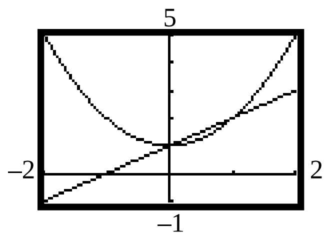
Chapter 12

Systems of Equations and Inequalities

12.7 Systems of Nonlinear Equations

1. $\begin{cases} y = x^2 + 1 \\ y = x + 1 \end{cases}$

Graph: $y_1 = x^2 + 1$; $y_2 = x + 1$



(0, 1) and (1, 2) are the intersection points.

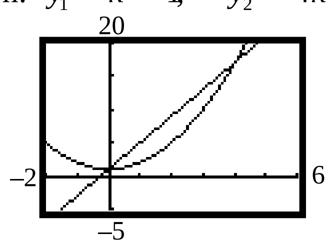
Solve by substitution:

$$x^2 + 1 = x + 1$$
$$x^2 - x = 0$$
$$x(x - 1) = 0$$
$$x = 0 \text{ or } x = 1$$
$$y = 1 \qquad y = 2$$

Solutions: (0, 1) and (1, 2)

2. $\begin{cases} y = x^2 + 1 \\ y = 4x + 1 \end{cases}$

Graph: $y_1 = x^2 + 1$; $y_2 = 4x + 1$



(0, 1) and (4, 17) are the intersection points.

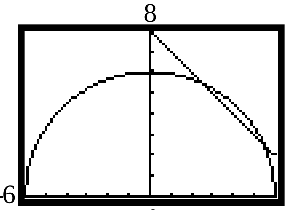
Solve by substitution:

$$x^2 + 1 = 4x + 1$$
$$x^2 - 4x = 0$$
$$x(x - 4) = 0$$
$$x = 0 \text{ or } x = 4$$
$$y = 1 \qquad y = 17$$

Solutions: (0, 1) and (4, 17)

3. $\begin{cases} y = \sqrt{36 - x^2} \\ y = 8 - x \end{cases}$

Graph: $y_1 = \sqrt{36 - x^2}$; $y_2 = 8 - x$



(2.59, 5.41) and (5.41, 2.59) are the intersection points.

Solve by substitution:

$$\sqrt{36 - x^2} = 8 - x$$
$$36 - x^2 = 64 - 16x + x^2$$
$$2x^2 - 16x + 28 = 0$$
$$x^2 - 8x + 14 = 0$$
$$x = \frac{8 \pm \sqrt{64 - 56}}{2}$$
$$x = \frac{8 \pm 2\sqrt{2}}{2}$$
$$x = 4 \pm \sqrt{2}$$

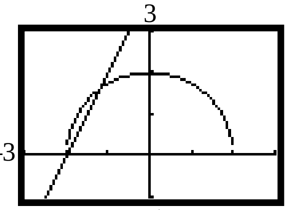
If $x = 4 + \sqrt{2}$, $y = 8 - \left(4 + \sqrt{2}\right) = 4 - \sqrt{2}$

If $x = 4 - \sqrt{2}$, $y = 8 - \left(4 - \sqrt{2}\right) = 4 + \sqrt{2}$

Solutions:
$\left(4 + \sqrt{2}, 4 - \sqrt{2}\right)$ and $\left(4 - \sqrt{2}, 4 + \sqrt{2}\right)$

4. $\begin{cases} y = \sqrt{4 - x^2} \\ y = 2x + 4 \end{cases}$

Graph: $y_1 = \sqrt{4 - x^2}$; $y_2 = 2x + 4$



(–2, 0) and (–1.2, 1.6) are the intersection points.
((–2, 0) is difficult to find using INTERSECT because it is at the end of the domain of the function.)

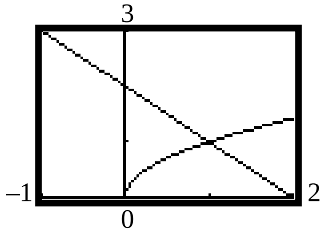
Solve by substitution:

$$\sqrt{4 - x^2} = 2x + 4$$
$$4 - x^2 = 4x^2 + 16x + 16$$
$$5x^2 + 16x + 12 = 0$$
$$(x + 2)(5x + 6) = 0$$
$$x = -2 \text{ or } x = -\frac{6}{5}$$
$$y = 0 \text{ or } y = \frac{8}{5}$$

Solutions: $(-2, 0)$ and $\left(-\frac{6}{5}, \frac{8}{5}\right)$

5. $\begin{cases} y = \sqrt{x} \\ y = 2 - x \end{cases}$

Graph: $y_1 = \sqrt{x}$; $y_2 = 2 - x$



(1, 1) is the intersection point.

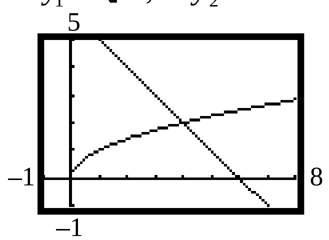
Solve by substitution:

$$\begin{aligned} \sqrt{x} &= 2 - x \\ x &= 4 - 4x + x^2 \\ x^2 - 5x + 4 &= 0 \\ (x-4)(x-1) &= 0 \\ x = 4 \quad &\text{or } x = 1 \\ y = -2 \quad &\text{or } y = 1 \end{aligned}$$

Eliminate (4, –2); it does not check.
Solution: (1, 1)

6. $\begin{cases} y = \sqrt{x} \\ y = 6 - x \end{cases}$

Graph: $y_1 = \sqrt{x}$; $y_2 = 6 - x$



(4, 2) is the intersection point.

Solve by substitution:

$$\begin{aligned} \sqrt{x} &= 6 - x \\ x &= 36 - 12x + x^2 \\ x^2 - 13x + 36 &= 0 \\ (x-4)(x-9) &= 0 \\ x = 4 \quad &\text{or } x = 9 \\ y = 2 \quad &\text{or } y = -3 \end{aligned}$$

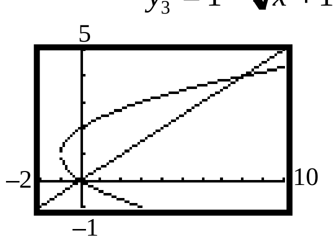
Eliminate (9, –3); it does not check.
Solution: (4, 2)

7. $\begin{cases} x = 2y \\ x = y^2 - 2y \end{cases}$

Solve each equation for y in order to enter it into the graphing utility:

$$\begin{aligned} y^2 - 2y + 1 &= x + 1 \\ (y-1)^2 &= x + 1 \\ y - 1 &= \pm\sqrt{x+1} \\ y &= 1 \pm \sqrt{x+1} \end{aligned}$$

Graph: $y_1 = \dfrac{x}{2}$; $y_2 = 1 + \sqrt{x+1}$;
$y_3 = 1 - \sqrt{x+1}$



(0, 0) and (8, 4) are the intersection points.

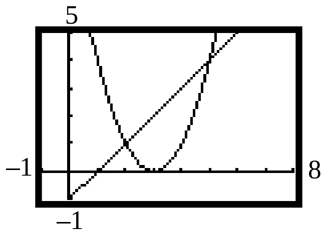
Solve by substitution:

$$\begin{aligned} 2y &= y^2 - 2y \\ y^2 - 4y &= 0 \\ y(y-4) &= 0 \\ y = 0 \quad &\text{or } y = 4 \\ x = 0 \quad &\text{or } x = 8 \end{aligned}$$

Solutions: (0, 0) and (8, 4)

8. $\begin{cases} y = x - 1 \\ y = x^2 - 6x + 9 \end{cases}$

Graph: $y_1 = x - 1$; $y_2 = x^2 - 6x + 9$



(2, 1) and (5, 4) are the intersection points.

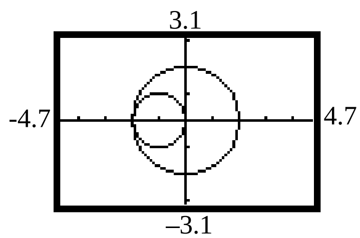
Solve by substitution:

$$x^2 - 6x + 9 = x - 1$$
$$x^2 - 7x + 10 = 0$$
$$(x - 2)(x - 5) = 0$$
$$x = 2 \text{ or } x = 5$$
$$y = 1 \text{ or } y = 4$$

Solutions: (2, 1) and (5, 4)

9. $\begin{cases} x^2 + y^2 = 4 \\ x^2 + 2x + y^2 = 0 \end{cases}$

Graph: $y_1 = \sqrt{4 - x^2}$; $y_2 = -\sqrt{4 - x^2}$; $y_3 = \sqrt{-x^2 - 2x}$; $y_4 = -\sqrt{-x^2 - 2x}$



(–2, 0) is the intersection point. (Note: This intersection point is impossible to find on your graphing utility unless you have just the right window and make an excellent guess.)

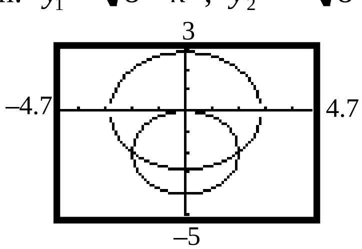
Substitute 4 for $x^2 + y^2$ in the second equation:

$$2x + 4 = 0$$
$$2x = -4$$
$$x = -2$$
$$y = \sqrt{4 - (-2)^2} = 0$$

Solution: (–2, 0)

10. $\begin{cases} x^2 + y^2 = 8 \\ x^2 + y^2 + 4y = 0 \end{cases}$

Graph: $y_1 = \sqrt{8 - x^2}$; $y_2 = -\sqrt{8 - x^2}$; $y_3 = -2 + \sqrt{4 - x^2}$; $y_4 = -2 - \sqrt{4 - x^2}$



(–2, –2) and (2, –2) are the intersection points.

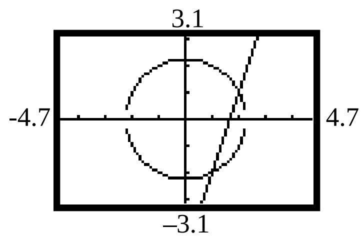
Substitute 8 for $x^2 + y^2$ in the second equation:

$$8 + 4y = 0$$
$$4y = -8$$
$$y = -2$$
$$x = \pm\sqrt{8 - (-2)^2} = \pm 2$$

Solution: (–2, –2) and (2, –2)

11. $\begin{cases} y = 3x - 5 \\ x^2 + y^2 = 5 \end{cases}$

Graph: $y_1 = 3x - 5$; $y_2 = \sqrt{5 - x^2}$; $y_3 = -\sqrt{5 - x^2}$



(1, –2) and (2, 1) are the intersection points.

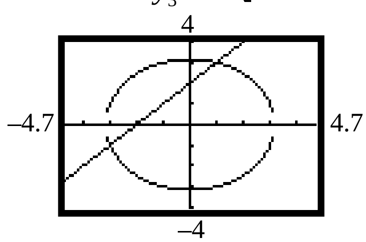
Solve by substitution:

$$x^2 + (3x - 5)^2 = 5$$
$$x^2 + 9x^2 - 30x + 25 = 5$$
$$10x^2 - 30x + 20 = 0$$
$$x^2 - 3x + 2 = 0$$
$$(x - 1)(x - 2) = 0$$
$$x = 1 \quad \text{or} \quad x = 2$$
$$y = 3(1) - 5 \qquad y = 3(2) - 5$$
$$y = -2 \qquad y = 1$$

Solutions: (1, –2) and (2, 1)

12. $\begin{cases} x^2 + y^2 = 10 \\ y = x + 2 \end{cases}$

Graph: $y_1 = x + 2$; $y_2 = \sqrt{10 - x^2}$; $y_3 = -\sqrt{10 - x^2}$



(1, 3) and (–3, –1) are the intersection points.

