Chapter 13

Sequences; Induction; The Binomial Theorem

13.2 Arithmetic Sequences

1. $d = a_{n+1} - a_n = (n+1+4) - (n+4) = n+5-n-4 = 1$
$a_1 = 1+4 = 5,\ \ a_2 = 2+4 = 6,\ \ a_3 = 3+4 = 7,\ \ a_4 = 4+4 = 8$

2. $d = a_{n+1} - a_n = (n+1-5) - (n-5) = n-4-n+5 = 1$
$a_1 = 1-5 = -4,\ \ a_2 = 2-5 = -3,\ \ a_3 = 3-5 = -2,\ \ a_4 = 4-5 = -1$

3. $d = a_{n+1} - a_n = \big(2(n+1)-5\big) - (2n-5) = 2n+2-5-2n+5 = 2$
$a_1 = 2\cdot 1-5 = -3,\ \ a_2 = 2\cdot 2-5 = -1,\ \ a_3 = 2\cdot 3-5 = 1,\ \ a_4 = 2\cdot 4-5 = 3$

4. $d = a_{n+1} - a_n = \big(3(n+1)+1\big) - (3n+1) = 3n+3+1-3n-1 = 3$
$a_1 = 3\cdot 1+1 = 4,\ \ a_2 = 3\cdot 2+1 = 7,\ \ a_3 = 3\cdot 3+1 = 10,\ \ a_4 = 3\cdot 4+1 = 13$

5. $d = a_{n+1} - a_n = \big(6-2(n+1)\big) - (6-2n) = 6-2n-2-6+2n = -2$
$a_1 = 6-2\cdot 1 = 4,\ \ a_2 = 6-2\cdot 2 = 2,\ \ a_3 = 6-2\cdot 3 = 0,\ \ a_4 = 6-2\cdot 4 = -2$

6. $d = a_{n+1} - a_n = \big(4-2(n+1)\big) - (4-2n) = 4-2n-2-4+2n = -2$
$a_1 = 4-2\cdot 1 = 2,\ \ a_2 = 4-2\cdot 2 = 0,\ \ a_3 = 4-2\cdot 3 = -2,\ \ a_4 = 4-2\cdot 4 = -4$

7. $d = a_{n+1} - a_n = \left(\frac{1}{2} - \frac{1}{3}(n+1)\right) - \left(\frac{1}{2} - \frac{1}{3}n\right) = \frac{1}{2} - \frac{1}{3}n - \frac{1}{3} - \frac{1}{2} + \frac{1}{3}n = -\frac{1}{3}$
$a_1 = \frac{1}{2} - \frac{1}{3}\cdot 1 = \frac{1}{6},\ \ a_2 = \frac{1}{2} - \frac{1}{3}\cdot 2 = -\frac{1}{6},\ \ a_3 = \frac{1}{2} - \frac{1}{3}\cdot 3 = -\frac{1}{2},\ \ a_4 = \frac{1}{2} - \frac{1}{3}\cdot 4 = -\frac{5}{6}$

8. $d = a_{n+1} - a_n = \left(\frac{2}{3} + \frac{1}{4}(n+1)\right) - \left(\frac{2}{3} + \frac{1}{4}n\right) = \frac{2}{3} + \frac{1}{4}n + \frac{1}{4} - \frac{2}{3} - \frac{1}{4}n = \frac{1}{4}$
$a_1 = \frac{2}{3} + \frac{1}{4}\cdot 1 = \frac{11}{12},\ \ a_2 = \frac{2}{3} + \frac{1}{4}\cdot 2 = \frac{7}{6},\ \ a_3 = \frac{2}{3} + \frac{1}{4}\cdot 3 = \frac{17}{12},\ \ a_4 = \frac{2}{3} + \frac{1}{4}\cdot 4 = \frac{5}{3}$

