Chapter 12

Systems of Equations and Inequalities

12.6 Partial Fraction Decomposition

1. The rational expression $\frac{x}{x^2-1}$ is proper, since the degree of the numerator is less than the degree of the denominator.

2. The rational expression $\frac{5x+2}{x^3-1}$ is proper, since the degree of the numerator is less than the degree of the denominator.

3. The rational expression $\frac{x^2+5}{x^2-4}$ is improper, so perform the division:

$$\begin{array}{r} 1 \\ x^2-4\overline{)\,x^2+5} \\ \underline{x^2-4} \\ 9 \end{array}$$

The proper rational expression is:

$$\frac{x^2+5}{x^2-4} = 1 + \frac{9}{x^2-4}$$

4. The rational expression $\frac{3x^2-2}{x^2-1}$ is improper, so perform the division:

$$\begin{array}{r} 3 \\ x^2-1\overline{)\,3x^2-2} \\ \underline{3x^2-3} \\ 1 \end{array}$$

The proper rational expression is:

$$\frac{3x^2-2}{x^2-1} = 3 + \frac{1}{x^2-1}$$

5. The rational expression $\frac{5x^3+2x-1}{x^2-4}$ is improper, so perform the division:

$$\begin{array}{r} 5x \\ x^2-4\overline{)\,5x^3+0x^2+2x-1} \\ \underline{5x^3\qquad -20x} \\ 22x-1 \end{array}$$

The proper rational expression is:

$$\frac{5x^3+2x-1}{x^2-4} = 5x + \frac{22x-1}{x^2-4}$$

6. The rational expression $\frac{3x^4+x^2-2}{x^3+8}$ is improper, so perform the division:

$$\begin{array}{r} 3x \\ x^3+8\overline{)\,3x^4+x^2+0x-2} \\ \underline{3x^4\qquad +24x} \\ x^2-24x-2 \end{array}$$

The proper rational expression is:

$$\frac{3x^4+x^2-2}{x^3+8} = 3x + \frac{x^2-24x-2}{x^3+8}$$

7. The rational expression $\frac{x(x-1)}{(x+4)(x-3)} = \frac{x^2-x}{x^2+x-12}$ is improper, so perform the division:

$$\begin{array}{r} 1 \\ x^2+x-12 \overline{)\, x^2-x+0} \\ \underline{x^2+x-12} \\ -2x+12 \end{array}$$

The proper rational expression is: $\frac{x(x-1)}{(x+4)(x-3)} = 1 + \frac{-2x+12}{x^2+x-12}$

8. The rational expression $\frac{2x(x^2+4)}{x^2+1} = \frac{2x^3+8x}{x^2+1}$ is improper, so perform the division:

$$\begin{array}{r} 2x \\ x^2+1 \overline{)\, 2x^3+8x} \\ \underline{2x^3+2x} \\ 6x \end{array}$$

The proper rational expression is: $\frac{2x(x^2+4)}{x^2+1} = 2x + \frac{6x}{x^2+1}$

9. Find the partial fraction decomposition:

$$\frac{4}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$4 = A(x-1) + Bx$ (Multiply both sides by $x(x-1)$.)

Let $x = 1$ then $4 = A(0) + B$, or $B = 4$

Let $x = 0$: then $4 = A(-1) + B(0)$, or $A = -4$

$$\frac{4}{x(x-1)} = \frac{-4}{x} + \frac{4}{x-1}$$

10. Find the partial fraction decomposition:

$$\frac{3x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$3x = A(x-1) + B(x+2)$ (Multiply both sides by $(x+2)(x-1)$.)

Let $x = 1$ then $3(1) = A(0) + B(3)$, or $B = 3 \rightarrow B = 1$

Let $x = -2$: then $3(-2) = A(-3) + B(0)$, or $-3A = -6 \rightarrow A = 2$

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{x+2} + \frac{1}{x-1}$$

11. Find the partial fraction decomposition:

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$1 = A(x^2+1) + (Bx+C)x$ (Multiply both sides by $x(x^2+1)$.)

Let $x = 0$: then $1 = A(1) + (B(0) + C)(0)$, or $A = 1$
Let $x = 1$: then $1 = A(1+1) + (B(1) + C)(1)$, or $1 = 2A + B + C$
or $1 = 2(1) + B + C$
or $-1 = B + C$
Let $x = -1$: then $1 = A(1+1) + (B(-1) + C)(-1)$, or $1 = 2A + B - C$
or $1 = 2(1) + B - C$
or $-1 = B - C$

Solve the system of equations:

$$\begin{array}{rr} B + C = -1 & \\ B - C = -1 & \\ \hline 2B = -2 & -1 + C = -1 \\ B = -1 & C = 0 \end{array}$$

$$\frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1}$$

12. Find the partial fraction decomposition:

$$\frac{1}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$$

$$1 = A(x^2+4) + (Bx+C)(x+1)$$

(Multiply both sides by $(x+1)(x^2+4)$.)

Let $x = -1$: then $1 = A(5) + (B(-1) + C)(0) \rightarrow 5A = 1 \rightarrow A = \frac{1}{5}$
Let $x = 1$: then $1 = A(1+4) + (B(1) + C)(1+1) \rightarrow 1 = 5A + 2B + 2C$
$\rightarrow 1 = 5\left(\frac{1}{5}\right) + 2B + 2C \rightarrow 0 = 2B + 2C$
$\rightarrow 0 = B + C$
Let $x = 0$: then $1 = A(0+4) + (B(0) + C)(0+1) \rightarrow 1 = 4A + C$
$\rightarrow 1 = 4\left(\frac{1}{5}\right) + C \rightarrow \frac{1}{5} = C$
$B + \frac{1}{5} = 0 \rightarrow B = -\frac{1}{5}$

$$\frac{1}{(x+1)(x^2+4)} = \frac{\frac{1}{5}}{x+1} + \frac{-\frac{1}{5}x + \frac{1}{5}}{x^2+4}$$

13. Find the partial fraction decomposition:

$$\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$x = A(x-2) + B(x-1)$ (Multiply both sides by $(x-1)(x-2)$.)
Let $x = 1$: then $1 = A(1-2) + B(1-1) \rightarrow 1 = -A \rightarrow A = -1$
Let $x = 2$: then $2 = A(2-2) + B(2-1) \rightarrow 2 = B \rightarrow B = 2$

$$\frac{x}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{2}{x-2}$$

14. Find the partial fraction decomposition:

$$\frac{3x}{(x+2)(x-4)} = \frac{A}{x+2} + \frac{B}{x-4}$$

$3x = A(x-4) + B(x+2)$ (Multiply both sides by $(x+2)(x-4)$.)