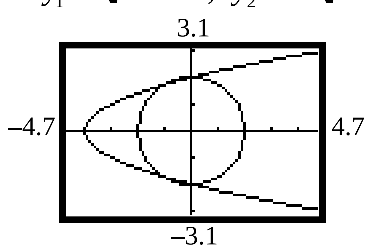
Solve by substitution:

$$x^2 + (x + 2)^2 = 10$$
$$x^2 + x^2 + 4x + 4 = 10$$
$$2x^2 + 4x - 6 = 0$$
$$x^2 + 2x - 3 = 0$$
$$(x + 3)(x - 1) = 0$$
$$x = -3 \quad \text{or} \quad x = 1$$
$$y = -1 \qquad y = 3$$

Solutions: (–3, –1) and (1, 3)

13. $\begin{cases} x^2 + y^2 = 4 \\ y^2 - x = 4 \end{cases}$

Graph: $y_1 = \sqrt{4 - x^2}$; $y_2 = -\sqrt{4 - x^2}$; $y_3 = \sqrt{x + 4}$; $y_4 = -\sqrt{x + 4}$



(–1, 1.73), (–1, –1.73), (0, 2), and (0, –2) are the intersection points.

Substitute $x + 4$ for y^2 in the first equation:

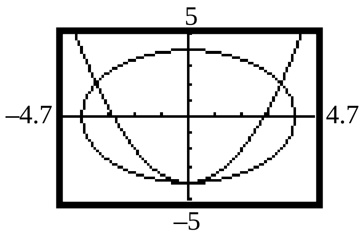
$$x^2 + x + 4 = 4$$
$$x^2 + x = 0$$
$$x(x + 1) = 0$$
$$x = 0 \quad \text{or} \quad x = -1$$
$$y^2 = 4 \qquad y^2 = 3$$
$$y = \pm 2 \qquad y^2 = \pm\sqrt{3}$$

Solutions:

(0, –2), (0, 2), $\left(-1, \sqrt{3}\right)$, $\left(-1, -\sqrt{3}\right)$

14. $\begin{cases} x^2 + y^2 = 16 \\ x^2 - 2y = 8 \end{cases}$

Graph: $y_1 = \sqrt{16 - x^2}$; $y_2 = -\sqrt{16 - x^2}$; $y_3 = (x^2 - 8)/2$



(–3.46, 2), (0, –4), and (3.46, 2) are the intersection points.

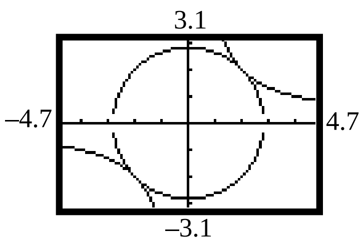
Substitute $2y + 8$ for x^2 in the first equation:

$$2y + 8 + y^2 = 16$$
$$y^2 + 2y - 8 = 0$$
$$(y + 4)(y - 2) = 0$$
$$y = -4 \quad \text{or} \quad y = 2$$
$$x^2 = 0 \quad \text{or} \quad x^2 = 12$$
$$x = 0 \qquad x = \pm 2\sqrt{3}$$

Solutions: $(0, -4)$, $\left(2\sqrt{3}, 2\right)$, $\left(-2\sqrt{3}, 2\right)$

15. $\begin{cases} xy = 4 \\ x^2 + y^2 = 8 \end{cases}$

Graph: $y_1 = \dfrac{4}{x}$; $y_2 = \sqrt{8 - x^2}$;
$y_3 = -\sqrt{8 - x^2}$



(–2, –2) and (2, 2) are the intersection points.

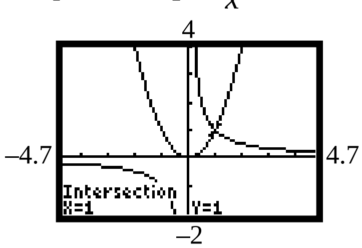
Solve by substitution:

$$x^2 + \left(\frac{4}{x}\right)^2 = 8$$
$$x^2 + \frac{16}{x^2} = 8$$
$$x^4 + 16 = 8x^2$$
$$x^4 - 8x^2 + 16 = 0$$
$$\left(x^2 - 4\right)^2 = 0$$
$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = 2 \quad \text{or} \quad x = -2$$
$$y = 2 \qquad y = -2$$

Solutions: (–2, –2) and (2, 2)

16. $\begin{cases} x^2 = y \\ xy = 1 \end{cases}$

Graph: $y_1 = x^2$; $y_2 = \dfrac{1}{x}$



(1, 1) is the intersection point.

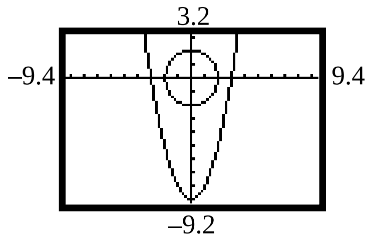
Solve by substitution:

$$x^2 = \frac{1}{x}$$
$$x^3 = 1$$
$$x = 1$$
$$y = 1$$

Solutions: (1, 1)

17. $\begin{cases} x^2 + y^2 = 4 \\ y = x^2 - 9 \end{cases}$

Graph: $y_1 = x^2 - 9$; $y_2 = \sqrt{4 - x^2}$; $y_3 = -\sqrt{4 - x^2}$



No solution; Inconsistent.

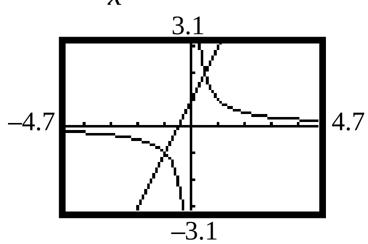
Solve by substitution:

$$x^2 + (x^2 - 9)^2 = 4$$
$$x^2 + x^4 - 18x^2 + 81 = 4$$
$$x^4 - 17x^2 + 77 = 0$$
$$x^2 = \frac{17 \pm \sqrt{289 - 4(77)}}{2}$$
$$x^2 = \frac{17 \pm \sqrt{-19}}{2}$$

There are no real solutions to this expression; Inconsistent.

18. $\begin{cases} xy = 1 \\ y = 2x + 1 \end{cases}$

Graph: $y_1 = \frac{1}{x}$; $y_2 = 2x + 1$



(–1, –1) and (0.5, 2) are the intersection points.

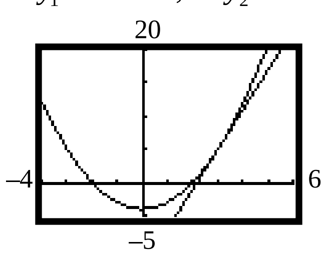
Solve by substitution:

$$x(2x+1) = 1$$
$$2x^2 + x - 1 = 0$$
$$(x+1)(2x-1) = 0$$
$$x = -1 \text{ or } x = \frac{1}{2}$$
$$y = -1 \qquad y = 2$$

Solutions: (–1, –1) and $\left(\frac{1}{2}, 2\right)$

19. $\begin{cases} y = x^2 - 4 \\ y = 6x - 13 \end{cases}$

Graph: $y_1 = x^2 - 4$; $y_2 = 6x - 13$



(3,5) is the intersection point.

Solve by substitution:

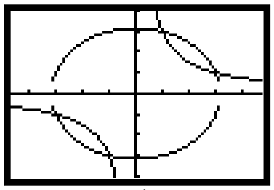
$$x^2 - 4 = 6x - 13$$
$$x^2 - 6x + 9 = 0$$
$$(x - 3)^2 = 0$$
$$x - 3 = 0$$
$$x = 3$$
$$y = 6(3) - 13 = 5$$

Solutions: (3,5)

20. $\begin{cases} x^2 + y^2 = 10 \\ xy = 3 \end{cases}$

Graph: $y_1 = \dfrac{3}{x}$; $y_2 = \sqrt{10 - x^2}$;

$y_3 = -\sqrt{10 - x^2}$



(1, 3), (3, 1), (–3, –1), and (–1, –3) are the intersection points.

Solve by substitution:

$$x^2 + \left(\frac{3}{x}\right)^2 = 10$$
$$x^2 + \frac{9}{x^2} = 10$$
$$x^4 + 9 = 10x^2$$
$$x^4 - 10x^2 + 9 = 0$$
$$(x^2 - 9)(x^2 - 1) = 0$$
$$(x-3)(x+3)(x-1)(x+1) = 0$$
$$x = 3 \text{ or } x = -3 \text{ or } x = 1 \text{ or } x = -1$$
$$y = 1 \quad y = -1 \quad y = 3 \quad y = -3$$

Solutions: (3, 1), (–3, –1), (1, 3), (–1, –3)

21. Solve the second equation for y, substitute into the first equation and solve:

$$\begin{cases} 2x^2 + y^2 = 18 \\ xy = 4 \rightarrow y = \dfrac{4}{x} \end{cases}$$

$$2x^2 + \left(\frac{4}{x}\right)^2 = 18$$
$$2x^2 + \frac{16}{x^2} = 18$$
$$2x^4 + 16 = 18x^2$$
$$2x^4 - 18x^2 + 16 = 0$$
$$x^4 - 9x^2 + 8 = 0$$
$$\left(x^2 - 8\right)\left(x^2 - 1\right) = 0$$
$$x^2 = 8 \quad \text{or} \quad x^2 = 1$$
$$x = \pm\sqrt{8} = \pm 2\sqrt{2} \quad \text{or} \quad x = \pm 1$$

If $x = 2\sqrt{2}$: $y = \dfrac{4}{2\sqrt{2}} = \sqrt{2}$

If $x = -2\sqrt{2}$: $y = \dfrac{4}{-2\sqrt{2}} = -\sqrt{2}$

If $x = 1$: $y = \dfrac{4}{1} = 4$

If $x = -1$: $y = \dfrac{4}{-1} = -4$

Solutions: $\left(2\sqrt{2}, \sqrt{2}\right), \left(-2\sqrt{2}, -\sqrt{2}\right), (1, 4), (-1, -4)$

22. Solve the second equation for y, substitute into the first equation and solve:

$$\begin{cases} x^2 - y^2 = 21 \\ x + y = 7 \quad \rightarrow \quad y = 7 - x \end{cases}$$

$$x^2 - (7-x)^2 = 21 \rightarrow x^2 - (49 - 14x + x^2) = 21$$

$$14x = 70 \rightarrow x = 5 \rightarrow y = 7 - 5 = 2$$

Solution: (5, 2)

23. Substitute the first equation into the second equation and solve:

$$\begin{cases} y = 2x + 1 \\ 2x^2 + y^2 = 1 \end{cases}$$

$$2x^2 + (2x+1)^2 = 1 \rightarrow 2x^2 + 4x^2 + 4x + 1 = 1$$

$$6x^2 + 4x = 0 \rightarrow 2x(3x+2) = 0$$

$$2x = 0 \quad \text{or} \quad 3x + 2 = 0$$

$$x = 0 \quad \text{or} \quad x = -\frac{2}{3}$$

If $x = 0$: $\quad y = 2(0) + 1 = 1$

If $x = -\frac{2}{3}$: $\quad y = 2\left(-\frac{2}{3}\right) + 1 = -\frac{4}{3} + 1 = -\frac{1}{3}$

Solutions: $(0,1), \left(-\frac{2}{3}, -\frac{1}{3}\right)$

24. Solve the second equation for x and substitute into the first equation and solve:

$$\begin{cases} x^2 - 4y^2 = 16 \\ 2y - x = 2 \quad \rightarrow \quad x = 2y - 2 \end{cases}$$

$$(2y-2)^2 - 4y^2 = 16 \rightarrow 4y^2 - 8y + 4 - 4y^2 = 16$$

$$-8y = 12 \rightarrow y = -\frac{3}{2} \rightarrow x = 2\left(-\frac{3}{2}\right) - 2 = -5$$

Solutions: $\left(-5, -\frac{3}{2}\right)$

25. Solve the first equation for y, substitute into the second equation and solve:

$$\begin{cases} x + y + 1 = 0 \quad \rightarrow \quad y = -x - 1 \\ x^2 + y^2 + 6y - x = -5 \end{cases}$$

$$x^2 + (-x-1)^2 + 6(-x-1) - x = -5$$

$$x^2 + x^2 + 2x + 1 - 6x - 6 - x = -5$$

$$2x^2 - 5x = 0 \rightarrow x(2x-5) = 0 \rightarrow x = 0 \quad \text{or} \quad x = \frac{5}{2}$$