9. $d = a_{n+1} - a_n = \ln 3^{n+1} - \ln 3^n = (n+1)\ln 3 - n\ln 3 = \ln 3(n+1-n) = \ln 3$
$a_1 = \ln 3^1 = \ln 3,\ \ a_2 = \ln 3^2 = 2\ln 3,\ \ a_3 = \ln 3^3 = 3\ln 3,\ \ a_4 = \ln 3^4 = 4\ln 3$

10. $d = a_{n+1} - a_n = e^{\ln(n+1)} - e^{\ln n} = (n+1) - n = 1$
$a_1 = e^{\ln 1} = 1,\ \ a_2 = e^{\ln 2} = 2,\ \ a_3 = e^{\ln 3} = 3,\ \ a_4 = e^{\ln 4} = 4$

11. $a_n = a + (n-1)d = 2 + (n-1)3 = 2 + 3n - 3 = 3n - 1$
$a_5 = 3 \; 5 - 1 = 14$

12. $a_n = a + (n-1)d = -2 + (n-1)4 = -2 + 4n - 4 = 4n - 6$
$a_5 = 4 \; 5 - 6 = 14$

13. $a_n = a + (n-1)d = 5 + (n-1)(-3) = 5 - 3n + 3 = 8 - 3n$
$a_5 = 8 - 3 \; 5 = -7$

14. $a_n = a + (n-1)d = 6 + (n-1)(-2) = 6 - 2n + 2 = 8 - 2n$
$a_5 = 8 - 2 \; 5 = -2$

15. $a_n = a + (n-1)d = 0 + (n-1)\frac{1}{2} = \frac{1}{2}n - \frac{1}{2}$
$a_5 = \frac{1}{2} \; 5 - \frac{1}{2} = 2$

16. $a_n = a + (n-1)d = 1 + (n-1)\left(-\frac{1}{3}\right) = 1 - \frac{1}{3}n + \frac{1}{3} = \frac{4}{3} - \frac{1}{3}n$
$a_5 = \frac{4}{3} - \frac{1}{3} \; 5 = \frac{4}{3} - \frac{5}{3} = -\frac{1}{3}$

17. $a_n = a + (n-1)d = \sqrt{2} + (n-1)\sqrt{2} = \sqrt{2} + \sqrt{2}n - \sqrt{2} = \sqrt{2}n$
$a_5 = 5\sqrt{2}$

18. $a_n = a + (n-1)d = 0 + (n-1)\pi = \pi n - \pi$
$a_5 = 5\pi - \pi = 4\pi$

19. $a_1 = 2, \quad d = 2, \quad a_n = a + (n-1)d$
$a_{12} = 2 + (12-1)2 = 2 + 11(2) = 2 + 22 = 24$

20. $a_1 = -1, \quad d = 2, \quad a_n = a + (n-1)d$
$a_8 = -1 + (8-1)2 = -1 + 7(2) = -1 + 14 = 13$

21. $a_1 = 1, \quad d = -2 - 1 = -3, \quad a_n = a + (n-1)d$
$a_{10} = 1 + (10-1)(-3) = 1 + 9(-3) = 1 - 27 = -26$

22. $a_1 = 5, \quad d = 0 - 5 = -5, \quad a_n = a + (n-1)d$
$a_9 = 5 + (9-1)(-5) = 5 + 8(-5) = 5 - 40 = -35$

23. $a_1 = a, \quad d = (a+b) - a = b, \quad a_n = a + (n-1)d$
$a_8 = a + (8-1)b = a + 7b$

24. $a_1 = 2\sqrt{5}, \quad d = 4\sqrt{5} - 2\sqrt{5} = 2\sqrt{5}, \quad a_n = a + (n-1)d$
$a_7 = 2\sqrt{5} + (7-1)2\sqrt{5} = 2\sqrt{5} + 6\left(2\sqrt{5}\right) = 2\sqrt{5} + 12\sqrt{5} = 14\sqrt{5}$