Let $x = -2$: then $3(-2) = A(-2-4) + B(-2+2) \rightarrow -6 = -6A \rightarrow A = 1$
Let $x = 4$: then $3(4) = A(4-4) + B(4+2) \rightarrow 12 = 6B \rightarrow B = 2$

$$\frac{3x}{(x+2)(x-4)} = \frac{1}{x+2} + \frac{2}{x-4}$$

15. Find the partial fraction decomposition:

$$\frac{x^2}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

(Multiply both sides by $(x-1)^2(x+1)$.)

$$x^2 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

Let $x = 1$: then $1^2 = A(1-1)(1+1) + B(1+1) + C(1-1)^2$

$$\rightarrow 1 = 2B \rightarrow B = \frac{1}{2}$$

Let $x = -1$: then $(-1)^2 = A(-1-1)(-1+1) + B(-1+1) + C(-1-1)^2$

$$\rightarrow 1 = 4C \rightarrow C = \frac{1}{4}$$

Let $x = 0$: then $0^2 = A(0-1)(0+1) + B(0+1) + C(0-1)^2$

$$\rightarrow 0 = -A + B + C \rightarrow A = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$\frac{x^2}{(x-1)^2(x+1)} = \frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{(x-1)^2} + \frac{\frac{1}{4}}{x+1}$$

16. Find the partial fraction decomposition:

$$\frac{x+1}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$ (Multiply both sides by $x^2(x-2)$.)

$$x+1 = Ax(x-2) + B(x-2) + Cx^2$$

Let $x = 0$: then $0+1 = A(0)(0-2) + B(0-2) + C(0)^2$

$$\rightarrow 1 = -2B \rightarrow B = -\frac{1}{2}$$

Let $x = 2$: then $2+1 = A(2)(2-2) + B(2-2) + C(2)^2$

$$\rightarrow 3 = 4C \rightarrow C = \frac{3}{4}$$

Let $x = 1$: then $1+1 = A(1)(1-2) + B(1-2) + C(1)^2$

$$\rightarrow 2 = -A - B + C \rightarrow A = -B + C - 2 = -\left(-\frac{1}{2}\right) + \frac{3}{4} - 2 = -\frac{3}{4}$$

$$\frac{x+1}{x^2(x-2)} = \frac{-\frac{3}{4}}{x} + \frac{-\frac{1}{2}}{x^2} + \frac{\frac{3}{4}}{x-2}$$

17. Find the partial fraction decomposition:

$$\frac{1}{x^3-8}=\frac{1}{(x-2)(x^2+2x+4)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+4}$$

(Multiply both sides by $(x-2)(x^2+2x+4)$.)

$$1=A(x^2+2x+4)+(Bx+C)(x-2)$$

Let $x=2$: then $1=A\left(2^2+2(2)+4\right)+(B(2)+C)(2-2)$

$$\rightarrow\ 1=12A\ \rightarrow\ A=\frac{1}{12}$$

Let $x=0$: then $1=A\left(0^2+2(0)+4\right)+(B(0)+C)(0-2)$

$$\rightarrow\ 1=4A-2C\ \rightarrow\ 1=4\left(\frac{1}{12}\right)-2C$$

$$\rightarrow\ -2C=\frac{2}{3}\ \rightarrow\ C=-\frac{1}{3}$$

Let $x=1$: then $1=A\left(1^2+2(1)+4\right)+(B(1)+C)(1-2)$

$$\rightarrow\ 1=7A-B-C\ \rightarrow\ 1=7\left(\frac{1}{12}\right)-B+\frac{1}{3}\ \rightarrow\ B=-\frac{1}{12}$$

$$\frac{1}{x^3-8}=\frac{\frac{1}{12}}{x-2}+\frac{-\left(\frac{1}{12}\right)x-\frac{1}{3}}{x^2+2x+4}$$

18. Find the partial fraction decomposition:

$$\frac{2x+4}{x^3-1}=\frac{2x+4}{(x-1)(x^2+x+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

(Multiply both sides by $(x-1)(x^2+x+1)$.)

$$2x+4=A(x^2+x+1)+(Bx+C)(x-1)$$

Let $x=1$ then $2(1)+4=A\left(1^2+1+1\right)+(B(1)+C)(1-1)$

$$\rightarrow\ 6=3A\ \rightarrow\ A=2$$

Let $x=0$: then $2(0)+4=A\left(0^2+0+1\right)+(B(0)+C)(0-1)$

$$\rightarrow\ 4=A-C\ \rightarrow\ 4=2-C\ \rightarrow\ C=-2$$

Let $x=-1$: then $2(-1)+4=A\left((-1)^2+(-1)+1\right)+(B(-1)+C)(-1-1)$

$$\rightarrow\ 2=A+2B-2C\ \rightarrow\ 2=2+2B-2(-2)$$

$$\rightarrow\ 2B=-4\ \rightarrow\ B=-2$$

$$\frac{2x+4}{x^3-1}=\frac{2}{x-1}+\frac{-2x-2}{x^2+x+1}$$

19. Find the partial fraction decomposition:

$$\frac{x^2}{(x-1)^2(x+1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}$$

(Multiply both sides by $(x-1)^2(x+1)^2$.)