If $x = 0$: $\quad y = -(0) - 1 = -1$

If $x = \frac{5}{2}$: $\quad y = -\frac{5}{2} - 1 = -\frac{7}{2}$

Solutions: $(0,-1), \left(\frac{5}{2}, -\frac{7}{2}\right)$

26. Solve the second equation for y, substitute into the first equation and solve:

$$\begin{cases} 2x^2 - xy + y^2 = 8 \\ xy = 4 \quad \rightarrow \quad y = \dfrac{4}{x} \end{cases}$$

$$2x^2 - x\left(\frac{4}{x}\right) + \left(\frac{4}{x}\right)^2 = 8 \rightarrow 2x^2 - 4 + \frac{16}{x^2} = 8$$

$$2x^4 + 16 = 12x^2 \rightarrow x^4 - 6x^2 + 8 = 0$$

$$\left(x^2 - 4\right)\left(x^2 - 2\right) = 0 \rightarrow (x-2)(x+2)\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right) = 0$$

$$x = 2 \text{ or } x = -2 \text{ or } x = \sqrt{2} \text{ or } x = -\sqrt{2}$$

$$y = 2 \qquad y = -2 \qquad y = 2\sqrt{2} \qquad y = -2\sqrt{2}$$

Solutions: $(2, 2), (-2, -2), \left(\sqrt{2}, 2\sqrt{2}\right), \left(-\sqrt{2}, -2\sqrt{2}\right)$

27. Solve the second equation for y, substitute into the first equation and solve:

$$\begin{cases} 4x^2 - 3xy + 9y^2 = 15 \\ 2x + 3y = 5 \quad \rightarrow \quad y = -\dfrac{2}{3}x + \dfrac{5}{3} \end{cases}$$

$$4x^2 - 3x\left(-\frac{2}{3}x + \frac{5}{3}\right) + 9\left(-\frac{2}{3}x + \frac{5}{3}\right)^2 = 15 \rightarrow 4x^2 + 2x^2 - 5x + 4x^2 - 20x + 25 = 15$$

$$10x^2 - 25x + 10 = 0 \rightarrow 2x^2 - 5x + 2 = 0$$

$$(2x-1)(x-2) = 0 \rightarrow x = \frac{1}{2} \text{ or } x = 2$$

If $x = \frac{1}{2}$: $\quad y = -\frac{2}{3}\left(\frac{1}{2}\right) + \frac{5}{3} = \frac{4}{3}$

If $x = 2$: $\quad y = -\frac{2}{3}(2) + \frac{5}{3} = \frac{1}{3}$

Solutions: $\left(\frac{1}{2}, \frac{4}{3}\right), \left(2, \frac{1}{3}\right)$

28. Solve the second equation for x, substitute into the first equation and solve:

$$\begin{cases} 2y^2 - 3xy + 6y + 2x + 4 = 0 \\ 2x - 3y + 4 = 0 \quad \rightarrow \quad 2x = 3y - 4 \quad \rightarrow \quad x = \dfrac{3y-4}{2} \end{cases}$$

$$2y^2 - 3\left(\frac{3y-4}{2}\right)y + 6y + 2\left(\frac{3y-4}{2}\right) = -4$$

$$2y^2 - \frac{9}{2}y^2 + 6y + 6y + 3y - 4 = -4 \rightarrow -\frac{5}{2}y^2 + 15y = 0$$

$$-5y^2 + 30y = 0 \rightarrow -5y(y-6) = 0 \rightarrow y = 0 \text{ or } y = 6 \rightarrow x = -2 \quad x = 7$$

Solutions: $(-2, 0), (7, 6)$

29. Multiply each side of the second equation by 4 and add the equations to eliminate y:

$$\begin{cases} x^2 - 4y^2 = -7 \\ 3x^2 + y^2 = 31 \end{cases} \quad \begin{array}{l} \longrightarrow \\ \xrightarrow{4} \end{array} \quad \begin{array}{r} x^2 - 4y^2 = -7 \\ \underline{12x^2 + 4y^2 = 124} \\ 13x^2 \qquad = 117 \\ x^2 = 9 \\ x = \pm 3 \end{array}$$

If $x = 3$: $3(3)^2 + y^2 = 31 \rightarrow y^2 = 4 \rightarrow y = \pm 2$

If $x = -3$: $3(-3)^2 + y^2 = 31 \rightarrow y^2 = 4 \rightarrow y = \pm 2$

Solutions: (3, 2), (3, –2), (–3, 2), (–3, –2)

30. Multiply each side of the second equation by –2 and add the equations to eliminate y:

$$\begin{cases} 3x^2 - 2y^2 = -5 \\ 2x^2 - y^2 = -2 \end{cases} \quad \begin{array}{l} \longrightarrow \\ \xrightarrow{-2} \end{array} \quad \begin{array}{r} 3x^2 - 2y^2 = -5 \\ \underline{-4x^2 + 2y^2 = 4} \\ -x^2 \qquad = -1 \\ x^2 = 1 \rightarrow x = \pm 1 \end{array}$$

If $x = 1$: $2(1)^2 - y^2 = -2 \rightarrow y^2 = 4 \rightarrow y = \pm 2$

If $x = -1$: $2(-1)^2 - y^2 = -2 \rightarrow y^2 = 4 \rightarrow y = \pm 2$

Solutions: (1, 2), (1, –2), (–1, 2), (–1, –2)

31. Multiply each side of the first equation by 5 and each side of the second equation by 3 to eliminate y:

$$\begin{cases} 7x^2 - 3y^2 = -5 \\ 3x^2 + 5y^2 = 12 \end{cases} \quad \begin{array}{l} \xrightarrow{5} \\ \xrightarrow{3} \end{array} \quad \begin{array}{r} 35x^2 - 15y^2 = -25 \\ \underline{9x^2 + 15y^2 = 36} \\ 44x^2 \qquad = 11 \\ x^2 = \dfrac{1}{4} \rightarrow x = \pm\dfrac{1}{2} \end{array}$$

If $x = \dfrac{1}{2}$: $3\left(\dfrac{1}{2}\right)^2 + 5y^2 = 12 \rightarrow 5y^2 = \dfrac{45}{4} \rightarrow y^2 = \dfrac{9}{4} \rightarrow y = \pm\dfrac{3}{2}$

If $x = -\dfrac{1}{2}$: $3\left(-\dfrac{1}{2}\right)^2 + 5y^2 = 12 \rightarrow 5y^2 = \dfrac{45}{4} \rightarrow y^2 = \dfrac{9}{4} \rightarrow y = \pm\dfrac{3}{2}$

Solutions: $\left(\dfrac{1}{2}, \dfrac{3}{2}\right), \left(\dfrac{1}{2}, -\dfrac{3}{2}\right), \left(-\dfrac{1}{2}, \dfrac{3}{2}\right), \left(-\dfrac{1}{2}, -\dfrac{3}{2}\right)$

32. Multiply each side of the first equation by –2 and add the equations to eliminate x:

$$\begin{cases} x^2 - 3y^2 = -1 \\ 2x^2 - 7y^2 = -5 \end{cases} \quad \begin{array}{l} \xrightarrow{-2} \\ \longrightarrow \end{array} \quad \begin{array}{r} -2x^2 + 6y^2 = 2 \\ \underline{2x^2 - 7y^2 = -5} \\ -y^2 = -3 \\ y^2 = 3 \\ y = \pm\sqrt{3} \end{array}$$

If $y=\sqrt{3}$: $\quad x^2-3\left(\sqrt{3}\right)^2=-1 \rightarrow x^2=8 \rightarrow x=\pm 2\sqrt{2}$

If $y=-\sqrt{3}$: $\quad x^2-3\left(-\sqrt{3}\right)^2=-1 \rightarrow x^2=8 \rightarrow x=\pm 2\sqrt{2}$

Solutions: $\left(2\sqrt{2},\sqrt{3}\right),\left(2\sqrt{2},-\sqrt{3}\right),\left(-2\sqrt{2},\sqrt{3}\right),\left(-2\sqrt{2},-\sqrt{3}\right)$

33. Multiply each side of the second equation by 2 and add to eliminate xy:

$$\begin{cases} x^2+2xy=10 \\ 3x^2-xy=2 \end{cases} \quad \begin{array}{l} \longrightarrow \\ \xrightarrow{2} \end{array} \quad \begin{array}{r} x^2+2xy=10 \\ \underline{6x^2-2xy=4} \\ 7x^2=14 \\ x^2=2 \\ x=\pm\sqrt{2} \end{array}$$

If $x=\sqrt{2}$: $\quad 3\left(\sqrt{2}\right)^2-\sqrt{2}\,y=2 \rightarrow -\sqrt{2}\,y=-4 \rightarrow y=\dfrac{4}{\sqrt{2}} \rightarrow y=2\sqrt{2}$

If $x=-\sqrt{2}$: $\quad 3\left(-\sqrt{2}\right)^2-\left(-\sqrt{2}\right)y=2 \rightarrow \sqrt{2}\,y=-4 \rightarrow y=\dfrac{-4}{\sqrt{2}} \rightarrow y=-2\sqrt{2}$

Solutions: $\left(\sqrt{2},2\sqrt{2}\right),\left(-\sqrt{2},-2\sqrt{2}\right)$

34. Multiply each side of the second equation by –5 and add to eliminate xy:

$$\begin{cases} 5xy+13y^2=-36 \\ xy+7y^2=6 \end{cases} \quad \begin{array}{l} \longrightarrow \\ \xrightarrow{-5} \end{array} \quad \begin{array}{r} 5xy+13y^2=-36 \\ \underline{-5xy-35y^2=-30} \\ -22y^2=-66 \\ y^2=3 \\ y=\pm\sqrt{3} \end{array}$$

If $y=\sqrt{3}$: $\quad x\left(\sqrt{3}\right)+7\left(\sqrt{3}\right)^2=6 \rightarrow \sqrt{3}\,x=-15 \rightarrow x=\dfrac{-15}{\sqrt{3}} \rightarrow x=-5\sqrt{3}$

If $y=-\sqrt{3}$: $\quad x\left(-\sqrt{3}\right)+7\left(-\sqrt{3}\right)^2=6 \rightarrow -\sqrt{3}\,x=-15 \rightarrow x=\dfrac{15}{\sqrt{3}} \rightarrow x=5\sqrt{3}$

Solutions: $\left(-5\sqrt{3},\sqrt{3}\right),\left(5\sqrt{3},-\sqrt{3}\right)$

35. Multiply each side of the first equation by 2 and add the equations to eliminate y:

$$\begin{cases} 2x^2+y^2=2 \\ x^2-2y^2=-8 \end{cases} \quad \begin{array}{l} \xrightarrow{2} \\ \longrightarrow \end{array} \quad \begin{array}{r} 4x^2+2y^2=4 \\ \underline{x^2-2y^2=-8} \\ 5x^2=-4 \\ x^2=-\dfrac{4}{5} \end{array}$$

No solution. The system is inconsistent.

36. Multiply each side of the first equation by 2 and add the equations to eliminate x:

$$\begin{cases} -x^2 + y^2 = -4 \xrightarrow{2} & -2x^2 + 2y^2 = -8 \\ 2x^2 + 3y^2 = 6 \longrightarrow & \underline{2x^2 + 3y^2 = 6} \end{cases}$$

$$5y^2 = -2$$

$$y^2 = -\frac{2}{5}$$

No solution. The system is inconsistent.

