Chapter 13

Sequences; Induction; The Binomial Theorem

13.1 Sequences

1. $a_1 = 1, \quad a_2 = 2, \quad a_3 = 3, \quad a_4 = 4, \quad a_5 = 5$

2. $a_1 = 2, \quad a_2 = 5, \quad a_3 = 10, \quad a_4 = 17, \quad a_5 = 26$

3. $a_1 = \frac{1}{1+2} = \frac{1}{3}, \quad a_2 = \frac{2}{2+2} = \frac{2}{4} = \frac{1}{2}, \quad a_3 = \frac{3}{3+2} = \frac{3}{5}, \quad a_4 = \frac{4}{4+2} = \frac{4}{6} = \frac{2}{3},$
$a_5 = \frac{5}{5+2} = \frac{5}{7}$

4. $a_1 = \frac{2(1)+1}{2(1)} = \frac{3}{2}, \quad a_2 = \frac{2(2)+1}{2(2)} = \frac{5}{4}, \quad a_3 = \frac{2(3)+1}{2(3)} = \frac{7}{6}, \quad a_4 = \frac{2(4)+1}{2(4)} = \frac{9}{8},$
$a_5 = \frac{2(5)+1}{2(5)} = \frac{11}{10}$

5. $a_1 = (-1)^{1+1}(1^2) = 1, \quad a_2 = (-1)^{2+1}(2^2) = -4, \quad a_3 = (-1)^{3+1}(3^2) = 9,$
$a_4 = (-1)^{4+1}(4^2) = -16, \quad a_5 = (-1)^{5+1}(5^2) = 25$

6. $a_1 = (-1)^{1-1}\left(\frac{1}{2(1)-1}\right) = 1, \quad a_2 = (-1)^{2-1}\left(\frac{2}{2(2)-1}\right) = -\frac{2}{3}, \quad a_3 = (-1)^{3-1}\left(\frac{3}{2(3)-1}\right) = \frac{3}{5},$
$a_4 = (-1)^{4-1}\left(\frac{4}{2(4)-1}\right) = -\frac{4}{7}, \quad a_5 = (-1)^{5-1}\left(\frac{5}{2(5)-1}\right) = \frac{5}{9}$

7. $a_1 = \frac{2^1}{3^1+1} = \frac{2}{4} = \frac{1}{2}, \quad a_2 = \frac{2^2}{3^2+1} = \frac{4}{10} = \frac{2}{5}, \quad a_3 = \frac{2^3}{3^3+1} = \frac{8}{28} = \frac{2}{7},$
$a_4 = \frac{2^4}{3^4+1} = \frac{16}{82} = \frac{8}{41}, \quad a_5 = \frac{2^5}{3^5+1} = \frac{32}{244} = \frac{8}{61}$

8. $a_1 = \left(\frac{4}{3}\right)^1 = \frac{4}{3}, \quad a_2 = \left(\frac{4}{3}\right)^2 = \frac{16}{9}, \quad a_3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27}, \quad a_4 = \left(\frac{4}{3}\right)^4 = \frac{256}{81},$
$a_5 = \left(\frac{4}{3}\right)^5 = \frac{1024}{243}$

9. $a_1 = \frac{(-1)^1}{(1+1)(1+2)} = \frac{-1}{2 \cdot 3} = \frac{-1}{6}$, $a_2 = \frac{(-1)^2}{(2+1)(2+2)} = \frac{1}{3 \cdot 4} = \frac{1}{12}$,

$a_3 = \frac{(-1)^3}{(3+1)(3+2)} = \frac{-1}{4 \cdot 5} = \frac{-1}{20}$, $a_4 = \frac{(-1)^4}{(4+1)(4+2)} = \frac{1}{5 \cdot 6} = \frac{1}{30}$,

$a_5 = \frac{(-1)^5}{(5+1)(5+2)} = \frac{-1}{6 \cdot 7} = -\frac{1}{42}$

10. $a_1 = \frac{3^1}{1} = \frac{3}{1} = 3$, $a_2 = \frac{3^2}{2} = \frac{9}{2}$, $a_3 = \frac{3^3}{3} = \frac{27}{3} = 9$, $a_4 = \frac{3^4}{4} = \frac{81}{4}$, $a_5 = \frac{3^5}{5} = \frac{243}{5}$