$$x^2=A(x-1)(x+1)^2+B(x+1)^2+C(x-1)^2(x+1)+D(x-1)^2$$

Let $x = 1$: then $1^2 = A(1-1)(1+1)^2 + B(1+1)^2 + C(1-1)^2(1+1) + D(1-1)^2$

$$\rightarrow\ 1 = 4B\ \rightarrow\ B = \frac{1}{4}$$

Let $x = -1$: then

$$(-1)^2 = A(-1-1)(-1+1)^2 + B(-1+1)^2 + C(-1-1)^2(-1+1) + D(-1-1)^2$$

$$\rightarrow\ 1 = 4D\ \rightarrow\ D = \frac{1}{4}$$

Let $x = 0$: then

$$0^2 = A(0-1)(0+1)^2 + B(0+1)^2 + C(0-1)^2(0+1) + D(0-1)^2$$

$$\rightarrow\ 0 = -A + B + C + D\ \rightarrow\ A - C = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Let $x = 2$: then

$$2^2 = A(2-1)(2+1)^2 + B(2+1)^2 + C(2-1)^2(2+1) + D(2-1)^2$$

$$\rightarrow\ 4 = 9A + 9B + 3C + D\ \rightarrow\ 9A + 3C = 4 - \frac{9}{4} - \frac{1}{4} = \frac{3}{2}$$

$$\rightarrow\ 3A + C = \frac{1}{2}$$

Solve the system of equations:

$$\begin{aligned} A - C &= \frac{1}{2} \\ 3A + C &= \frac{1}{2} \\ \hline 4A &= 1 \end{aligned}$$

$$A = \frac{1}{4}\ \rightarrow\ \frac{3}{4} + C = \frac{1}{2}\ \rightarrow\ C = -\frac{1}{4}$$

$$\frac{x^2}{(x-1)^2(x+1)^2} = \frac{\left(\frac{1}{4}\right)}{x-1} + \frac{\left(\frac{1}{4}\right)}{(x-1)^2} + \frac{\left(-\frac{1}{4}\right)}{x+1} + \frac{\left(\frac{1}{4}\right)}{(x+1)^2}$$

20. Find the partial fraction decomposition:

$$\frac{x+1}{x^2(x-2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

(Multiply both sides by $x^2(x-2)^2$.)

$$x + 1 = Ax(x-2)^2 + B(x-2)^2 + Cx^2(x-2) + Dx^2$$

Let $x = 0$: then $0 + 1 = A(0)(0-2)^2 + B(0-2)^2 + C(0)^2(0-2) + D(0)^2$

$$\rightarrow\ 1 = 4B\ \rightarrow\ B = \frac{1}{4}$$

Let $x = 2$: then $2 + 1 = A(2)(2-2)^2 + B(2-2)^2 + C(2)^2(2-2) + D(2)^2$

$$\rightarrow\ 3 = 4D\ \rightarrow\ D = \frac{3}{4}$$

Let $x = 1$: then $1 + 1 = A(1)(1-2)^2 + B(1-2)^2 + C(1)^2(1-2) + D(1)^2$

$\rightarrow\ 2 = A + B - C + D\ \rightarrow\ A - C = 2 - \frac{1}{4} - \frac{3}{4} = 1$

Let $x = 3$: then $3 + 1 = A(3)(3-2)^2 + B(3-2)^2 + C(3)^2(3-2) + D(3)^2$

$\rightarrow\ 4 = 3A + B + 9C + 9D\ \rightarrow\ 3A + 9C = 4 - \frac{1}{4} - \frac{27}{4} = -3$

$\rightarrow\ A + 3C = -1$

Solve the system of equations:

$A - C = 1\ \rightarrow\ A = C + 1$

$A + 3C = -1$

$C + 1 + 3C = -1\ \rightarrow\ 4C = -2\ \rightarrow\ C = -\frac{1}{2}$

$A = -\frac{1}{2} + 1 = \frac{1}{2}$

$$\frac{x+1}{x^2(x-2)^2} = \frac{\frac{1}{2}}{x} + \frac{-\frac{1}{4}}{x^2} + \frac{-\frac{1}{2}}{x-2} + \frac{\frac{3}{4}}{(x-2)^2}$$

21. Find the partial fraction decomposition:

$$\frac{x-3}{(x+2)(x+1)^2} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

(Multiply both sides by $(x+2)(x+1)^2$.)

$x - 3 = A(x+1)^2 + B(x+2)(x+1) + C(x+2)$

Let $x = -2$: then $-2 - 3 = A(-2+1)^2 + B(-2+2)(-2+1) + C(-2+2)$

$\rightarrow\ -5 = A\ \rightarrow\ A = -5$

Let $x = -1$: then $-1 - 3 = A(-1+1)^2 + B(-1+2)(-1+1) + C(-1+2)$

$\rightarrow\ -4 = C\ \rightarrow\ C = -4$

Let $x = 0$: then

$0 - 3 = A(0+1)^2 + B(0+2)(0+1) + C(0+2)\ \rightarrow\ -3 = A + 2B + 2C$

$\rightarrow\ -3 = -5 + 2B + 2(-4)\ \rightarrow\ 2B = 10\ \rightarrow\ B = 5$

$$\frac{x-3}{(x+2)(x+1)^2} = \frac{-5}{x+2} + \frac{5}{x+1} + \frac{-4}{(x+1)^2}$$

22. Find the partial fraction decomposition:

$$\frac{x^2+x}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

(Multiply both sides by $(x+2)(x-1)^2$.)

$x^2 + x = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$

Let $x = -2$: then $(-2)^2 + (-2) = A(-2-1)^2 + B(-2+2)(-2-1) + C(-2+2)$

$\rightarrow \; 2 = 9A \; \rightarrow \; A = \frac{2}{9}$

Let $x = 1$: then $1^2 + 1 = A(1-1)^2 + B(1+2)(1-1) + C(1+2)$

$\rightarrow \; 2 = 3C \; \rightarrow \; C = \frac{2}{3}$

Let $x = 0$: then $0^2 + 0 = A(0-1)^2 + B(0+2)(0-1) + C(0+2)$

$\rightarrow \; 0 = A - 2B + 2C \; \rightarrow \; 2B = \frac{2}{9} + 2\left(\frac{2}{3}\right) = \frac{14}{9} \; \rightarrow \; B = \frac{7}{9}$

$$\frac{x^2 + x}{(x+2)(x-1)^2} = \frac{\frac{2}{9}}{x+2} + \frac{\frac{7}{9}}{x-1} + \frac{\frac{2}{3}}{(x-1)^2}$$

23. Find the partial fraction decomposition:

$$\frac{x+4}{x^2(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4}$$ (Multiply both sides by $x^2(x^2+4)$.)