25. $a_8 = a + 7d = 8 \qquad a_{20} = a + 19d = 44$
Solve the system of equations:
$8 - 7d + 19d = 44$
$12d = 36 \rightarrow d = 3 \rightarrow a = 8 - 7(3) = 8 - 21 = -13$
Recursive formula: $a_1 = -13 \qquad a_n = a_{n-1} + 3$

26. $a_4 = a + 3d = 3 \qquad a_{20} = a + 19d = 35$
Solve the system of equations:
$3 - 3d + 19d = 35$
$16d = 32 \rightarrow d = 2 \rightarrow a = 3 - 3(2) = 3 - 6 = -3$
Recursive formula: $a_1 = -3 \qquad a_n = a_{n-1} + 2$

27. $a_9 = a + 8d = -5 \qquad a_{15} = a + 14d = 31$
Solve the system of equations:
$-5 - 8d + 14d = 31 \rightarrow 6d = 36 \rightarrow d = 6$
$a = -5 - 8(6) = -5 - 48 = -53$
Recursive formula: $a_1 = -53 \qquad a_n = a_{n-1} + 6$

28. $a_8 = a + 7d = 4 \qquad a_{18} = a + 17d = -96$
Solve the system of equations:
$4 - 7d + 17d = -96 \rightarrow 10d = -100 \rightarrow d = -10$
$a = 4 - 7(-10) = 4 + 70 = 74$
Recursive formula: $a_1 = 74 \qquad a_n = a_{n-1} - 10$

29. $a_{15} = a + 14d = 0 \qquad a_{40} = a + 39d = -50$
Solve the system of equations:
$-14d + 39d = -50 \rightarrow 25d = -50 \rightarrow d = -2$
$a = -14(-2) = 28$
Recursive formula: $a_1 = 28 \qquad a_n = a_{n-1} - 2$

30. $a_5 = a + 4d = -2 \qquad a_{13} = a + 12d = 30$
Solve the system of equations:
$-2 - 4d + 12d = 30 \rightarrow 8d = 32 \rightarrow d = 4$
$a = -2 - 4(4) = -18$
Recursive formula: $a_1 = -18 \qquad a_n = a_{n-1} + 4$

31. $a_{14} = a + 13d = -1 \qquad a_{18} = a + 17d = -9$

Solve the system of equations:

$$-1 - 13d + 17d = -9$$
$$4d = -8 \rightarrow d = -2 \rightarrow a = -1 - 13(-2) = -1 + 26 = 25$$

Recursive formula: $a_1 = 25 \qquad a_n = a_{n-1} - 2$

32. $a_{12} = a + 11d = 4 \qquad a_{18} = a + 17d = 28$

Solve the system of equations:

$$4 - 11d + 17d = 28$$
$$6d = 24 \rightarrow d = 4 \rightarrow a = 4 - 11(4) = 4 - 44 = -40$$

Recursive formula: $a_1 = -40 \qquad a_n = a_{n-1} + 4$

33. $S_n = \frac{n}{2}(a + a_n) = \frac{n}{2}(1 + (2n - 1)) = \frac{n}{2}(2n) = n^2$

34. $S_n = \frac{n}{2}(a + a_n) = \frac{n}{2}(2 + 2n) = n + n^2$

35. $S_n = \frac{n}{2}(a + a_n) = \frac{n}{2}(7 + (2 + 5n)) = \frac{n}{2}(9 + 5n) = \frac{9}{2}n + \frac{5}{2}n^2$

36. $S_n = \frac{n}{2}(a + a_n) = \frac{n}{2}(-1 + (4n - 5)) = \frac{n}{2}(4n - 6) = 2n^2 - 3n$

37. $a_1 = 2, \quad d = 4 - 2 = 2, \quad a_n = a + (n - 1)d$

$$70 = 2 + (n - 1)2 \rightarrow 70 = 2 + 2n - 2 \rightarrow 70 = 2n \rightarrow n = 35$$
$$S_n = \frac{n}{2}(a + a_n) = \frac{35}{2}(2 + 70) = \frac{35}{2}(72) = 35(36) = 1260$$