11. $a_1 = \frac{1}{e^1} = \frac{1}{e}$, $a_2 = \frac{2}{e^2}$, $a_3 = \frac{3}{e^3}$, $a_4 = \frac{4}{e^4}$, $a_5 = \frac{5}{e^5}$

12. $a_1 = \frac{1^2}{2^1} = \frac{1}{2}$, $a_2 = \frac{2^2}{2^2} = 1$, $a_3 = \frac{3^2}{2^3} = \frac{9}{8}$, $a_4 = \frac{4^2}{2^4} = \frac{16}{16} = 1$, $a_5 = \frac{5^2}{2^5} = \frac{25}{32}$

13. $\frac{n}{n+1}$

14. $\frac{1}{n(n+1)}$

15. $\frac{1}{2^{n-1}}$

16. $\left(\frac{2}{3}\right)^n$

17. $(-1)^{n+1}$

18. $\left(\frac{1}{n}\right)^{(-1)^n}$

19. $(-1)^{n+1} n$

20. $(-1)^{n+1} 2n$

21. $a_1 = 2$, $a_2 = 3 + 2 = 5$, $a_3 = 3 + 5 = 8$, $a_4 = 3 + 8 = 11$, $a_5 = 3 + 11 = 14$

22. $a_1 = 3$, $a_2 = 4 - 3 = 1$, $a_3 = 4 - 1 = 3$, $a_4 = 4 - 3 = 1$, $a_5 = 4 - 1 = 3$

23. $a_1 = -2$, $a_2 = 2 + (-2) = 0$, $a_3 = 3 + 0 = 3$, $a_4 = 4 + 3 = 7$, $a_5 = 5 + 7 = 12$

24. $a_1 = 1$, $a_2 = 2 - 1 = 1$, $a_3 = 3 - 1 = 2$, $a_4 = 4 - 2 = 2$, $a_5 = 5 - 2 = 3$

25. $a_1 = 5$, $a_2 = 2 \cdot 5 = 10$, $a_3 = 2 \cdot 10 = 20$, $a_4 = 2 \cdot 20 = 40$, $a_5 = 2 \cdot 40 = 80$

26. $a_1 = 2$, $a_2 = -2$, $a_3 = -(-2) = 2$, $a_4 = -2$, $a_5 = -(-2) = 2$

27. $a_1 = 3$, $a_2 = \frac{3}{2}$, $a_3 = \frac{\left(\frac{3}{2}\right)}{3} = \frac{1}{2}$, $a_4 = \frac{\left(\frac{1}{2}\right)}{4} = \frac{1}{8}$, $a_5 = \frac{\left(\frac{1}{8}\right)}{5} = \frac{1}{40}$

28. $a_1 = -2$, $a_2 = 2 + 3(-2) = -4$, $a_3 = 3 + 3(-4) = -9$, $a_4 = 4 + 3(-9) = -23$,
$a_5 = 5 + 3(-23) = -64$

29. $a_1 = 1$, $a_2 = 2$, $a_3 = 2 \cdot 1 = 2$, $a_4 = 2 \cdot 2 = 4$, $a_5 = 4 \cdot 2 = 8$

30. $a_1 = -1,\ \ a_2 = 1,\ \ a_3 = -1 + 3\ \ 1 = 2,\ \ a_4 = 1 + 4\ \ 2 = 9,\ \ a_5 = 2 + 5\ \ 9 = 47$

31. $a_1 = A,\ \ a_2 = A + d,\ \ a_3 = (A + d) + d = A + 2d,\ \ a_4 = (A + 2d) + d = A + 3d,$
$a_5 = (A + 3d) + d = A + 4d$

32. $a_1 = A,\ \ a_2 = rA,\ \ a_3 = r(rA) = r^2 A,\ \ a_4 = r\left(r^2 A\right) = r^3 A,\ \ a_5 = r\left(r^3 A\right) = r^4 A$

33. $a_1 = \sqrt{2},\ \ a_2 = \sqrt{2 + \sqrt{2}},\ \ a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}},\ \ a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}},$
$a_5 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}$