$$x + 4 = Ax(x^2+4) + B(x^2+4) + (Cx+D)x^2$$

Let $x = 0$: then $0 + 4 = A(0)(0^2+4) + B(0^2+4) + (C0+D)(0)^2$

$\rightarrow \; 4 = 4B \; \rightarrow \; B = 1$

Let $x = 1$: then $1 + 4 = A(1)(1^2+4) + B(1^2+4) + (C(1)+D)(1)^2$

$\rightarrow \; 5 = 5A + 5B + C + D \; \rightarrow \; 5 = 5A + 5 + C + D$

$\rightarrow \; 5A + C + D = 0$

Let $x = -1$: then

$-1 + 4 = A(-1)((-1)^2+4) + B((-1)^2+4) + (C(-1)+D)(-1)^2$

$\rightarrow \; 3 = -5A + 5B - C + D \; \rightarrow \; 3 = -5A + 5 - C + D$

$\rightarrow \; -5A - C + D = -2$

Let $x = 2$: then $2 + 4 = A(2)(2^2+4) + B(2^2+4) + (C(2)+D)(2)^2$

$\rightarrow \; 6 = 16A + 8B + 8C + 4D \; \rightarrow \; 6 = 16A + 8 + 8C + 4D$

$\rightarrow \; 16A + 8C + 4D = -2$

Solve the system of equations:

$$\begin{array}{r} 5A + C + D = 0 \\ -5A - C + D = -2 \\ \hline 2D = -2 \\ D = -1 \end{array}$$

$$\begin{array}{r} 5A + C - 1 = 0 \\ C = 1 - 5A \end{array}$$

$$\begin{array}{r} 16A + 8(1-5A) + 4(-1) = -2 \\ 16A + 8 - 40A - 4 = -2 \\ -24A = -6 \\ A = \frac{1}{4} \end{array}$$

$$C = 1 - 5\left(\frac{1}{4}\right)$$

$$C = 1 - \frac{5}{4} = -\frac{1}{4}$$

$$\frac{x+4}{x^2(x^2+4)} = \frac{\frac{1}{4}}{x} + \frac{1}{x^2} + \frac{-\frac{1}{4}x - 1}{x^2+4}$$

24. Find the partial fraction decomposition:

$$\frac{10x^2+2x}{(x-1)^2(x^2+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2}$$

(Multiply both sides by $(x-1)^2(x^2+2)$.)

$$10x^2+2x = A(x-1)(x^2+2) + B(x^2+2) + (Cx+D)(x-1)^2$$

Let $x = 1$ then $10(1)^2+2(1) = A(1-1)(1^2+2) + B(1^2+2) + (C(1)+D)(1-1)^2$

$\rightarrow 12 = 3B \rightarrow B = 4$

Let $x = 0$: then $10(0)^2+2\ 0 = A(0-1)(0^2+2) + B(0^2+2) + (C(0)+D)(0-1)^2$

$\rightarrow 0 = -2A+2B+D \rightarrow 0 = -2A+8+D$

$\rightarrow 2A - D = 8$

Let $x = -1$: then

$10(-1)^2+2(-1) = A(-1-1)((-1)^2+2) + B((-1)^2+2) + (C(-1)+D)(-1-1)^2$

$\rightarrow 8 = -6A+3B-4C+4D \rightarrow 8 = -6A+12-4C+4D$

$\rightarrow -6A-4C+4D = -4$

Let $x = 2$: then $10(2)^2+2(2) = A(2-1)(2^2+2) + B(2^2+2) + (C(2)+D)(2-1)^2$

$\rightarrow 44 = 6A+6B+2C+D \rightarrow 44 = 6A+24+2C+D$

$\rightarrow 6A+2C+D = 20$

Solve the system of equations (Substitute for D):

$D = 2A-8$

$-6A-4C+4D = -4 \rightarrow -6A-4C+4(2A-8) = -4$

$\rightarrow 2A-4C = 28 \rightarrow A-2C = 14$

$6A+2C+D = 20 \rightarrow 6A+2C+2A-8 = 20 \rightarrow 8A+2C = 28$

Add the equations and solve:

$A-2C = 14$

$\underline{8A+2C = 28}$

$9A = 42 \rightarrow A = \frac{14}{3}$

$2C = A-14$

$2C = \frac{14}{3} - 14 = -\frac{28}{3} = -\frac{14}{3}$

$D = 2A-8$

$D = 2\left(\frac{14}{3}\right) - 8 = \frac{4}{3}$

$$\frac{10x^2+2x}{(x-1)^2(x^2+2)} = \frac{\frac{14}{3}}{x-1} + \frac{4}{(x-1)^2} + \frac{-\frac{14}{3}x + \frac{4}{3}}{x^2+2}$$

25. Find the partial fraction decomposition:

$$\frac{x^2+2x+3}{(x+1)(x^2+2x+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+4}$$

(Multiply both sides by $(x+1)(x^2+2x+4)$.)

$$x^2+2x+3 = A(x^2+2x+4) + (Bx+C)(x+1)$$

Let $x = -1$: then

$$(-1)^2+2(-1)+3 = A((-1)^2+2(-1)+4) + (B(-1)+C)(-1+1)$$

$\rightarrow 2 = 3A \rightarrow A = \frac{2}{3}$

Let $x = 0$: then $0^2 + 2(0) + 3 = A(0^2 + 2(0) + 4) + (B(0) + C)(0 + 1)$

$$\rightarrow \; 3 = 4A + C \; \rightarrow \; 3 = 4\left(\frac{2}{3}\right) + C \; \rightarrow \; C = \frac{1}{3}$$

Let $x = 1$: then

$$1^2 + 2(1) + 3 = A(1^2 + 2(1) + 4) + (B(1) + C)(1 + 1)$$

$$\rightarrow \; 6 = 7A + 2B + 2C \; \rightarrow \; 6 = 7\left(\frac{2}{3}\right) + 2B + 2\left(\frac{1}{3}\right)$$

$$\rightarrow \; 2B = 6 - \frac{14}{3} - \frac{2}{3} \; \rightarrow \; 2B = \frac{2}{3} \; \rightarrow \; B = \frac{1}{3}$$

$$\frac{x^2 + 2x + 3}{(x+1)(x^2 + 2x + 4)} = \frac{\left(\frac{2}{3}\right)}{x+1} + \frac{\left(\frac{1}{3}\right)x + \frac{1}{3}}{x^2 + 2x + 4}$$