38. $a_1 = 1, \quad d = 3 - 1 = 2, \quad a_n = a + (n - 1)d$

$$59 = 1 + (n - 1)2 \rightarrow 59 = 1 + 2n - 2 \rightarrow 60 = 2n \rightarrow n = 30$$
$$S_n = \frac{n}{2}(a + a_n) = \frac{30}{2}(1 + 59) = 15(60) = 900$$

39. $a_1 = 5, \quad d = 9 - 5 = 4, \quad a_n = a + (n - 1)d$

$$49 = 5 + (n - 1)4 \rightarrow 49 = 5 + 4n - 4 \rightarrow 48 = 4n \rightarrow n = 12$$
$$S_n = \frac{n}{2}(a + a_n) = \frac{12}{2}(5 + 49) = 6(54) = 324$$

40. $a_1 = 2, \quad d = 5 - 2 = 3, \quad a_n = a + (n - 1)d$

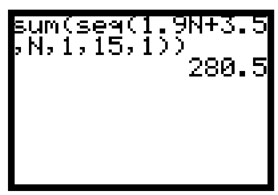
$$41 = 2 + (n - 1)3 \rightarrow 41 = 2 + 3n - 3 \rightarrow 42 = 3n \rightarrow n = 14$$
$$S_n = \frac{n}{2}(a + a_n) = \frac{14}{2}(2 + 41) = 7(43) = 301$$

41. Using the sum of the sequence feature:

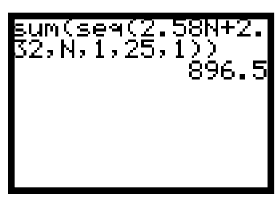
42. Using the sum of the sequence feature:

43.
$$\begin{aligned} d &= 5.2 - 2.8 = 2.4 \\ a &= 2.8 \\ 36.4 &= 2.8 + (n-1)2.4 \\ 36.4 &= 2.8 + 2.4n - 2.4 \\ 36 &= 2.4n \\ n &= 15 \\ a_n &= 2.8 + (n-1)2.4 = 2.8 + 2.4n - 2.4 = 2.4n + 0.4 \end{aligned}$$

44.
$$\begin{aligned} d &= 7.3 - 5.4 = 1.9 \\ a &= 5.4 \\ 32 &= 5.4 + (n-1)1.9 \\ 32 &= 5.4 + 1.9n - 1.9 \\ 28.5 &= 1.9n \\ n &= 15 \\ a_n &= 5.4 + (n-1)1.9 = 5.4 + 1.9n - 1.9 = 1.9n + 3.5 \end{aligned}$$



45.
$$\begin{aligned} d &= 7.48 - 4.9 = 2.58 \\ a &= 4.9 \\ 66.82 &= 4.9 + (n-1)2.58 \\ 66.82 &= 4.9 + 2.58n - 2.58 \\ 64.5 &= 2.58n \\ n &= 25 \\ a_n &= 4.9 + (n-1)2.58 = 4.9 + 2.58n - 2.58 \\ a_n &= 2.58n + 2.32 \end{aligned}$$



46. $d = 6.9 - 3.71 = 3.19$

$a = 3.71$

$80.27 = 3.71 + (n-1)3.19$

$80.27 = 3.71 + 3.19n - 3.19$

$79.75 = 3.19n$

$n = 25$

$a_n = 3.71 + (n-1)3.19 \quad = 3.71 + 3.19n - 3.19$

$a_n = 3.19n + 0.52$

47. Find the common difference of the terms and solve the system of equations:

$$(2x+1)-(x+3) = d \;\rightarrow\; x-2 = d$$
$$(5x+2)-(2x+1) = d \;\rightarrow\; 3x+1 = d$$
$$3x+1 = x-2 \rightarrow 2x = -3 \rightarrow x = -1.5$$

48. Find the common difference of the terms and solve the system of equations:

$$(3x+2)-(2x) = d \;\rightarrow\; x+2 = d$$
$$(5x+3)-(3x+2) = d \;\rightarrow\; 2x+1 = d$$
$$2x+1 = x+2 \rightarrow x = 1$$

49. The total number of seats is: $S = 25 + 26 + 27 + \cdots$

This is the sum of an arithmetic sequence with $d = 1$, $a = 25$, and $n = 30$.