34. $a_1 = \sqrt{2},\ \ a_2 = \sqrt{\dfrac{2^{\frac{1}{2}}}{2}} = \left(2^{-1/2}\right)^{1/2} = 2^{-1/4} = \dfrac{1}{2^{1/4}},\ \ a_3 = \sqrt{\dfrac{2^{-1/4}}{2}} = \left(2^{-5/4}\right)^{1/2} = 2^{-5/8} = \dfrac{1}{2^{5/8}},$

$a_4 = \sqrt{\dfrac{2^{-5/8}}{2}} = \left(2^{-13/8}\right)^{1/2} = 2^{-13/16} = \dfrac{1}{2^{13/16}},\ \ a_5 = \sqrt{\dfrac{2^{-13/16}}{2}} = \left(2^{-29/16}\right)^{1/2} = 2^{-29/32} = \dfrac{1}{2^{29/32}}$

35. $\sum_{k=1}^{10} 5 = \underbrace{5 + 5 + 5 + \ldots + 5}_{\text{10 times}} = 50$

36. $\sum_{k=1}^{20} 8 = \underbrace{8 + 8 + 8 + \ldots + 8}_{\text{20 times}} = 20(8) = 160$

37. $\sum_{k=1}^{6} k = 1 + 2 + 3 + 4 + 5 + 6 = 21$

38. $\sum_{k=1}^{4} (-k) = (-1) + (-2) + (-3) + (-4) = -10$

39. $\sum_{k=1}^{5} (5k + 3) = 8 + 13 + 18 + 23 + 28 = 90$

40. $\sum_{k=1}^{6} (3k - 7) = -4 + (-1) + 2 + 5 + 8 + 11 = 21$

41. $\sum_{k=1}^{3} (k^2 + 4) = 5 + 8 + 13 = 26$

42. $\sum_{k=0}^{4} (k^2 - 4) = -4 + (-3) + 0 + 5 + 12 = 10$

43. $\displaystyle\sum_{k=1}^{6}(-1)^k 2^k = (-1)^1\ 2^1 + (-1)^2\ 2^2 + (-1)^3\ 2^3 + (-1)^4\ 2^4 + (-1)^5\ 2^5 + (-1)^6\ 2^6$

$= -2 + 4 - 8 + 16 - 32 + 64 = 42$

44. $\displaystyle\sum_{k=1}^{4}(-1)^k 3^k = (-1)^1\ 3^1 + (-1)^2\ 3^2 + (-1)^3\ 3^3 + (-1)^4\ 3^4 = -3 + 9 - 27 + 81 = 60$

45. $\displaystyle\sum_{k=1}^{4}(k^3 - 1) = 0 + 7 + 26 + 63 = 96$

46. $\displaystyle\sum_{k=0}^{3}(k^3 + 2) = 2 + 3 + 10 + 29 = 44$

47. $\displaystyle\sum_{k=1}^{n}(k+2) = 3 + 4 + 5 + 6 + \cdots + (n+2)$

48. $\displaystyle\sum_{k=1}^{n}(2k+1) = 3 + 5 + 7 + 9 + \cdots + (2n+1)$

49. $\displaystyle\sum_{k=1}^{n}\frac{k^2}{2} = \frac{1}{2} + 2 + \frac{9}{2} + 8 + \frac{25}{2} + \cdots + \frac{n^2}{2}$

50. $\displaystyle\sum_{k=1}^{n}(k+1)^2 = 2^2 + 3^2 + 4^2 + \cdots + (n+1)^2 = 4 + 9 + 16 + \cdots + (n+1)^2$

51. $\displaystyle\sum_{k=0}^{n}\frac{1}{3^k} = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n}$

52. $\displaystyle\sum_{k=0}^{n}\left(\frac{3}{2}\right)^k = \left(\frac{3}{2}\right)^0 + \left(\frac{3}{2}\right)^1 + \left(\frac{3}{2}\right)^2 + \cdots + \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \cdots + \left(\frac{3}{2}\right)^n$

53. $\displaystyle\sum_{k=0}^{n-1}\frac{1}{3^{k+1}} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n}$

54. $\displaystyle\sum_{k=0}^{n-1}(2k+1) = 1 + 3 + 5 + 7 + \cdots + \big(2(n-1)+1\big) = 1 + 3 + 5 + 7 + \cdots + (2n-1)$

55. $\displaystyle\sum_{k=2}^{n}(-1)^k \ln k = \ln 2 - \ln 3 + \ln 4 - \ln 5 + \cdots + (-1)^n \ln n$