37. Multiply each side of the second equation by 2 and add the equations to eliminate y:

$$\begin{cases} x^2 + 2y^2 = 16 \longrightarrow & x^2 + 2y^2 = 16 \\ 4x^2 - y^2 = 24 \xrightarrow{2} & \underline{8x^2 - 2y^2 = 48} \end{cases}$$

$$9x^2 = 64$$

$$x^2 = \frac{64}{9} \rightarrow x = \pm\frac{8}{3}$$

If $x = \frac{8}{3}$: $\left(\frac{8}{3}\right)^2 + 2y^2 = 16 \rightarrow 2y^2 = \frac{80}{9} \rightarrow y^2 = \frac{40}{9} \rightarrow y = \pm\frac{2\sqrt{10}}{3}$

If $x = -\frac{8}{3}$: $\left(-\frac{8}{3}\right)^2 + 2y^2 = 16 \rightarrow 2y^2 = \frac{80}{9} \rightarrow y^2 = \frac{40}{9} \rightarrow y = \pm\frac{2\sqrt{10}}{3}$

Solutions: $\left(\frac{8}{3}, \frac{2\sqrt{10}}{3}\right), \left(\frac{8}{3}, \frac{-2\sqrt{10}}{3}\right), \left(-\frac{8}{3}, \frac{2\sqrt{10}}{3}\right), \left(-\frac{8}{3}, \frac{-2\sqrt{10}}{3}\right)$

38. Multiply each side of the first equation by 2 and add the equations to eliminate y:

$$\begin{cases} 4x^2 + 3y^2 = 4 \xrightarrow{2} & 8x^2 + 6y^2 = 8 \\ 2x^2 - 6y^2 = -3 \longrightarrow & \underline{2x^2 - 6y^2 = -3} \end{cases}$$

$$10x^2 = 5$$

$$x^2 = \frac{1}{2} \rightarrow x = \pm\frac{\sqrt{2}}{2}$$

If $x = \frac{\sqrt{2}}{2}$: $4\left(\frac{\sqrt{2}}{2}\right)^2 + 3y^2 = 4 \rightarrow 3y^2 = 2 \rightarrow y^2 = \frac{2}{3} \rightarrow y = \pm\frac{\sqrt{6}}{3}$

If $x = -\frac{\sqrt{2}}{2}$: $4\left(-\frac{\sqrt{2}}{2}\right)^2 + 3y^2 = 4 \rightarrow 3y^2 = 2 \rightarrow y^2 = \frac{2}{3} \rightarrow y = \pm\frac{\sqrt{6}}{3}$

Solutions: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{3}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{3}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{3}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{3}\right)$

39. Multiply each side of the second equation by 2 and add the equations to eliminate y:

$$\begin{cases} \dfrac{5}{x^2} - \dfrac{2}{y^2} = -3 & \longrightarrow \quad \dfrac{5}{x^2} - \dfrac{2}{y^2} = -3 \\[2ex] \dfrac{3}{x^2} + \dfrac{1}{y^2} = 7 & \xrightarrow{2} \quad \underline{\dfrac{6}{x^2} + \dfrac{2}{y^2} = 14} \end{cases}$$

$$\dfrac{11}{x^2} = 11$$

$$11 = 11x^2 \to x^2 = 1 \to x = \pm 1$$

If $x = 1$: $\quad \dfrac{3}{(1)^2} + \dfrac{1}{y^2} = 7 \;\to\; \dfrac{1}{y^2} = 4 \;\to\; y^2 = \dfrac{1}{4} \;\to\; y = \pm\dfrac{1}{2}$

If $x = -1$: $\quad \dfrac{3}{(-1)^2} + \dfrac{1}{y^2} = 7 \;\to\; \dfrac{1}{y^2} = 4 \;\to\; y^2 = \dfrac{1}{4} \;\to\; y = \pm\dfrac{1}{2}$

Solutions: $\left(1, \dfrac{1}{2}\right), \left(1, -\dfrac{1}{2}\right), \left(-1, \dfrac{1}{2}\right), \left(-1, -\dfrac{1}{2}\right)$

40. Multiply each side of the first equation by –3 and add the equations to eliminate x:

$$\begin{cases} \dfrac{2}{x^2} - \dfrac{3}{y^2} = -1 & \xrightarrow{-3} \quad \dfrac{-6}{x^2} + \dfrac{9}{y^2} = 3 \\[2ex] \dfrac{6}{x^2} - \dfrac{7}{y^2} = -2 & \longrightarrow \quad \underline{\dfrac{6}{x^2} - \dfrac{7}{y^2} = -2} \end{cases}$$

$$\dfrac{2}{y^2} = 1$$

$$2 = y^2 \to y = \pm\sqrt{2}$$

If $y = \sqrt{2}$: $\quad \dfrac{2}{x^2} - \dfrac{3}{\left(\sqrt{2}\right)^2} = -1 \;\to\; \dfrac{2}{x^2} = \dfrac{1}{2} \;\to\; x^2 = 4 \;\to\; x = \pm 2$

If $y = -\sqrt{2}$: $\quad \dfrac{2}{x^2} - \dfrac{3}{\left(-\sqrt{2}\right)^2} = -1 \;\to\; \dfrac{2}{x^2} = \dfrac{1}{2} \;\to\; x^2 = 4 \;\to\; x = \pm 2$

Solutions: $\left(2, \sqrt{2}\right), \left(2, -\sqrt{2}\right), \left(-2, \sqrt{2}\right), \left(-2, -\sqrt{2}\right)$

41. Multiply each side of the first equation by –2 and add the equations to eliminate x:

$$\begin{cases} \dfrac{1}{x^4} + \dfrac{6}{y^4} = 6 & \xrightarrow{-2} \quad \dfrac{-2}{x^4} - \dfrac{12}{y^4} = -12 \\[2ex] \dfrac{2}{x^4} - \dfrac{2}{y^4} = 19 & \longrightarrow \quad \underline{\dfrac{2}{x^4} - \dfrac{2}{y^4} = 19} \end{cases}$$

$$\dfrac{-14}{y^4} = 7$$

$$-14 = 7y^4 \to y^4 = -2$$

There are no real solutions. The system is inconsistent.

42. Add the equations to eliminate y:

$$\begin{cases} \dfrac{1}{x^4} - \dfrac{1}{y^4} = 1 \\ \dfrac{1}{x^4} + \dfrac{1}{y^4} = 4 \end{cases}$$

$$\frac{2}{x^4} = 5 \quad \to \quad 2 = 5x^4 \quad \to \quad x^4 = \frac{2}{5} \quad \to \quad x = \pm\sqrt[4]{\frac{2}{5}}$$

If $x = \sqrt[4]{\frac{2}{5}}$: $\quad \dfrac{1}{\left(\sqrt[4]{\frac{2}{5}}\right)^4} + \dfrac{1}{y^4} = 4 \;\to\; \dfrac{1}{y^4} = \dfrac{3}{2} \;\to\; y^4 = \dfrac{2}{3} \;\to\; y = \pm\sqrt[4]{\dfrac{2}{3}}$

If $x = -\sqrt[4]{\frac{2}{5}}$: $\quad \dfrac{1}{\left(-\sqrt[4]{\frac{2}{5}}\right)^4} + \dfrac{1}{y^4} = 4 \;\to\; \dfrac{1}{y^4} = \dfrac{3}{2} \;\to\; y^4 = \dfrac{2}{3} \;\to\; y = \pm\sqrt[4]{\dfrac{2}{3}}$

Solutions: $\left(\sqrt[4]{\frac{2}{5}}, \sqrt[4]{\frac{2}{3}}\right)\left(\sqrt[4]{\frac{2}{5}}, -\sqrt[4]{\frac{2}{3}}\right)\left(-\sqrt[4]{\frac{2}{5}}, \sqrt[4]{\frac{2}{3}}\right)\left(-\sqrt[4]{\frac{2}{5}}, -\sqrt[4]{\frac{2}{3}}\right)$

43. Factor the first equation, solve for x, substitute into the second equation and solve:

$$\begin{cases} x^2 - 3xy + 2y^2 = 0 \quad \to \quad (x-2y)(x-y) = 0 \quad \to \quad x = 2y \text{ or } x = y \\ x^2 + xy = 6 \end{cases}$$

Substitute $x = 2y$ and solve:

$$x^2 + xy = 6$$
$$(2y)^2 + (2y)y = 6$$
$$4y^2 + 2y^2 = 6 \to 6y^2 = 6$$
$$y^2 = 1 \to y = \pm 1$$

If $y = 1$: $\quad x = 2 \cdot 1 = 2$

If $y = -1$: $\quad x = 2(-1) = -2$

Substitute $x = y$ and solve:

$$x^2 + xy = 6$$
$$y^2 + y \cdot y = 6$$
$$y^2 + y^2 = 6 \to 2y^2 = 6$$
$$y^2 = 3 \to y = \pm\sqrt{3}$$

If $y = \sqrt{3}$: $\quad x = \sqrt{3}$

If $y = -\sqrt{3}$: $\quad x = -\sqrt{3}$

Solutions: $(2, 1), (-2, -1), \left(\sqrt{3}, \sqrt{3}\right), \left(-\sqrt{3}, -\sqrt{3}\right)$

44. Factor the first equation, solve for x, substitute into the second equation and solve:

$$\begin{cases} x^2 - xy - 2y^2 = 0 \quad \to \quad (x-2y)(x+y) = 0 \quad \to \quad x = 2y \text{ or } x = -y \\ xy + x = -6 \end{cases}$$

Substitute $x = 2y$ and solve:

$$xy + x = -6$$
$$(2y)y + 2y = -6$$
$$2y^2 + 2y + 6 = 0$$
$$2(y^2 + y + 6) = 0$$
$$y = \frac{-1 \pm \sqrt{1^2 - 4(1)(6)}}{2(1)}$$

No real solution

Substitute $x = -y$ and solve:

$$xy + x = -6$$
$$-y \cdot y + (-y) = -6$$
$$-y^2 - y + 6 = 0$$
$$(-y-3)(y-2) = 0$$
$$y = -3 \text{ or } y = 2$$

If $y = -3$: $\quad x = 3$

If $y = 2$: $\quad x = -2$

Solutions: $(3, -3), (-2, 2)$

45. Multiply each side of the second equation by –y and add the equations to eliminate y:

$$\begin{cases} y^2+y+x^2-x-2=0 \\ y+1+\dfrac{x-2}{y}=0 \end{cases} \quad \begin{array}{l} \longrightarrow \quad y^2+y+x^2-x-2=0 \\ \xrightarrow{\cdot(-y)} \quad \underline{-y^2-y \qquad -x+2=0} \end{array}$$

$$x^2-2x \quad =0 \rightarrow x(x-2)=0$$

$$x=0 \text{ or } x=2$$

If $x=0$: $\quad y^2+y+0^2-0-2=0 \rightarrow y^2+y-2=0 \rightarrow (y+2)(y-1)=0$

$\rightarrow \; y=-2$ or $y=1$

If $x=2$: $\quad y^2+y+2^2-2-2=0 \rightarrow y^2+y=0 \rightarrow y(y+1)=0$

$\rightarrow \; y=0$ or $y=-1$

Solutions: (0, –2), (0, 1), (2, 0), (2, –1)

46. Multiply each side of the second equation by $-x^2$ and add the equations to eliminate x:

$$\begin{cases} x^3-2x^2+y^2+3y-4=0 \\ x-2+\dfrac{y^2-y}{x^2}=0 \end{cases} \quad \begin{array}{l} \longrightarrow \quad x^3-2x^2+y^2+3y=4 \\ \xrightarrow{\cdot(-x^2)} \quad \underline{-x^3+2x^2-y^2+\;y=0} \end{array}$$

$$4y=4 \rightarrow y=1$$

If $y=1$: $\quad x^3-2x^2+1^2+3\cdot1-4=0 \rightarrow x^3-2x^2=0 \rightarrow x^2(x-2)=0$

$\rightarrow \; x=0$ or $x=2$ (Note: $x\neq 0$ - division by zero)

Solutions: (2, 1)

47. Rewrite each equation in exponential form:

$$\begin{cases} \log_x y=3 \rightarrow y=x^3 \\ \log_x(4y)=5 \rightarrow 4y=x^5 \end{cases}$$

Substitute the first equation into the second and solve:

$$4x^3=x^5$$

$$x^5-4x^3=0 \rightarrow x^3(x^2-4)=0 \rightarrow x^3=0 \text{ or } x^2=4 \rightarrow x=0 \text{ or } x=\pm 2$$

The base of a logarithm must be positive, thus $x\neq 0$ and $x\neq -2$.