26. Find the partial fraction decomposition:

$$\frac{x^2 - 11x - 18}{x(x^2 + 3x + 3)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 3x + 3} \quad \text{(Multiply both sides by } x(x^2 + 3x + 3).\text{)}$$

$$x^2 - 11x - 18 = A(x^2 + 3x + 3) + (Bx + C)(x)$$

Let $x = 0$: then $0^2 - 11(0) - 18 = A(0^2 + 3(0) + 3) + (B(0) + C)(0)$

$$\rightarrow \; -18 = 3A \; \rightarrow \; A = -6$$

Let $x = 1$: then $1^2 - 11(1) - 18 = A(1^2 + 3(1) + 3) + (B(1) + C)(1)$

$$\rightarrow \; -28 = 7A + B + C \; \rightarrow \; -28 = 7(-6) + B + C \; \rightarrow \; B + C = 14$$

Let $x = -1$: then $(-1)^2 - 11(-1) - 18 = A((-1)^2 + 3(-1) + 3) + (B(-1) + C)(-1)$

$$\rightarrow \; -6 = A + B - C \; \rightarrow \; -6 = -6 + B - C \; \rightarrow \; B - C = 0$$

Add the last two equations and solve:

$$2B = 14 \; \rightarrow \; B = 7$$

$$C = 14 - B = 14 - 7 = 7$$

$$\frac{x^2 - 11x - 18}{x(x^2 + 3x + 3)} = \frac{-6}{x} + \frac{7x + 7}{x^2 + 3x + 3}$$

27. Find the partial fraction decomposition:

$$\frac{x}{(3x-2)(2x+1)} = \frac{A}{3x-2} + \frac{B}{2x+1} \quad \text{(Multiply both sides by } (3x-2)(2x+1).\text{)}$$

$$x = A(2x + 1) + B(3x - 2)$$

Let $x = -\frac{1}{2}$: then $-\frac{1}{2} = A\left(2\left(-\frac{1}{2}\right) + 1\right) + B\left(3\left(-\frac{1}{2}\right) - 2\right)$

$$\rightarrow \; -\frac{1}{2} = -\frac{7}{2}B \; \rightarrow \; B = \frac{1}{7}$$

Let $x = \frac{2}{3}$: then $\frac{2}{3} = A\left(2\left(\frac{2}{3}\right) + 1\right) + B\left(3\left(\frac{2}{3}\right) - 2\right) \; \rightarrow \frac{2}{3} = \frac{7}{3}A \; \rightarrow \; A = \frac{2}{7}$

$$\frac{x}{(3x-2)(2x+1)} = \frac{\frac{2}{7}}{3x-2} + \frac{\frac{1}{7}}{2x+1}$$

28. Find the partial fraction decomposition:

$$\frac{1}{(2x+3)(4x-1)} = \frac{A}{2x+3} + \frac{B}{4x-1}$$

(Multiply both sides by $(2x+3)(4x-1)$.)

$$1 = A(4x-1) + B(2x+3)$$

Let $x = -\frac{3}{2}$: then $1 = A\left(4\left(-\frac{3}{2}\right)-1\right) + B\left(2\left(-\frac{3}{2}\right)+3\right)$

$\rightarrow \quad 1 = -7A \quad \rightarrow \quad A = -\frac{1}{7}$

Let $x = \frac{1}{4}$: then $1 = A\left(4\left(\frac{1}{4}\right)-1\right) + B\left(2\left(\frac{1}{4}\right)+3\right) \quad \rightarrow \quad 1 = \frac{7}{2}B \quad \rightarrow \quad B = \frac{2}{7}$

$$\frac{1}{(2x+3)(4x-1)} = \frac{-\frac{1}{7}}{2x+3} + \frac{\frac{2}{7}}{4x-1}$$

29. Find the partial fraction decomposition:

$$\frac{x}{x^2+2x-3} = \frac{x}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

(Multiply both sides by $(x+3)(x-1)$.)

$$x = A(x-1) + B(x+3)$$

Let $x = 1$ then $1 = A(1-1) + B(1+3) \quad \rightarrow \quad 1 = 4B \quad \rightarrow \quad B = \frac{1}{4}$

Let $x = -3$: then $-3 = A(-3-1) + B(-3+3) \quad \rightarrow \quad -3 = -4A \quad \rightarrow \quad A = \frac{3}{4}$

$$\frac{x}{x^2+2x-3} = \frac{\frac{3}{4}}{x+3} + \frac{\frac{1}{4}}{x-1}$$

30. Find the partial fraction decomposition:

$$\frac{x^2-x-8}{(x+1)(x^2+5x+6)} = \frac{x^2-x-8}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

(Multiply both sides by $(x+1)(x+2)(x+3)$.)

$$x^2 - x - 8 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

Let $x = -1$: then

$$(-1)^2 - (-1) - 8 = A(-1+2)(-1+3) + B(-1+1)(-1+3) + C(-1+1)(-1+2)$$

$$\rightarrow \quad -6 = 2A \quad \rightarrow \quad A = -3$$

Let $x = -2$: then

$$(-2)^2 - (-2) - 8 = A(-2+2)(-2+3) + B(-2+1)(-2+3) + C(-2+1)(-2+2)$$

$$\rightarrow \quad -2 = -B \quad \rightarrow \quad B = 2$$

Let $x = -3$: then

$$(-3)^2 - (-3) - 8 = A(-3+2)(-3+3) + B(-3+1)(-3+3) + C(-3+1)(-3+2)$$

$$\rightarrow \quad 4 = 2C \quad \rightarrow \quad C = 2$$

$$\frac{x^2-x-8}{(x+1)(x^2+5x+6)} = \frac{-3}{x+1} + \frac{2}{x+2} + \frac{2}{x+3}$$

31. Find the partial fraction decomposition:

$$\frac{x^2+2x+3}{(x^2+4)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{(x^2+4)^2}$$

(Multiply both sides by $(x^2+4)^2$.)