56. $\displaystyle\sum_{k=3}^{n}(-1)^{k+1} 2^k = (-1)^4 2^3 + (-1)^5 2^4 + (-1)^6 2^5 + \cdots + (-1)^{n+1} 2^n$

$= 2^3 - 2^4 + 2^5 - 2^6 + \cdots + (-1)^{n+1} 2^n$

57. $1+2+3+\cdots+20=\sum_{k=1}^{20} k$

58. $1^3+2^3+3^3+\cdots+8^3=\sum_{k=1}^{8} k^3$

59. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots+\frac{13}{13+1}=\sum_{k=1}^{13}\frac{k}{k+1}$

60. $1+3+5+7+\cdots+\left[2(12)-1\right]=\sum_{k=1}^{12}(2k-1)$

61. $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\cdots+(-1)^6\left(\frac{1}{3^6}\right)=\sum_{k=0}^{6}(-1)^k\left(\frac{1}{3^k}\right)$

62. $\frac{2}{3}-\frac{4}{9}+\frac{8}{27}+\cdots+(-1)^{11+1}\left(\frac{2}{3}\right)^{11}=\sum_{k=1}^{11}(-1)^{k+1}\left(\frac{2}{3}\right)^k$

63. $3+\frac{3^2}{2}+\frac{3^3}{3}+\cdots+\frac{3^n}{n}=\sum_{k=1}^{n}\frac{3^k}{k}$

64. $\frac{1}{e}+\frac{2}{e^2}+\frac{3}{e^3}+\cdots+\frac{n}{e^n}=\sum_{k=1}^{n}\frac{k}{e^k}$

65. $a+(a+d)+(a+2d)+\cdots+(a+nd)=\sum_{k=0}^{n}(a+kd)$

66. $a+ar+ar^2+\cdots+ar^{n-1}=\sum_{k=1}^{n}ar^{k-1}$

67. $B_1=1.01(3000)-100=\$2930$

68. $B_1=1.005(18500)-534.47=\$18{,}058.03$

69. $p_1=1.03(2000)+20=2080;\qquad p_2=1.03(2080)+20=2162.4$

70. $p_1=0.9(250)+15=240;\qquad p_2=0.9(240)+15=231$

71. $a_1=1,\ a_2=1,\ a_3=2,\ a_4=3,\ a_5=5,\ a_6=8,\ a_7=13,\ a_8=21,\ a_n=a_{n-1}+a_{n-2}$
$a_8=a_7+a_6=13+8=21$
After 7 months there are 21 mature pairs of rabbits.

72. (a) $u_1 = \dfrac{\left(1+\sqrt{5}\right)^1 - \left(1-\sqrt{5}\right)^1}{2^1\sqrt{5}} = \dfrac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}} = \dfrac{2\sqrt{5}}{2\sqrt{5}} = 1$

$u_2 = \dfrac{\left(1+\sqrt{5}\right)^2 - \left(1-\sqrt{5}\right)^2}{2^2\sqrt{5}} = \dfrac{1+2\sqrt{5}+5-1+2\sqrt{5}-5}{4\sqrt{5}} = \dfrac{4\sqrt{5}}{4\sqrt{5}} = 1$

(b) $$u_{n+1}+u_n = \frac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}\sqrt{5}} + \frac{\left(1+\sqrt{5}\right)^{n} - \left(1-\sqrt{5}\right)^{n}}{2^{n}\sqrt{5}}$$

$$= \frac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1} + 2\left(1+\sqrt{5}\right)^{n} - 2\left(1-\sqrt{5}\right)^{n}}{2^{n+1}\sqrt{5}}$$

$$= \frac{\left(1+\sqrt{5}\right)^{n}\left[1+\sqrt{5}+2\right] - \left(1-\sqrt{5}\right)^{n}\left[1-\sqrt{5}+2\right]}{2^{n+1}\sqrt{5}}$$

$$= \frac{\left(1+\sqrt{5}\right)^{n}\left[3+\sqrt{5}\right] - \left(1-\sqrt{5}\right)^{n}\left[3-\sqrt{5}\right]}{2^{n+1}\sqrt{5}}$$