If $x=2$: $\quad y=2^3=8$

Solution: (2, 8)

48. Rewrite each equation in exponential form:

$$\begin{cases} \log_x(2y)=3 \rightarrow 2y=x^3 \\ \log_x(4y)=2 \rightarrow 4y=x^2 \end{cases}$$

Substitute the first equation into the second and solve:

$$2x^3=x^2 \rightarrow 2x^3-x^2=0 \rightarrow x^2(2x-1)=0 \rightarrow x^2=0 \text{ or } x=\frac{1}{2} \rightarrow x=\frac{1}{2} \text{ or } x=0$$

The base of a logarithm must be positive, thus $x\neq 0$.

If $x=\frac{1}{2}$: $\quad 4y=\left(\frac{1}{2}\right)^2=\frac{1}{4} \rightarrow y=\frac{1}{16}$ $\qquad$ Solution: $\left(\frac{1}{2},\frac{1}{16}\right)$

49. Rewrite each equation in exponential form:

$$\begin{cases} \ln x = 4\ln y \;\rightarrow\; x = e^{4\ln y} = e^{\ln y^4} = y^4 \\ \log_3 x = 2 + 2\log_3 y \;\rightarrow\; x = 3^{2+2\log_3 y} = 3^2 \; 3^{2\log_3 y} = 3^2 \; 3^{\log_3 y^2} = 9y^2 \end{cases}$$

So we have the system

$$\begin{cases} x = y^4 \\ x = 9y^2 \end{cases}$$

Therefore we have

$$\begin{aligned} 9y^2 &= y^4 \\ 9y^2 - y^4 &= 0 \\ y^2(9-y^2) &= 0 \\ y^2(3+y)(3-y) &= 0 \\ y = 0 \text{ or } y &= -3 \text{ or } y = 3 \end{aligned}$$

Since $\ln y$ is undefined when $y \le 0$, the only solution is $y = 3$.

If $y = 3$: $x = y^4 \rightarrow x = 3^4 = 81$

Solution: $(81, 3)$

50. Rewrite each equation in exponential form:

$$\begin{cases} \ln x = 5\ln y \;\rightarrow\; x = e^{5\ln y} = e^{\ln y^5} = y^5 \\ \log_2 x = 3 + 2\log_2 y \;\rightarrow\; x = 2^{3+2\log_2 y} = 2^3 \; 2^{2\log_2 y} = 2^3 \; 2^{\log_2 y^2} = 8y^2 \end{cases}$$

So we have the system $\begin{cases} x = y^5 \\ x = 8y^2 \end{cases}$

Therefore we have

$$\begin{aligned} 8y^2 &= y^5 \\ 8y^2 - y^5 &= 0 \\ y^2(8-y^3) &= 0 \\ y = 0 \text{ or } 8 - y^3 &= 0 \rightarrow 8 = y^3 \rightarrow 2 = y \end{aligned}$$

Since $\ln y$ is undefined when $y \le 0$, the only solution is $y = 2$.

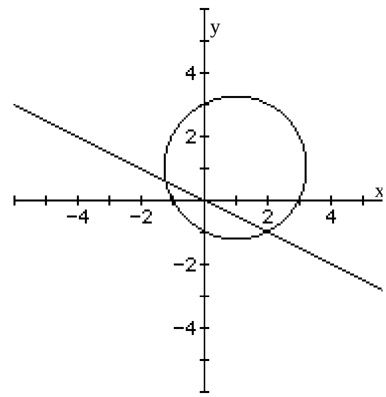
If $y = 2$: $x = y^5 \rightarrow x = 2^5 = 32$

Solution: $(32, 2)$

51. Solve the first equation for x, substitute into the second equation and solve:

$$\begin{cases} x + 2y = 0 \;\rightarrow\; x = -2y \\ (x-1)^2 + (y-1)^2 = 5 \end{cases}$$

$$(-2y-1)^2 + (y-1)^2 = 5$$
$$4y^2 + 4y + 1 + y^2 - 2y + 1 = 5$$
$$5y^2 + 2y - 3 = 0$$
$$(5y-3)(y+1) = 0$$
$$y = \frac{3}{5} = 0.6 \quad \text{or } y = -1$$
$$x = -\frac{6}{5} = -1.2 \quad \text{or } x = 2$$

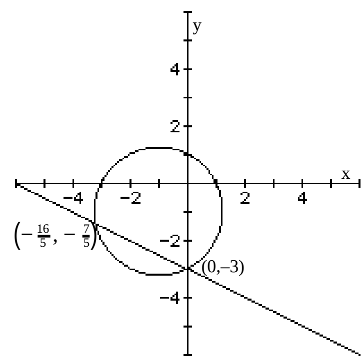


The points of intersection are $(-1.2, 0.6)$, $(2, -1)$.

52. Solve the first equation for x, substitute into the second equation and solve:

$$\begin{cases} x + 2y = -6 \;\rightarrow\; x = -2y - 6 \\ (x+1)^2 + (y+1)^2 = 5 \end{cases}$$

$$(-2y - 6 + 1)^2 + (y+1)^2 = 5$$
$$4y^2 + 20y + 25 + y^2 + 2y + 1 = 5$$
$$5y^2 + 22y + 21 = 0$$
$$(5y+7)(y+3) = 0$$
$$y = -\frac{7}{5} \quad \text{or } y = -3$$
$$x = -\frac{16}{5} \quad \text{or} \quad x = 0$$



The points of intersection are $\left(-\frac{16}{5}, -\frac{7}{5}\right)$, $(0, -3)$.

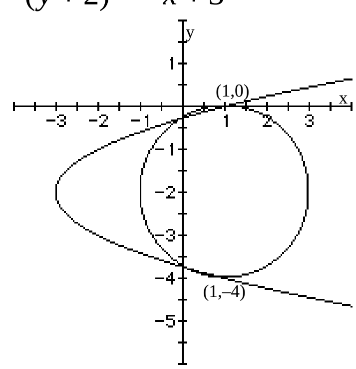
53. Complete the square on the second equation, substitute into the first equation and solve:

$$\begin{cases} (x-1)^2 + (y+2)^2 = 4 \\ y^2 + 4y - x + 1 = 0 \;\rightarrow\; y^2 + 4y + 4 = x - 1 + 4 \;\rightarrow\; (y+2)^2 = x + 3 \end{cases}$$

$$(x-1)^2 + x + 3 = 4$$
$$x^2 - 2x + 1 + x + 3 = 4$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0 \quad \text{or } x = 1$$

If $x = 0$: $\quad (y+2)^2 = 0 + 3 \;\rightarrow\; y + 2 = \pm\sqrt{3}$
$$\rightarrow\; y = -2 \pm \sqrt{3}$$

If $x = 1$: $\quad (y+2)^2 = 1 + 3 \;\rightarrow\; y + 2 = \pm 2$
$$\rightarrow\; y = -2 \pm 2$$



The points of intersection are:

$$\left(0, -2-\sqrt{3}\right), \left(0, -2+\sqrt{3}\right), (1, -4), (1, 0).$$

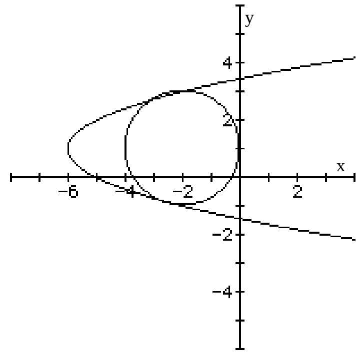
54. Complete the square on the second equation, substitute into the first equation and solve:

$$\begin{cases} (x+2)^2+(y-1)^2=4 \\ y^2-2y-x-5=0 \ \rightarrow\ y^2-2y+1=x+5+1 \rightarrow\ (y-1)^2=x+6 \end{cases}$$

$$(x+2)^2+x+6=4$$
$$x^2+4x+4+x+6=4$$
$$x^2+5x+6=0$$
$$(x+2)(x+3)=0$$
$$x=-2 \quad \text{or } x=-3$$

If $x=-2$: $(y-1)^2=-2+6 \ \rightarrow\ y-1=\pm 2$
$\rightarrow\ y=-1$ or $y=3$

If $x=-3$: $(y-1)^2=-3+6 \ \rightarrow\ y-1=\pm\sqrt{3}$
$\rightarrow\ y=1\pm\sqrt{3}$



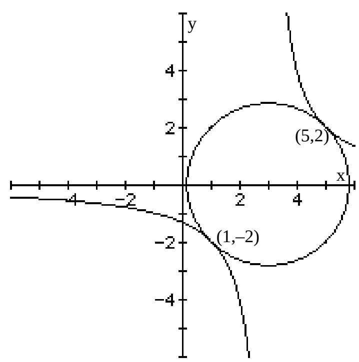
The points of intersection are:

$\left(-3, 1-\sqrt{3}\right), \left(-3, 1+\sqrt{3}\right), (-2, -1), (-2, 3)$.

55. Solve the first equation for x, substitute into the second equation and solve:

$$\begin{cases} y=\dfrac{4}{x-3} \ \rightarrow\ x-3=\dfrac{4}{y} \ \rightarrow\ x=\dfrac{4}{y}+3 \\ x^2-6x+y^2+1=0 \end{cases}$$

$$\left(\frac{4}{y}+3\right)^2-6\left(\frac{4}{y}+3\right)+y^2+1=0$$
$$\frac{16}{y^2}+\frac{24}{y}+9-\frac{24}{y}-18+y^2+1=0$$
$$\frac{16}{y^2}+y^2-8=0$$
$$16+y^4-8y^2=0$$
$$y^4-8y^2+16=0$$
$$(y^2-4)^2=0$$
$$y^2-4=0$$
$$y^2=4$$
$$y=\pm 2$$



If $y=2$: $x=\dfrac{4}{2}+3=5$

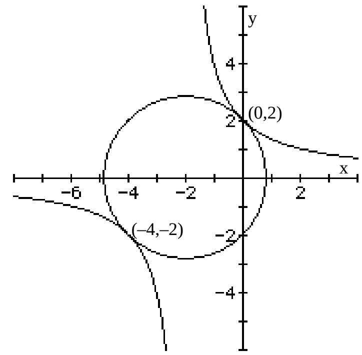
If $y=-2$: $x=\dfrac{4}{-2}+3=1$

The points of intersection are: (1, –2), (5, 2).

56. Substitute the first equation into the second equation and solve:

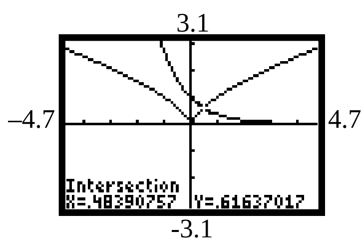
$$\begin{cases} y = \dfrac{4}{x+2} \\ x^2 + 4x + y^2 - 4 = 0 \end{cases}$$

$$\begin{aligned}
x^2 + 4x + \left(\frac{4}{x+2}\right)^2 - 4 &= 0 \\
x^2 + 4x - 4 &= -\left(\frac{4}{x+2}\right)^2 \\
(x+2)^2\left(x^2 + 4x - 4\right) &= -16 \\
\left(x^2 + 4x + 4\right)\left(x^2 + 4x - 4\right) &= -16 \\
x^4 + 8x^3 + 16x^2 - 16 &= -16 \\
x^4 + 8x^3 + 16x^2 &= 0 \\
x^2\left(x^2 + 8x + 16\right) &= 0 \\
x^2(x+4)^2 &= 0 \\
x = 0 \quad &\text{or} \quad x = -4 \\
y = 2 \quad &\qquad y = -2
\end{aligned}$$



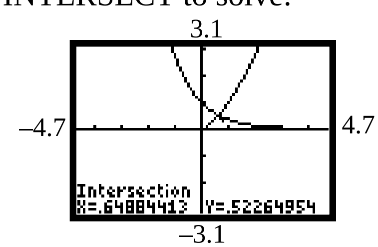
The points of intersection are: (0, 2), (–4, –2).