$$x^2+2x+3=(Ax+B)(x^2+4)+Cx+D$$
$$x^2+2x+3=Ax^3+Bx^2+4Ax+4B+Cx+D$$
$$x^2+2x+3=Ax^3+Bx^2+(4A+C)x+4B+D$$
$$A=0$$
$$B=1$$
$$4A+C=2 \rightarrow 4(0)+C=2 \rightarrow C=2$$
$$4B+D=3 \rightarrow 4(1)+D=3 \rightarrow D=-1$$
$$\frac{x^2+2x+3}{(x^2+4)^2}=\frac{1}{x^2+4}+\frac{2x-1}{(x^2+4)^2}$$

32. Find the partial fraction decomposition:

$$\frac{x^3+1}{(x^2+16)^2}=\frac{Ax+B}{x^2+16}+\frac{Cx+D}{(x^2+16)^2}$$ (Multiply both sides by $(x^2+16)^2$.)

$$x^3+1=(Ax+B)(x^2+16)+Cx+D$$
$$x^3+1=Ax^3+Bx^2+16Ax+16B+Cx+D$$
$$x^3+1=Ax^3+Bx^2+(16A+C)x+16B+D$$
$$A=1$$
$$B=0$$
$$16A+C=0 \rightarrow 16(1)+C=0 \rightarrow C=-16$$
$$16B+D=1 \rightarrow 16(0)+D=1 \rightarrow D=1$$
$$\frac{x^3+1}{(x^2+16)^2}=\frac{x}{x^2+16}+\frac{-16x+1}{(x^2+16)^2}$$

33. Find the partial fraction decomposition:

$$\frac{7x+3}{x^3-2x^2-3x}=\frac{7x+3}{x(x-3)(x+1)}=\frac{A}{x}+\frac{B}{x-3}+\frac{C}{x+1}$$

(Multiply both sides by $x(x-3)(x+1)$.)

$$7x+3=A(x-3)(x+1)+Bx(x+1)+Cx(x-3)$$

Let $x=0$: then $7(0)+3=A(0-3)(0+1)+B(0)(0+1)+C(0)(0-3)$

$$\rightarrow 3=-3A \rightarrow A=-1$$

Let $x=3$: then $7(3)+3=A(3-3)(3+1)+B(3)(3+1)+C(3)(3-3)$

$$\rightarrow 24=12B \rightarrow B=2$$

Let $x=-1$: then

$$7(-1)+3=A(-1-3)(-1+1)+B(-1)(-1+1)+C(-1)(-1-3)$$
$$\rightarrow -4=4C \rightarrow C=-1$$
$$\frac{7x+3}{x^3-2x^2-3x}=\frac{-1}{x}+\frac{2}{x-3}+\frac{-1}{x+1}$$

34. Find the partial fraction decomposition:

$$\frac{x^5+1}{x^6-x^4}=\frac{(x+1)(x^4-x^3+x^2-x+1)}{x^4(x-1)(x+1)}=\frac{x^4-x^3+x^2-x+1}{x^4(x-1)}$$

$$=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x^4}+\frac{E}{x-1}\quad\text{(Multiply both sides by } x^4(x-1).)$$

$$x^4-x^3+x^2-x+1=Ax^3(x-1)+Bx^2(x-1)+Cx(x-1)+D(x-1)+Ex^4$$

Let $x=0$: then

$$0^4-0^3+0^2-0+1=A\ 0^3(0-1)+B\ 0^2(0-1)+C\ 0(0-1)+D(0-1)+E\ 0^4$$

$$\rightarrow\ 1=-D\ \rightarrow\ D=-1$$

Let $x=1$: then

$$1^4-1^3+1^2-1+1=A\ 1^3(1-1)+B\ 1^2(1-1)+C\ 1(1-1)+D(1-1)+E\ 1^4$$

$$\rightarrow\ 1=E\ \rightarrow\ E=1$$

Let $x=-1$: then

$$(-1)^4-(-1)^3+(-1)^2-(-1)+1$$

$$=A(-1)^3(-1-1)+B(-1)^2(-1-1)+C(-1)(-1-1)+D(-1-1)+E(-1)^4$$

$$\rightarrow\ 5=2A-2B+2C-2D+E\ \rightarrow\ 5=2A-2B+2C+2+1$$

$$\rightarrow\ 2A-2B+2C=2\ \rightarrow\ A-B+C=1$$

Let $x=2$: then

$$2^4-2^3+2^2-2+1=A\ 2^3(2-1)+B\ 2^2(2-1)+C\ 2(2-1)+D(2-1)+E\ 2^4$$

$$\rightarrow\ 11=8A+4B+2C+D+16E\ \rightarrow\ 11=8A+4B+2C-1+16$$

$$\rightarrow\ 8A+4B+2C=-4\ \rightarrow\ 4A+2B+C=-2$$

Let $x=-2$: then

$$(-2)^4-(-2)^3+(-2)^2-(-2)+1$$

$$=A(-2)^3(-2-1)+B(-2)^2(-2-1)+C(-2)(-2-1)+D(-2-1)+E(-2)^4$$

$$\rightarrow\ 31=24A-12B+6C-3D+16E$$

$$\rightarrow\ 31=24A-12B+6C+3+16$$

$$\rightarrow\ 24A-12B+6C=12\ \rightarrow\ 4A-2B+C=2$$

$$(4A+2B+C)-(4A-2B+C)=-2-2\ \rightarrow\ 4B=-4\ \rightarrow\ B=-1$$

$$A+1+C=1\ \rightarrow\ A+C=0\ \rightarrow\ A=-C$$

$$4(-C)-2(-1)+C=2\ \rightarrow\ -3C=0\ \rightarrow\ C=0$$

$$A=-0=0$$

$$\frac{x^5+1}{x^6-x^4}=\frac{x^4-x^3+x^2-x+1}{x^4(x-1)}=\frac{-1}{x^2}+\frac{-1}{x^4}+\frac{1}{x-1}$$

35. Perform synthetic division to find a factor:

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 5 & -2 \\ & & 2 & -4 & 2 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

$$x^3 - 4x^2 + 5x - 2 = (x-2)(x^2 - 2x + 1) = (x-2)(x-1)^2$$

Find the partial fraction decomposition:

$$\frac{x^2}{x^3 - 4x^2 + 5x - 2} = \frac{x^2}{(x-2)(x-1)^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

(Multiply both sides by $(x-2)(x-1)^2$.)