$$= \frac{\left(1+\sqrt{5}\right)^{n+2}\dfrac{\left(3+\sqrt{5}\right)}{\left(1+\sqrt{5}\right)^2} - \left(1-\sqrt{5}\right)^{n+2}\dfrac{\left(3-\sqrt{5}\right)}{\left(1-\sqrt{5}\right)^2}}{2^{n+1}\sqrt{5}}$$

$$= \frac{\left(1+\sqrt{5}\right)^{n+2}\dfrac{\left(3+\sqrt{5}\right)}{\left(6+2\sqrt{5}\right)} - \left(1-\sqrt{5}\right)^{n+2}\dfrac{\left(3-\sqrt{5}\right)}{\left(6-2\sqrt{5}\right)}}{2^{n+1}\sqrt{5}}$$

$$= \frac{\left(1+\sqrt{5}\right)^{n+2}\dfrac{1}{2} - \left(1-\sqrt{5}\right)^{n+2}\dfrac{1}{2}}{2^{n+1}\sqrt{5}} = \frac{\left(1+\sqrt{5}\right)^{n+2} - \left(1-\sqrt{5}\right)^{n+2}}{2^{n+2}\sqrt{5}}$$

$$= u_{n+2}$$

(c) Since $u_1 = 1$, $u_2 = 1$, $u_{n+2} = u_{n+1} + u_n$, $\{u_n\}$ is the Fibonacci sequence.

73. 1, 1, 2, 3, 5, 8, 13 This is the Fibonacci sequence.

74. a) $u_1 = 1$, $u_2 = 1$, $u_3 = 2$, $u_4 = 3$, $u_5 = 5$, $u_6 = 8$, $u_7 = 13$, $u_8 = 21$, $u_9 = 34$, $u_{10} = 55$

b) $\dfrac{u_2}{u_1} = \dfrac{1}{1} = 1$, $\dfrac{u_3}{u_2} = \dfrac{2}{1} = 1$, $\dfrac{u_4}{u_3} = \dfrac{3}{2} = 1.5$, $\dfrac{u_5}{u_4} = \dfrac{5}{3} = 1.67$, $\dfrac{u_6}{u_5} = \dfrac{8}{5} = 1.6$,

$\dfrac{u_7}{u_6} = \dfrac{13}{8} = 1.625$, $\dfrac{u_8}{u_7} = \dfrac{21}{13} = 1.615$, $\dfrac{u_9}{u_8} = \dfrac{34}{21} = 1.619$,

$\dfrac{u_{10}}{u_9} = \dfrac{55}{34} = 1.618$, $\dfrac{u_{11}}{u_{10}} = \dfrac{89}{55} = 1.618$

c) 1.618

d) $\dfrac{u_1}{u_2} = \dfrac{1}{1} = 1$, $\dfrac{u_2}{u_3} = \dfrac{1}{2} = 0.5$, $\dfrac{u_3}{u_4} = \dfrac{2}{3} = 0.667$, $\dfrac{u_4}{u_5} = \dfrac{3}{5} = 0.6$,

$$\frac{u_5}{u_6} = \frac{5}{8} = 0.625, \quad \frac{u_6}{u_7} = \frac{8}{13} = 0.615, \quad \frac{u_7}{u_8} = \frac{13}{21} = 0.619,$$

$$\frac{u_8}{u_9} = \frac{21}{34} = 0.618, \quad \frac{u_9}{u_{10}} = \frac{34}{55} = 0.618, \quad \frac{u_{10}}{u_{11}} = \frac{55}{89} = 0.618$$

e) 0.618

75. To show that $1 + 2 + 3 + \ldots + (n-1) + n = \dfrac{n(n+1)}{2}$

Let

$S = 1 + 2 + 3 + \ldots\ldots\ldots\ldots + (n-1) + n$, we can reverse the order to get

$+S = n + (n-1) + (n-2) + \ldots + 2 + 1$, now add these two lines to get

$$2S = \underbrace{[1 + n] + [2 + (n-1)] + [3 + (n-2)] + \ldots\ldots + [(n-1) + 2] + [n + 1]}_{\text{n terms}}$$

So we have

$$2S = \underbrace{[1 + n] + [1 + n] + [1 + n] + \ldots. + [n + 1] + [n + 1]}_{\text{n terms}}$$

$$2S = n\ [n+1] \rightarrow S = \frac{n\ (n+1)}{2}$$

76. Answers will vary.