57. Graph: $y_1 = x^{(2/3)}$; $y_2 = e^{(-x)}$
Use INTERSECT to solve:



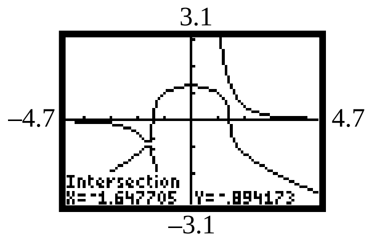
Solution: (0.48, 0.62)

58. Graph: $y_1 = x^{(3/2)}$; $y_2 = e^{(-x)}$
Use INTERSECT to solve:



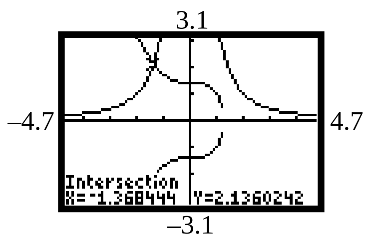
Solution: (0.65, 0.52)

59. Graph: $y_1 = \sqrt[3]{(2 - x^2)}$; $y_2 = 4 / x^3$
Use INTERSECT to solve:



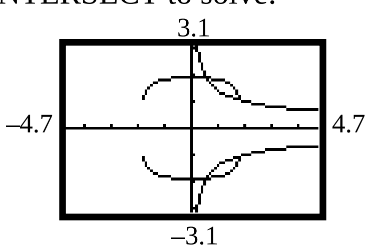
Solution: (–1.65, –0.89)

60. Graph: $y_1 = \sqrt{2 - x^3}$; $y_2 = -\sqrt{2 - x^3}$; $y_3 = 4 / x^2$
Use INTERSECT to solve:



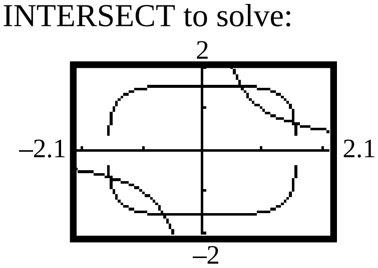
Solution: (–1.37, 2.14)

61. Graph: $y_1 = \sqrt[4]{(12 - x^4)}$; $y_2 = -\sqrt[4]{(12 - x^4)}$; $y_3 = \sqrt{2 / x}$; $y_4 = -\sqrt{2 / x}$
Use INTERSECT to solve:



Solutions: (0.58, 1.86), (1.81, 1.05), (1.81, –1.05), (0.58, –1.86)

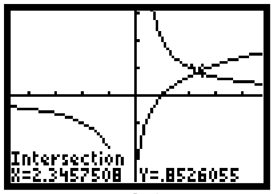
62. Graph: $y_1 = \sqrt[4]{(6 - x^4)}$; $y_2 = -\sqrt[4]{(6 - x^4)}$; $y_3 = 1 / x$
Use INTERSECT to solve:



Solutions: (0.64, 1.55), (1.55, 0.64), (–0.64, –1.55), (–1.55, –0.64)

63. Graph: $y_1 = 2/x$; $y_2 = \ln x$
Use INTERSECT to solve:

3.1

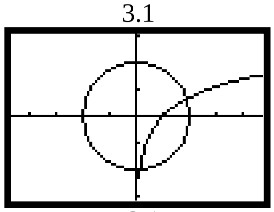
–4.7  4.7

–3.1

Solution: (2.35, 0.85)

64. Graph: $y_1 = \sqrt{4-x^2}$; $y_2 = -\sqrt{4-x^2}$; $y_3 = \ln x$
Use INTERSECT to solve:

3.1

–4.7  4.7

–3.1

Solution: (1.90, 0.64), (0.14, –2.00)

65. Let x and y be the two numbers. The system of equations is:

$$\begin{cases} x - y = 2 \\ x^2 + y^2 = 10 \end{cases}$$

Solve the first equation for x, substitute into the second equation and solve:

$$(y+2)^2 + y^2 = 10 \rightarrow y^2 + 4y + 4 + y^2 = 10$$
$$2y^2 + 4y - 6 = 0 \rightarrow y^2 + 2y - 3 = 0 \rightarrow (y+3)(y-1) = 0 \rightarrow y = -3 \text{ or } y = 1$$

If $y = -3$: $x = -3 + 2 = -1$

If $y = 1$: $x = 1 + 2 = 3$

The two numbers are 1 and 3 or –1 and –3.

66. Let x and y be the two numbers. The system of equations is:

$$\begin{cases} x + y = 7 \\ x^2 - y^2 = 21 \end{cases}$$

Solve the first equation for x, substitute into the second equation and solve:

$$(7-y)^2 - y^2 = 21 \rightarrow 49 - 14y + y^2 - y^2 = 21$$
$$-14y = -28 \rightarrow y = 2 \rightarrow x = 7 - 2 = 5$$

The two numbers are 2 and 5.

67. Let x and y be the two numbers. The system of equations is:

$$\begin{cases} xy = 4 \\ x^2 + y^2 = 8 \end{cases}$$

Solve the first equation for x, substitute into the second equation and solve:

$$\left(\frac{4}{y}\right)^2 + y^2 = 8 \rightarrow \frac{16}{y^2} + y^2 = 8 \rightarrow 16 + y^4 = 8y^2$$

$$y^4 - 8y^2 + 16 = 0 \rightarrow (y^2 - 4)^2 = 0 \rightarrow y^2 - 4 = 0 \rightarrow y^2 = 4 \rightarrow y = \pm 2$$

If $y = 2$: $\quad x = \frac{4}{2} = 2$

If $y = -2$: $\quad x = \frac{4}{-2} = -2$

The two numbers are 2 and 2 or –2 and –2.

68. Let x and y be the two numbers. The system of equations is:

$$\begin{cases} xy = 10 \\ x^2 - y^2 = 21 \end{cases}$$

Solve the first equation for x, substitute into the second equation and solve:

$$\left(\frac{10}{y}\right)^2 - y^2 = 21 \rightarrow \frac{100}{y^2} - y^2 = 21$$

$$100 - y^4 = 21y^2 \rightarrow y^4 + 21y^2 - 100 = 0 \rightarrow (y^2 - 4)(y^2 + 25) = 0$$

$$y^2 = 4 \rightarrow y = \pm 2$$

or $y^2 = -25$ which is impossible

If $y = 2$: $\quad x = \frac{10}{2} = 5$

If $y = -2$: $\quad x = \frac{10}{-2} = -5$

The two numbers are 2 and 5 or –2 and –5.

69. Let x and y be the two numbers. The system of equations is:

$$\begin{cases} x - y = xy \\ \frac{1}{x} + \frac{1}{y} = 5 \end{cases}$$

Solve the first equation for x, substitute into the second equation and solve:

$$x - xy = y \rightarrow x(1 - y) = y \rightarrow x = \frac{y}{1 - y}$$

$$\frac{1}{\left(\frac{y}{1-y}\right)} + \frac{1}{y} = 5 \rightarrow \frac{1-y}{y} + \frac{1}{y} = 5$$

$$\frac{2-y}{y} = 5 \rightarrow 2 - y = 5y \rightarrow 6y = 2 \rightarrow y = \frac{1}{3}$$

If $y = \frac{1}{3}$: $\quad x = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$

The two numbers are $\frac{1}{2}$ and $\frac{1}{3}$.

70. Let x and y be the two numbers. The system of equations is:

$$\begin{cases} x + y = xy \\ \dfrac{1}{x} - \dfrac{1}{y} = 3 \end{cases}$$

Solve the first equation for x, substitute into the second equation and solve:

$$xy - x = y \rightarrow x(y-1) = y \rightarrow x = \frac{y}{y-1} \rightarrow \frac{1}{\left(\frac{y}{y-1}\right)} - \frac{1}{y} = 3$$

$$\frac{y-1}{y} - \frac{1}{y} = 3 \rightarrow \frac{y-2}{y} = 3 \rightarrow y - 2 = 3y \rightarrow 2y = -2 \rightarrow y = -1$$

If $y = -1$: $\quad x = \dfrac{-1}{-1-1} = \dfrac{-1}{-2} = \dfrac{1}{2}$

The two numbers are $\frac{1}{2}$ and -1.

71. $\begin{cases} \dfrac{a}{b} = \dfrac{2}{3} \\ a + b = 10 \end{cases}$

Solve the second equation for a, substitute into the first equation and solve:

$$\frac{10-b}{b} = \frac{2}{3} \rightarrow 3(10 - b) = 2b \rightarrow 30 - 3b = 2b \rightarrow 30 = 5b$$

$b = 6 \rightarrow a = 4$

$a + b = 10; \quad b - a = 2$

The ratio of $a + b$ to $b - a$ is $\dfrac{10}{2} = 5$.

72. $\begin{cases} \dfrac{a}{b} = \dfrac{4}{3} \\ a + b = 14 \end{cases}$

Solve the second equation for a, substitute into the first equation and solve:

$$\frac{14-b}{b} = \frac{4}{3} \rightarrow 3(14 - b) = 4b \rightarrow 42 - 3b = 4b \rightarrow 42 = 7b$$

$b = 6 \rightarrow a = 8$

$a - b = 2; \quad a + b = 14$

The ratio of $a - b$ to a +b is $\dfrac{2}{14} = \dfrac{1}{7}$.

73. Let x = the width of the rectangle.
Let y = the length of the rectangle.

$$\begin{cases} 2x+2y=16 \\ xy=15 \end{cases}$$

Solve the first equation for y, substitute into the second equation and solve:

$$2x+2y=16$$
$$2y=16-2x$$
$$y=8-x$$

$$x(8-x)=15$$
$$8x-x^2=15$$
$$x^2-8x+15=0 \to (x-5)(x-3)=0$$
$$x=5 \text{ or } x=3$$
$$y=3 \qquad y=5$$

The dimensions of the rectangle are 3 inches by 5 inches.

74. Let $2x$ = the side of the first square.
Let $3x$ = the side of the second square.

$$(2x)^2+(3x)^2=52 \to 4x^2+9x^2=52 \to 13x^2=52 \to x^2=4 \to x=2$$

The sides of the first square are 4 feet and the sides of the second square are 6 feet.

75. Let x = the radius of the first circle.
Let y = the radius of the second circle.

$$\begin{cases} 2\pi x+2\pi y=12\pi \\ \pi x^2+\pi y^2=20\pi \end{cases}$$

Solve the first equation for y, substitute into the second equation and solve:

$$2\pi x+2\pi y=12\pi$$
$$x+y=6$$
$$y=6-x$$

$$\pi x^2+\pi y^2=20\pi$$
$$x^2+y^2=20$$
$$x^2+(6-x)^2=20$$
$$x^2+36-12x+x^2=20$$
$$2x^2-12x+16=0$$
$$x^2-6x+8=0$$
$$(x-4)(x-2)=0$$
$$x=4 \text{ or } x=2$$
$$y=2 \qquad y=4$$

The radii of the circles are 2 centimeters and 4 centimeters.

76. Let x = the length of each of the two equal sides in the isosceles triangle.
Let y = the length of the base.
The perimeter of the triangle: $x+x+y=18$
Since the altitude to the base y is 3, the Pythagorean theorem will produce another equation: $\left(\frac{y}{2}\right)^2+3^2=x^2$

Solve the system of equations:

$$\begin{cases} 2x + y = 18 \quad \rightarrow \quad y = 18 - 2x \\ \dfrac{y^2}{4} + 9 = x^2 \end{cases}$$

Solve the first equation for y, substitute into the second equation and solve:

$$\frac{(18-2x)^2}{4} + 9 = x^2$$
$$\frac{324 - 72x + 4x^2}{4} + 9 = x^2$$
$$81 - 18x + x^2 + 9 = x^2$$
$$-18x = -90 \quad \rightarrow \quad x = 5$$
$$y = 18 - 2(5) = 8$$

The base of the triangle is 8 centimeters.