$$x^2 = A(x-1)^2 + B(x-2)(x-1) + C(x-2)$$

Let $x = 2$: then $2^2 = A(2-1)^2 + B(2-2)(2-1) + C(2-2)$

$\rightarrow 4 = A \rightarrow A = 4$

Let $x = 1$: then $1^2 = A(1-1)^2 + B(1-2)(1-1) + C(1-2)$

$\rightarrow 1 = -C \rightarrow C = -1$

Let $x = 0$: then $0^2 = A(0-1)^2 + B(0-2)(0-1) + C(0-2)$

$\rightarrow 0 = A + 2B - 2C \rightarrow 0 = 4 + 2B - 2(-1)$

$\rightarrow 2B = -6 \rightarrow B = -3$

$$\frac{x^2}{x^3 - 4x^2 + 5x - 2} = \frac{4}{x-2} + \frac{-3}{x-1} + \frac{-1}{(x-1)^2}$$

36. Perform synthetic division to find a factor:

$$\begin{array}{r|rrrr} 1 & 1 & 1 & -5 & 3 \\ & & 1 & 2 & -3 \\ \hline & 1 & 2 & -3 & 0 \end{array}$$

$$x^3 + x^2 - 5x + 3 = (x-1)(x^2 + 2x - 3) = (x+3)(x-1)^2$$

Find the partial fraction decomposition:

$$\frac{x^2 + 1}{x^3 + x^2 - 5x + 3} = \frac{x^2 + 1}{(x+3)(x-1)^2} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

(Multiply both sides by $(x+3)(x-1)^2$.)

$$x^2 + 1 = A(x-1)^2 + B(x+3)(x-1) + C(x+3)$$

Let $x = -3$: then $(-3)^2 + 1 = A(-3-1)^2 + B(-3+3)(-3-1) + C(-3+3)$

$\rightarrow 10 = 16A \rightarrow A = \frac{5}{8}$

Let $x = 1$: then $1^2 + 1 = A(1-1)^2 + B(1+3)(1-1) + C(1+3)$

$\rightarrow 2 = 4C \rightarrow C = \frac{1}{2}$

Let $x = 0$: then $0^2 + 1 = A(0-1)^2 + B(0+3)(0-1) + C(0+3)$

$\rightarrow 1 = A - 3B + 3C \rightarrow 1 = \frac{5}{8} - 3B + 3\left(\frac{1}{2}\right)$

$\rightarrow 3B = \frac{9}{8} \rightarrow B = \frac{3}{8}$

$$\frac{x^2 + 1}{x^3 + x^2 - 5x + 3} = \frac{\frac{5}{8}}{x+3} + \frac{\frac{3}{8}}{x-1} + \frac{\frac{1}{2}}{(x-1)^2}$$

37. Find the partial fraction decomposition:

$$\frac{x^3}{(x^2+16)^3} = \frac{Ax+B}{x^2+16} + \frac{Cx+D}{(x^2+16)^2} + \frac{Ex+F}{(x^2+16)^3}$$

(Multiply both sides by $(x^2+16)^3$.)

$$x^3 = (Ax+B)(x^2+16)^2 + (Cx+D)(x^2+16) + Ex + F$$
$$x^3 = (Ax+B)(x^4+32x^2+256) + Cx^3 + Dx^2 + 16Cx + 16D + Ex + F$$
$$x^3 = Ax^5 + Bx^4 + 32Ax^3 + 32Bx^2 + 256Ax + 256B + Cx^3 + Dx^2 + 16Cx + 16D + Ex + F$$
$$x^3 = Ax^5 + Bx^4 + (32A+C)x^3 + (32B+D)x^2 + (256A+16C+E)x + (256B+16D+F)$$

$A = 0$

$B = 0$

$32A + C = 1 \rightarrow 32(0) + C = 1 \rightarrow C = 1$

$32B + D = 0 \rightarrow 32(0) + D = 0 \rightarrow D = 0$

$256A + 16C + E = 0 \rightarrow 256(0) + 16(1) + E = 0 \rightarrow E = -16$

$256B + 16D + F = 0 \rightarrow 256(0) + 16(0) + F = 0 \rightarrow F = 0$

$$\frac{x^3}{(x^2+16)^3} = \frac{x}{(x^2+16)^2} + \frac{-16x}{(x^2+16)^3}$$

38. Find the partial fraction decomposition:

$$\frac{x^2}{(x^2+4)^3} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{Ex+F}{(x^2+4)^3}$$

(Multiply both sides by $(x^2+4)^3$.)

$$x^2 = (Ax+B)(x^2+4)^2 + (Cx+D)(x^2+4) + Ex + F$$
$$x^2 = (Ax+B)(x^4+8x^2+16) + Cx^3 + Dx^2 + 4Cx + 4D + Ex + F$$
$$x^2 = Ax^5 + Bx^4 + 8Ax^3 + 8Bx^2 + 16Ax + 16B + Cx^3 + Dx^2 + 4Cx + 4D + Ex + F$$
$$x^2 = Ax^5 + Bx^4 + (8A+C)x^3 + (8B+D)x^2 + (16A+4C+E)x + (16B+4D+F)$$

$A = 0$

$B = 0$

$8A + C = 0 \rightarrow 8(0) + C = 0 \rightarrow C = 0$

$8B + D = 1 \rightarrow 8(0) + D = 1 \rightarrow D = 1$

$16A + 4C + E = 0 \rightarrow 16(0) + 4(0) + E = 0 \rightarrow E = 0$

$16B + 4D + F = 0 \rightarrow 16(0) + 4(1) + F = 0 \rightarrow F = -4$

$$\frac{x^2}{(x^2+4)^3} = \frac{1}{(x^2+4)^2} + \frac{-4}{(x^2+4)^3}$$

39. Find the partial fraction decomposition:

$$\frac{4}{2x^2-5x-3} = \frac{4}{(x-3)(2x+1)} = \frac{A}{x-3} + \frac{B}{2x+1}$$

(Multiply both sides by $(x-3)(2x+1)$.)

$$4 = A(2x+1) + B(x-3)$$

Let $x = -\frac{1}{2}$: then $4 = A\left(2\left(-\frac{1}{2}\right)+1\right) + B\left(-\frac{1}{2}-3\right)$

