80 \end{bmatrix}$

59. $\begin{bmatrix} 0.02 & -0.04 & -0.01 & 0.01 \\ -0.02 & 0.05 & 0.03 & -0.03 \\ 0.02 & 0.01 & -0.04 & 0.00 \\ -0.02 & 0.06 & 0.07 & 0.06 \end{bmatrix}$

60. $\begin{bmatrix} 0.01 & 0.04 & -0.00 & 0.03 \\ 0.02 & -0.02 & 0.01 & 0.01 \\ -0.04 & 0.02 & 0.04 & 0.06 \\ 0.05 & -0.02 & -0.00 & -0.09 \end{bmatrix}$

61. $x = 4.57,\ y = -6.44,\ z = -24.07$

62. $x = 4.56,\ y = -6.06,\ z = -22.55$

63. $x = -1.19,\ y = 2.46,\ z = 8.27$

64. $x = -2.05,\ y = 3.88,\ z = 13.36$

65. (a) The rows of the 2 by 3 matrix represent stainless steel and aluminum. The columns represent 10-gallon, 5-gallon, and 1-gallon.

The 2 by 3 matrix is: $\begin{bmatrix} 500 & 350 & 400 \\ 700 & 500 & 850 \end{bmatrix}$

The 3 by 2 matrix is: $\begin{bmatrix} 500 & 700 \\ 350 & 500 \\ 400 & 850 \end{bmatrix}$

(b) The 3 by 1 matrix representing the amount of material is:

$$\begin{bmatrix} 15 \\ 8 \\ 3 \end{bmatrix}$$

(c) The days usage of materials is:

$$\begin{bmatrix} 500 & 350 & 400 \\ 700 & 500 & 850 \end{bmatrix}\begin{bmatrix} 15 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 11,500 \\ 17,050 \end{bmatrix}$$

11,500 pounds of stainless steel and 17,050 pounds of aluminum are used each day.

(d) The 1 by 2 matrix representing cost is:

$$\begin{bmatrix} 0.10 & 0.05 \end{bmatrix}$$

(e) The total cost of the days production was:

$$\begin{bmatrix} 0.10 & 0.05 \end{bmatrix}\begin{bmatrix} 11,500 \\ 17,050 \end{bmatrix} = \begin{bmatrix} 2002.50 \end{bmatrix}$$

The total cost of the days production was \$2,002.50.

66. (a) The rows of the 2 by 3 matrix represent the location. The columns represent the type of car sold.

The 2 by 3 matrix for January is:

$$\begin{bmatrix} 400 & 250 & 50 \\ 450 & 200 & 140 \end{bmatrix}$$

The 2 by 3 matrix for February is:

$$\begin{bmatrix} 350 & 100 & 30 \\ 350 & 300 & 100 \end{bmatrix}$$

(b) Adding the matrices:

$$\begin{bmatrix} 400 & 250 & 50 \\ 450 & 200 & 140 \end{bmatrix} + \begin{bmatrix} 350 & 100 & 30 \\ 350 & 300 & 100 \end{bmatrix} = \begin{bmatrix} 750 & 350 & 80 \\ 800 & 500 & 240 \end{bmatrix}$$

(c) The 3 by 1 matrix representing profit:

$$\begin{bmatrix} 100 \\ 150 \\ 200 \end{bmatrix}$$

(d) Multiplying to find the profit at each location:

$$\begin{bmatrix} 750 & 350 & 80 \\ 800 & 500 & 240 \end{bmatrix}\begin{bmatrix} 100 \\ 150 \\ 200 \end{bmatrix} = \begin{bmatrix} 143,500 \\ 203,000 \end{bmatrix}$$

The city location has a two month profit of \$143,500. The suburban location has a two month profit of \$203,000.

67. We need to consider 2 different cases.

Case 1: $a \neq 0$, $D = ad - bc \neq 0$.

$$A = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{R_1 = \frac{1}{a} r_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{R_2 = -c r_1 + r_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d - \frac{cb}{a} & -\frac{c}{a} & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & \frac{b}{b} & \frac{1}{a} & 0 \\ c & \frac{ad - cb}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\xrightarrow{R_2 = \left(\frac{a}{ad - bc}\right) r_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad - bc} & \frac{a}{ad - bc} \end{array}\right] \xrightarrow{R_1 = \left(-\frac{b}{a}\right) r_2 + r_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} + \frac{bc}{a(ad - bc)} & \frac{-b}{ad - bc} \\ 0 & 1 & -\frac{c}{ad - bc} & \frac{a}{ad - bc} \end{array}\right]$$

Note that $\dfrac{1}{a} + \dfrac{bc}{a(ad-bc)} = \dfrac{1\,(ad-bc)+bc}{a(ad-bc)} = \dfrac{ad-bc+bc}{a(ad-bc)} = \dfrac{ad}{a(ad-bc)} = \dfrac{d}{ad-bc}$

So, we have $\left[\begin{array}{cc|cc} 1 & 0 & \dfrac{1}{a} + \dfrac{bc}{a(ad-bc)} & \dfrac{-b}{ad-bc} \\ 0 & 1 & -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ 0 & 1 & -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{array}\right].$

Therefore $A^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix} = \left(\dfrac{1}{ad-bc}\right)\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \left(\dfrac{1}{D}\right)\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$

where $D = ad - bc$.

Case 2: $a = 0$, $D = ad - bc \neq 0$.

First note that $D = ad - bc = 0\ d - bc = -bc \neq 0$ $b \neq 0$ and $c \neq 0$.

$$A = \left[\begin{array}{cc|cc} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{\text{interchange } r_1 \text{ and } r_2} \left[\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{array}\right] \xrightarrow{R_1 = \frac{1}{c} r_1} \left[\begin{array}{cc|cc} 1 & \dfrac{d}{c} & 0 & \dfrac{1}{c} \\ 0 & b & 1 & 0 \end{array}\right] \xrightarrow{R_2 = \frac{1}{b} r_2} \left[\begin{array}{cc|cc} 1 & \dfrac{d}{c} & 0 & \dfrac{1}{c} \\ 0 & 1 & \dfrac{1}{b} & 0 \end{array}\right]$$

$$\xrightarrow{R_1 = \left(\frac{-d}{c}\right) r_2 + r_1} \left[\begin{array}{cc|cc} 1 & 0 & \dfrac{-d}{bc} & \dfrac{1}{c} \\ 0 & 1 & \dfrac{1}{b} & 0 \end{array}\right]$$

Therefore $A^{-1} = \begin{bmatrix} \dfrac{-d}{bc} & \dfrac{1}{c} \\ \dfrac{1}{b} & 0 \end{bmatrix} = \left(\dfrac{1}{-bc}\right)\begin{bmatrix} d & -b \\ -c & 0 \end{bmatrix} = \left(\dfrac{1}{D}\right)\begin{bmatrix} d & -b \\ -c & 0 \end{bmatrix},$

where $D = ad - bc = 0\ d - bc = -bc$.