77. The tortoise takes 9 + 3 = 12 minutes or 0.2 hour longer to complete the race than the hare.
Let r = the rate of the hare.
Let t = the time for the hare to complete the race.
Then $t + 0.2$ = the time for the tortoise.
$r - 0.5$ = the rate for the tortoise.
Since the length of the race is 21 meters, the distance equations are:

$$\begin{cases} rt = 21 \\ (r - 0.5)(t + 0.2) = 21 \end{cases}$$

Solve the first equation for r, substitute into the second equation and solve:

$$\left(\frac{21}{t} - 0.5\right)(t + 0.2) = 21 \rightarrow 21 + \frac{4.2}{t} - 0.5t - 0.1 = 21$$
$$10t\left(21 + \frac{4.2}{t} - 0.5t - 0.1\right) = 10t\,(21)$$
$$210t + 42 - 5t^2 - t = 210t \rightarrow 5t^2 + t - 42 = 0 \rightarrow (5t - 14)(t + 3) = 0$$
$$t = \frac{14}{5} = 2.8 \quad \text{or } t = -3$$

$t = -3$ makes no sense, since time cannot be negative.

Solve for r:

$$r = \frac{21}{2.8} = 7.5$$

The average speed of the hare is 7.5 meters per hour, and the average speed for the tortoise is 7 meters per hour.

78. Let $v_1, v_2, v_3 =$ the speeds of runners 1, 2, 3.
Let $t_1, t_2, t_3 =$ the times of runners 1, 2, 3.
Then by the conditions of the problem, we have the following system:

$$\begin{cases} 5280 = v_1 t_1 \\ 5270 = v_2 t_1 \\ 5260 = v_3 t_1 \\ 5280 = v_2 t_2 \end{cases}$$

Distance between the second runner and the third runner after t_2 seconds is:

$$5280 - v_3 t_2 = 5280 - v_3 t_1 \left(\frac{v_2 t_2}{v_2 t_1}\right) = 5280 - 5260\left(\frac{5280}{5270}\right) = 10.02$$

The second place runner beats the third place runner by 10.02 feet.

79. Let x = the width of the cardboard.
Let y = the length of the cardboard.
The width of the box will be $x-4$, the length of the box will be $y-4$, and the height is 2.
The volume is $V = (x-4)(y-4)(2)$.
Solve the system of equations:

$$\begin{cases} xy = 216 \\ 2(x-4)(y-4) = 224 \end{cases}$$

Solve the first equation for y, substitute into the second equation and solve:

$$\begin{aligned} (2x-8)\left(\frac{216}{x} - 4\right) &= 224 \\ 432 - 8x - \frac{1728}{x} + 32 &= 224 \\ 432x - 8x^2 - 1728 + 32x &= 224x \\ -8x^2 + 240x - 1728 &= 0 \\ x^2 - 30x + 216 &= 0 \\ (x-12)(x-18) &= 0 \end{aligned}$$

$$x = 12 \quad \text{or} \quad x = 18$$
$$y = 18 \qquad\quad y = 12$$

The cardboard should be 12 centimeters by 18 centimeters.

80. Let x = the width of the cardboard.
Let y = the length of the cardboard.
The area of the cardboard is: $xy = 216$
The volume of the tube is: $V = \pi r^2 h = 224$ where $h = y$ and $2\pi r = x$ or $r = \frac{x}{2\pi}$.
Solve the system of equations:

$$\begin{cases} xy = 216 \quad \rightarrow \quad y = \dfrac{216}{x} \\ \pi\left(\dfrac{x}{2\pi}\right)^2 y = 224 \quad \rightarrow \quad \dfrac{x^2 y}{4\pi} = 224 \end{cases}$$

Solve the first equation for y, substitute into the second equation and solve:

$$\frac{x^2\left(\frac{216}{x}\right)}{4\pi}=224$$

$$216x=896\pi \rightarrow x=\frac{896\pi}{216}\approx 13.03 \rightarrow y=\frac{216}{13.03}\approx 16.58$$

The cardboard should be 13.03 centimeters by 16.58 centimeters.

81. Find equations relating area and perimeter:

$$\begin{cases} x^2+y^2=4500 \\ 3x+3y+(x-y)=300 \end{cases}$$

Solve the second equation for y, substitute into the first equation and solve:

$$\begin{aligned} 4x+2y&=300 \\ 2y&=300-4x \\ y&=150-2x \end{aligned}$$

$$\begin{aligned} x^2+(150-2x)^2&=4500 \\ x^2+22500-600x+4x^2&=4500 \\ 5x^2-600x+18000&=0 \\ x^2-120x+3600&=0 \\ (x-60)^2&=0 \\ x-60&=0 \\ x&=60 \\ y=150-2(60)&=30 \end{aligned}$$

The sides of the squares are 30 feet and 60 feet.

82. Let x = the length of a side of the square.
Let r = the radius of the circle.
The area of the square is x^2 and the area of the circle is πr^2.
The perimeter of the square is $4x$ and the circumference of the circle is $2\pi r$.
Find equations relating area and perimeter:

$$\begin{cases} x^2+\pi r^2=100 \\ 4x+2\pi r=60 \end{cases}$$

Solve the second equation for x, substitute into the first equation and solve:

$$\begin{aligned} 4x+2\pi r&=60 \\ 4x&=60-2\pi r \\ x&=15-\frac{1}{2}\pi r \end{aligned}$$

$$\begin{aligned} \left(15-\tfrac{1}{2}\pi r\right)^2+\pi r^2&=100 \\ 225-15\pi r+\tfrac{1}{4}\pi^2 r^2+\pi r^2&=100 \\ \left(\tfrac{1}{4}\pi^2+\pi\right)r^2-15\pi r+125&=0 \end{aligned}$$

$$\begin{aligned} b^2-4ac&=(-15\pi)^2-4\left(\tfrac{1}{4}\pi^2+\pi\right)(125)=225\pi^2-500\left(\tfrac{1}{4}\pi^2+\pi\right) \\ &=100\pi^2-500\pi<0 \end{aligned}$$

Since the discriminant is less than zero, it is impossible to cut the wire into two pieces which have a total are of 100 square feet.

83. Solve the system for l and w:

$$\begin{cases} 2l + 2w = P \\ l\,w = A \end{cases}$$

Solve the first equation for l, substitute into the second equation and solve:

$$2l = P - 2w \;\rightarrow\; l = \frac{P}{2} - w$$

$$\left(\frac{P}{2} - w\right)w = A$$

$$\frac{P}{2}w - w^2 = A$$

$$w^2 - \frac{P}{2}w + A = 0$$

$$w = \frac{\frac{P}{2} \pm \sqrt{\frac{P^2}{4} - 4A}}{2} = \frac{\frac{P}{2} \pm \sqrt{\frac{P^2 - 16A}{4}}}{2} = \frac{\frac{P}{2} \pm \frac{\sqrt{P^2 - 16A}}{2}}{2}$$

$$w = \frac{P \pm \sqrt{P^2 - 16A}}{4}$$

If $w = \dfrac{P + \sqrt{P^2 - 16A}}{4}$ then $l = \dfrac{P}{2} - \dfrac{P + \sqrt{P^2 - 16A}}{4} = \dfrac{P - \sqrt{P^2 - 16A}}{4}$

If $w = \dfrac{P - \sqrt{P^2 - 16A}}{4}$ then $l = \dfrac{P}{2} - \dfrac{P - \sqrt{P^2 - 16A}}{4} = \dfrac{P + \sqrt{P^2 - 16A}}{4}$

If it is required that length be greater than width, then the solution is:

$w = \dfrac{P - \sqrt{P^2 - 16A}}{4}$ and $l = \dfrac{P + \sqrt{P^2 - 16A}}{4}$

84. Solve the system for l and b:

$$\begin{cases} P = b + 2l \quad \rightarrow \quad b = P - 2l \\ h^2 + \dfrac{b^2}{4} = l^2 \end{cases}$$

Solve the first equation for b, substitute into the second equation and solve:

$$4h^2 + b^2 = 4l^2 \rightarrow 4h^2 + (P - 2l)^2 = 4l^2$$

$$4h^2 + P^2 - 4Pl + 4l^2 = 4l^2 \rightarrow 4h^2 + P^2 = 4Pl$$

$$l = \frac{4h^2 + P^2}{4P} \rightarrow b = P - \frac{4h^2 + P^2}{2P}$$

85. Solve the equation:

$$m^2 - 4(2m - 4) = 0 \rightarrow m^2 - 8m + 16 = 0 \rightarrow (m - 4)^2 = 0 \rightarrow m - 4 = 0 \rightarrow m = 4$$

Use the point-slope equation with slope 4 and the point (2, 4) to obtain the equation of the tangent line:

$$y - 4 = 4(x - 2) \rightarrow y - 4 = 4x - 8 \rightarrow y = 4x - 4$$

86. Solve the system:

$$\begin{cases} x^2 + y^2 = 10 \\ y = mx + b \end{cases}$$

Solve the system by substitution:

$$x^2 + (mx+b)^2 = 10 \rightarrow x^2 + m^2x^2 + 2bmx + b^2 - 10 = 0$$

$$\left(1+m^2\right)x^2 + 2bmx + b^2 - 10 = 0$$

Note that the tangent line passes through (1, 3). Find the relation between m and b:

$$3 = m(1)+b$$
$$b = 3-m$$

There is one solution to the quadratic if the discriminant is zero.

$$(2bm)^2 - 4\left(m^2+1\right)\left(b^2-10\right) = 0$$
$$4b^2m^2 - 4b^2m^2 + 40m^2 - 4b^2 + 40 = 0$$
$$40m^2 - 4b^2 + 40 = 0$$

Substitute for b and solve:

$$40m^2 - 4(3-m)^2 + 40 = 0$$
$$40m^2 - 4m^2 + 24m - 36 + 40 = 0 \rightarrow 36m^2 + 24m + 4 = 0$$
$$9m^2 + 6m + 1 = 0 \rightarrow (3m+1)^2 = 0 \rightarrow m = -\frac{1}{3}$$
$$b = 3 - \left(-\frac{1}{3}\right) = \frac{10}{3}$$

The equation of the tangent line is $y = -\dfrac{1}{3}x + \dfrac{10}{3}$.

87. Solve the system:

$$\begin{cases} y = x^2 + 2 \\ y = mx + b \end{cases}$$

Solve the system by substitution:

$$x^2 + 2 = mx + b$$
$$x^2 - mx + 2 - b = 0$$

Note that the tangent line passes through (1, 3). Find the relation between m and b:

$$3 = m(1)+b$$
$$b = 3-m$$

Substitute into the quadratic to eliminate b:

$$x^2 - mx + 2 - (3-m) = 0$$
$$x^2 - mx + (m-1) = 0$$

Find when the discriminant is 0:

$$(-m)^2 - 4(1)(m-1) = 0 \rightarrow m^2 - 4m + 4 = 0 \rightarrow (m-2)^2 = 0$$
$$m - 2 = 0 \rightarrow m = 2 \rightarrow b = 3 - 2 = 1$$

The equation of the tangent line is $y = 2x + 1$.

88. Solve the system:

$$\begin{cases} x^2 + y = 5 \\ y = mx + b \end{cases}$$

Solve the system by substitution:

$$x^2 + mx + b = 5$$
$$x^2 + mx + b - 5 = 0$$

Note that the tangent line passes through (–2, 1). Find the relation between m and b:

$$1 = m(-2) + b$$
$$b = 2m + 1$$

Substitute into the quadratic to eliminate b:

$$x^2 + mx + 2m + 1 - 5 = 0 \rightarrow x^2 + mx + (2m - 4) = 0$$

Find when the discriminant is 0:

$$(m)^2 - 4(1)(2m - 4) = 0 \rightarrow m^2 - 8m + 16 = 0 \rightarrow (m - 4)^2 = 0$$
$$m - 4 = 0 \rightarrow m = 4 \rightarrow b = 2(4) + 1 = 9$$

The equation of the tangent line is $y = 4x + 9$.