$\rightarrow \quad 4 = -\frac{7}{2}B \quad \rightarrow \quad B = -\frac{8}{7}$

Let $x = 3$: then $4 = A(2(3)+1) + B(3-3) \quad \rightarrow \quad 4 = 7A \quad \rightarrow \quad A = \frac{4}{7}$

$$\frac{4}{2x^2-5x-3} = \frac{4}{(x-3)(2x+1)} = \frac{\frac{4}{7}}{x-3} + \frac{-\frac{8}{7}}{2x+1}$$

40. Find the partial fraction decomposition:

$$\frac{4x}{2x^2+3x-2} = \frac{4x}{(x+2)(2x-1)} = \frac{A}{x+2} + \frac{B}{2x-1}$$

(Multiply both sides by $(x+2)(2x-1)$.)

$$4x = A(2x-1) + B(x+2)$$

Let $x = \frac{1}{2}$: then $4\left(\frac{1}{2}\right) = A\left(2\left(\frac{1}{2}\right)-1\right) + B\left(\frac{1}{2}+2\right) \quad \rightarrow \quad 2 = \frac{5}{2}B \quad \rightarrow \quad B = \frac{4}{5}$

Let $x = -2$: then $4(-2) = A(2(-2)-1) + B(-2+2)$

$\rightarrow \quad -8 = -5A \quad \rightarrow \quad A = \frac{8}{5}$

$$\frac{4x}{2x^2+3x-2} = \frac{4x}{(x+2)(2x-1)} = \frac{\frac{8}{5}}{x+2} + \frac{\frac{4}{5}}{2x-1}$$

41. Find the partial fraction decomposition:

$$\frac{2x+3}{x^4-9x^2} = \frac{2x+3}{x^2(x-3)(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3} + \frac{D}{x+3}$$

(Multiply both sides by $x^2(x-3)(x+3)$.)

$$2x+3 = Ax(x-3)(x+3) + B(x-3)(x+3) + Cx^2(x+3) + Dx^2(x-3)$$

Let $x = 0$: then

$$2(0)+3 = A(0)(0-3)(0+3) + B(0-3)(0+3) + C(0)^2(0+3) + D(0)^2(0-3)$$

$\rightarrow \quad 3 = -9B \quad \rightarrow \quad B = -\frac{1}{3}$

Let $x = 3$: then

$$2(3)+3 = A(3)(3-3)(3+3) + B(3-3)(3+3) + C(3)^2(3+3) + D(3)^2(3-3)$$

$\rightarrow \quad 9 = 54C \quad \rightarrow \quad C = \frac{1}{6}$

Let $x = -3$: then

$$2(-3)+3 = A(-3)(-3-3)(-3+3) + B(-3-3)(-3+3) + C(-3)^2(-3+3) + D(-3)^2(-3-3)$$

$\rightarrow \quad -3 = -54D \quad \rightarrow \quad D = \frac{1}{18}$

Let $x = 1$: then

$$2(1)+3 = A(1)(1-3)(1+3) + B(1-3)(1+3) + C(1)^2(1+3) + D(1)^2(1-3)$$

$$\rightarrow \quad 5 = -8A - 8B + 4C - 2D$$

$$\rightarrow \quad 5 = -8A - 8\left(-\tfrac{1}{3}\right) + 4\left(\tfrac{1}{6}\right) - 2\left(\tfrac{1}{18}\right)$$

$$\rightarrow \quad 5 = -8A + \tfrac{8}{3} + \tfrac{2}{3} - \tfrac{1}{9} \quad \rightarrow \quad -8A = \tfrac{16}{9} \quad \rightarrow \quad A = -\tfrac{2}{9}$$

$$\frac{2x+3}{x^4-9x^2} = \frac{2x+3}{x^2(x-3)(x+3)} = \frac{-\frac{2}{9}}{x} + \frac{-\frac{1}{3}}{x^2} + \frac{\frac{1}{6}}{x-3} + \frac{\frac{1}{18}}{x+3}$$

42. Find the partial fraction decomposition:

$$\frac{x^2+9}{x^4-2x^2-8} = \frac{x^2+9}{(x^2+2)(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+2}$$

(Multiply both sides by $(x^2+2)(x-2)(x+2)$.)

$$x^2+9 = A(x^2+2)(x+2) + B(x-2)(x^2+2) + (Cx+D)(x-2)(x+2)$$

Let $x = 2$: then

$$2^2+9 = A(2^2+2)(2+2) + B(2-2)(2^2+2) + (C(2)+D)(2-2)(2+2)$$

$$\rightarrow\ 13 = 24A\ \rightarrow\ A = \tfrac{13}{24}$$

Let $x = -2$: then

$$(-2)^2+9 = A((-2)^2+2)(-2+2) + B(-2-2)((-2)^2+2) + (C(-2)+D)(-2-2)(-2+2)$$

$$\rightarrow\ 13 = -24B\ \rightarrow\ B = -\tfrac{13}{24}$$

Let $x = 0$: then

$$0^2+9 = A(0^2+2)(0+2) + B(0-2)(0^2+2) + (C(0)+D)(0-2)(0+2)$$

$$\rightarrow\ 9 = 4A - 4B - 4D\ \rightarrow\ 9 = \tfrac{13}{6} + \tfrac{13}{6} - 4D$$

$$\rightarrow\ 4D = -\tfrac{14}{3}\ \rightarrow\ D = -\tfrac{7}{6}$$

Let $x = 1$: then

$$1^2+9 = A(1^2+2)(1+2) + B(1-2)(1^2+2) + (C(1)+D)(1-2)(1+2)$$

$$\rightarrow\ 10 = 9A - 3B - 3C - 3D$$

$$\rightarrow\ 10 = \tfrac{39}{8} + \tfrac{13}{8} - 3C + \tfrac{7}{2}$$

$$\rightarrow\ 3C = 0\ \rightarrow\ C = 0$$

$$\frac{x^2+9}{x^4-2x^2-8} = \frac{x^2+9}{(x^2+2)(x-2)(x+2)} = \frac{\frac{13}{24}}{x-2} + \frac{-\frac{13}{24}}{x+2} + \frac{-\frac{7}{6}}{x^2+2}$$