89. Solve the system:

$$\begin{cases} 2x^2 + 3y^2 = 14 \\ y = mx + b \end{cases}$$

Solve the system by substitution:

$$2x^2 + 3(mx + b)^2 = 14$$
$$2x^2 + 3m^2x^2 + 6mbx + 3b^2 = 14$$
$$(3m^2 + 2)x^2 + 6mbx + 3b^2 - 14 = 0$$

Note that the tangent line passes through (1, 2). Find the relation between m and b:

$$2 = m(1) + b$$
$$b = 2 - m$$

Substitute into the quadratic to eliminate b:

$$(3m^2 + 2)x^2 + 6m(2 - m)x + 3(2 - m)^2 - 14 = 0$$
$$(3m^2 + 2)x^2 + (12m - 6m^2)x + 12 - 12m + 3m^2 - 14 = 0$$
$$(3m^2 + 2)x^2 + (12m - 6m^2)x + (3m^2 - 12m - 2) = 0$$

Find when the discriminant is 0:

$$(12m - 6m^2)^2 - 4(3m^2 + 2)(3m^2 - 12m - 2) = 0$$
$$144m^2 - 144m^3 + 36m^4 - 4(9m^4 - 36m^3 - 24m - 4) = 0$$
$$144m^2 - 144m^3 + 36m^4 - 36m^4 + 144m^3 + 96m + 16 = 0$$
$$144m^2 + 96m + 16 = 0$$
$$9m^2 + 6m + 1 = 0$$
$$(3m + 1)^2 = 0$$
$$3m + 1 = 0$$
$$m = -\frac{1}{3} \qquad b = 2 - \left(-\frac{1}{3}\right) = \frac{7}{3}$$

The equation of the tangent line is $y = -\frac{1}{3}x + \frac{7}{3}$.

90. Solve the system:
$$\begin{cases} 3x^2 + y^2 = 7 \\ y = mx + b \end{cases}$$
Solve the system by substitution:
$$3x^2 + (mx+b)^2 = 7$$
$$3x^2 + m^2x^2 + 2mbx + b^2 = 7$$
$$(m^2+3)x^2 + 2mbx + b^2 - 7 = 0$$
Note that the tangent line passes through (–1, 2).

Find the relation between m and b:
$$2 = m(-1) + b$$
$$b = m + 2$$
There is one solution to the quadratic if the discriminant is zero.
$$(2bm)^2 - 4(m^2+3)(b^2-7) = 0$$
$$4b^2m^2 - 4b^2m^2 + 28m^2 - 12b^2 + 84 = 0$$
$$28m^2 - 12b^2 + 84 = 0$$
$$7m^2 - 3b^2 + 21 = 0$$
Substitute for b and solve:
$$7m^2 - 3(m+2)^2 + 21 = 0 \rightarrow 7m^2 - 3m^2 - 12m - 12 + 21 = 0$$
$$4m^2 - 12m + 9 = 0 \rightarrow (2m-3)^2 = 0 \rightarrow m = \frac{3}{2}$$
$$b = \frac{3}{2} + 2 = \frac{7}{2}$$
The equation of the tangent line is $y = \frac{3}{2}x + \frac{7}{2}$.

91. Solve the system:
$$\begin{cases} x^2 - y^2 = 3 \\ y = mx + b \end{cases}$$
Solve the system by substitution:
$$x^2 - (mx+b)^2 = 3 \rightarrow x^2 - m^2x^2 - 2mbx - b^2 = 3 \rightarrow (1-m^2)x^2 - 2mbx - b^2 - 3 = 0$$
Note that the tangent line passes through (2, 1). Find the relation between m and b:
$$1 = m(2) + b$$
$$b = 1 - 2m$$
Substitute into the quadratic to eliminate b:
$$(1-m^2)x^2 - 2m(1-2m)x - (1-2m)^2 - 3 = 0$$
$$(1-m^2)x^2 + (-2m + 4m^2)x - 1 + 4m - 4m^2 - 3 = 0$$
$$(1-m^2)x^2 + (-2m + 4m^2)x + (-4m^2 + 4m - 4) = 0$$
Find when the discriminant is 0:
$$(-2m + 4m^2)^2 - 4(1-m^2)(-4m^2 + 4m - 4) = 0$$
$$4m^2 - 16m^3 + 16m^4 - 4(4m^4 - 4m^3 + 4m - 4) = 0$$
$$4m^2 - 16m^3 + 16m^4 - 16m^4 + 16m^3 - 16m + 16 = 0$$

$$4m^2 - 16m + 16 = 0 \rightarrow m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0 \rightarrow m - 2 = 0 \rightarrow m = 2 \rightarrow b = 1 - 2(2) = -3$$

The equation of the tangent line is $y = 2x - 3$.

92. Solve the system:

$$\begin{cases} 2y^2 - x^2 = 14 \\ y = mx + b \end{cases}$$

Solve the system by substitution:

$$2(mx+b)^2 - x^2 = 14 \rightarrow 2m^2x^2 + 4mbx + 2b^2 - x^2 = 14$$

$$(2m^2 - 1)x^2 + 4mbx + 2b^2 - 14 = 0$$

Note that the tangent line passes through (2, 3). Find the relation between m and b:

$$3 = m(2) + b$$
$$b = 3 - 2m$$

There is one solution to the quadratic if the discriminant is zero.

$$(4bm)^2 - 4(2m^2 - 1)(2b^2 - 14) = 0$$

$$16b^2m^2 - 16b^2m^2 + 112m^2 + 8b^2 - 56 = 0 \rightarrow 112m^2 + 8b^2 - 56 = 0 \rightarrow 14m^2 + b^2 - 7 = 0$$

Substitute for b and solve:

$$14m^2 + (3 - 2m)^2 - 7 = 0$$

$$14m^2 + 4m^2 - 12m + 9 - 7 = 0 \rightarrow 18m^2 - 12m + 2 = 0 \rightarrow 9m^2 - 6m + 1 = 0$$

$$(3m-1)^2 = 0 \rightarrow m = \frac{1}{3} \rightarrow b = 3 - 2\left(\frac{1}{3}\right) = \frac{7}{3}$$

The equation of the tangent line is $y = \frac{1}{3}x + \frac{7}{3}$.

93. Solve for r_1 and r_2:

$$\begin{cases} r_1 + r_2 = -\dfrac{b}{a} \\ r_1 r_2 = \dfrac{c}{a} \end{cases}$$

Substitute and solve:

$$r_1 = -r_2 - \frac{b}{a} \rightarrow \left(-r_2 - \frac{b}{a}\right) r_2 = \frac{c}{a}$$

$$-r_2^2 - \frac{b}{a} r_2 - \frac{c}{a} = 0 \rightarrow a r_2^2 + b r_2 + c = 0$$

$$r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r_1 = -r_2 - \frac{b}{a} = -\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right) - \frac{2b}{2a} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

The solutions are: $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

94. Consider the circle with equation $(x-h)^2+(y-k)^2=r^2$ and the third degree polynomial with equation $y=ax^3+bx^2+cx+d$.

Substituting the first equation into the first equation yields

$$(x-h)^2+\left(ax^3+bx^2+cx+d-k\right)^2=r^2.$$

In order to find the roots for this equation we can expand the terms on the left hand side of the equation.

Notice that $(x-h)^2$ yields a 2nd degree polynomial, and $\left(ax^3+bx^2+cx+d-k\right)^2$ yields a 6th degree polynomial.

Therefore, we need to find the roots of a 6th degree equation, and the Fundamental Theorem of Algebra states that there will be at most six real roots. Thus, the circle and the 3rd degree polynomial will intersect at most six times.

Now consider the circle with equation $(x-h)^2+(y-k)^2=r^2$ and the polynomial of degree *n* with equation $y=a_0+a_1x+a_2x^2+a_3x^3+...+a_nx^n$.

Substituting the first equation into the first equation yields

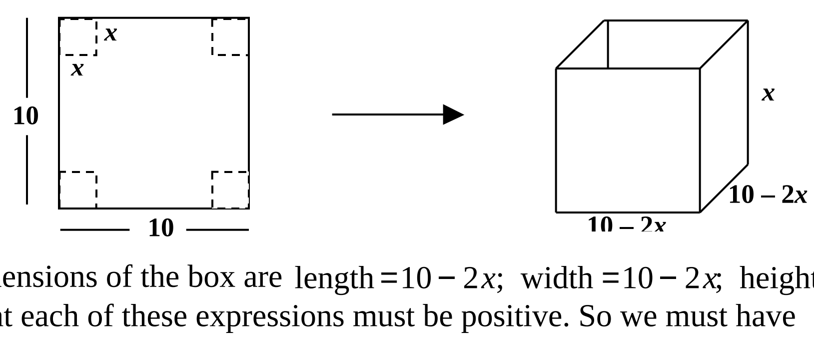
$$(x-h)^2+\left(a_0+a_1x+a_2x^2+a_3x^3+...+a_nx^n\right)^2=r^2.$$

In order to find the roots for this equation we can expand the terms on the left hand side of the equation.

Notice that $(x-h)^2$ yields a 2nd degree polynomial, and $\left(a_0+a_1x+a_2x^2+a_3x^3+...+a_nx^n\right)^2$ yields a polynomial of degree *2n*.

Therefore, we need to find the roots of an equation of degree *2n*, and the Fundamental Theorem of Algebra states that there will be at most *2n* real roots. Thus, the circle and the *n*th degree polynomial will intersect at most *2n* times.

95. Since the area of the square piece of sheet metal is 100 square feet, the sheet's dimensions are 10 feet by 10 feet. Let $x=$ the length of the cut.



The dimensions of the box are $\text{length}=10-2x;\ \text{width}=10-2x;\ \text{height}=x$

Note that each of these expressions must be positive. So we must have

$$x>0 \text{ and } 10-2x>0 \rightarrow x<5, \text{ that is, } 0<x<5.$$

So the volume of the box is given by

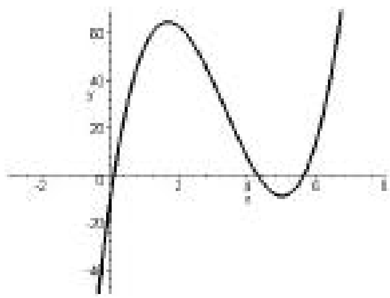
$$V = (length)\ (width)\ (height) = (10-2x)(10-2x)(x) = (10-2x)^2(x)$$

(a) In order to get a volume equal to 9 cubic feet, we solve $(10-2x)^2(x) = 9$.

$$(10-2x)^2(x) = 9 \rightarrow (100-40x+4x^2)x = 9 \rightarrow 100x - 40x^2 + 4x^3 = 9$$

So we need to solve the equation $4x^3 - 40x^2 + 100x - 9 = 0$.

Graphing the function $y_1 = 4x^3 - 40x^2 + 100x - 9$ on a calculator yields the graph



The graph indicates that there three real zeros on the interval [0,6].

Using the ZERO feature of a graphing calculator, we find that the three roots shown occur at $x \approx 0.09$, $x \approx 4.27$ and $x \approx 5.63$.

But we've already noted that we must have $0 < x < 5$, so the only practical values for the cut are $x \approx 0.09$ feet and $x \approx 4.27$ feet.

(b) If the sheet metal has dimensions *k* feet by *k* feet, then the volume equation becomes

$$V = (k-2x)(k-2x)(x) = (k-2x)^2(x) = 9$$

Solving for *k* we get the quadratic equation

$$xk^2 - 4x^2k + 4x^3 - 9 = 0$$

$$k = \frac{-(-4x^2) \pm \sqrt{(-4x^2)^2 - 4(x)(4x^3-9)}}{2x} = \frac{4x^2 \pm \sqrt{16x^4 - 16x^4 + 36x}}{2x}$$

$$= \frac{4x^2 \pm \sqrt{36x}}{2x} = \frac{4x^2 \pm 6\sqrt{x}}{2x}$$

Therefore, we get a real solution for k provided $x \geq 0$ and $4x^2 \pm 6\sqrt{x} \geq 0$.

$$4x^2 \pm 6\sqrt{x} \geq 0 \rightarrow 4x^2 \geq 6\sqrt{x} \rightarrow 16x^4 \geq 36x$$

$$16x^4 - 36x \geq 0 \rightarrow 4x(4x^3 - 9) \geq 0$$

This last inequality holds provided $x \geq \sqrt[3]{\frac{9}{4}}$.