Find the sum of the sequence:

$$S_{30} = \frac{30}{2}\left[2(25) + (30-1)(1)\right] = 15(50+29) = 15(79) = 1185$$

There are 1185 seats in the theater.

50. The total number of seats is: $S = 15 + 17 + 19 + \cdots$

This is the sum of an arithmetic sequence with $d = 2$, $a = 15$, and $n = 40$.

Find the sum of the sequence:

$$S_{40} = \frac{40}{2}\left[2(15) + (40-1)(2)\right] = 20(30+78) = 20(108) = 2160$$

The corner section has 2160 seats.

51. The lighter colored tiles have 20 tiles in the bottom row and 1 tile in the top row. The number decreases by 1 as we move up the triangle. This is an arithmetic sequence with $a_1 = 20$, $d = -1$ and $n = 20$. Find the sum:

$$S = \frac{20}{2}\left[2(20) + (20-1)(-1)\right] = 10(40-19) = 10(21) = 210 \text{ lighter tiles.}$$

The darker colored tiles have 19 tiles in the bottom row and 1 tile in the top row. The number decreases by 1 as we move up the triangle. This is an arithmetic sequence with $a_1 = 19$, $d = -1$, and $n = 19$. Find the sum:

$$S = \frac{19}{2}\left[2(19) + (19-1)(-1)\right] = \frac{19}{2}(38-18) = \frac{19}{2}(20) = 190 \text{ darker tiles.}$$

52. The number of bricks required decreases by 2 on each successive step. This is an arithmetic sequence with $a_1 = 100$, $d = -2$, and $n = 30$.

(a) The number of bricks for the top step is:

$$a_{30} = a_1 + (n-1)d = 100 + (30-1)(-2) = 100 + 29(-2) = 100 - 58 = 42$$

42 bricks are required for the top step.

(b) The total number of bricks required is the sum of the sequence:

$$S = \frac{30}{2}[100 + 42] = 15(142) = 2130$$

2130 bricks are required to build the staircase.

53. Find n in an arithmetic sequence with $a_1 = 10$, $d = 4$, $s_n = 2040$.

$$s_n = \frac{n}{2}[2a_1 + (n-1)d] \rightarrow 2040 = \frac{n}{2}[2(10) + (n-1)4]$$

$$4080 = n[20 + 4n - 4] \rightarrow 4080 = n(4n + 16)$$

$$4080 = 4n^2 + 16n \rightarrow 1020 = n^2 + 4n$$

$$n^2 + 4n - 1020 = 0 \rightarrow (n+34)(n-30) = 0 \rightarrow n = -34 \text{ or } n = 30$$

There are 30 rows in the corner section of the stadium.

54. The yearly salaries form an arithmetic sequence with:

$a_1 = 35000$, $d = 1400$, $s_n = 280{,}000$.

Find the number of years for the total salary to equal \$280,000.

$$s_n = \frac{n}{2}[2a_1 + (n-1)d]$$

$$280{,}000 = \frac{n}{2}[2(35000) + (n-1)1400] \rightarrow 280{,}000 = n[35000 + 700n - 700]$$

$$280{,}000 = n(700n + 34300)$$

$$280{,}000 = 700n^2 + 34300n \rightarrow 400 = n^2 + 49n \rightarrow n^2 + 49n - 400 = 0$$

$$n = \frac{-49 \pm \sqrt{49^2 - 4(1)(-400)}}{2(1)} = \frac{-49 \pm \sqrt{4001}}{2} \approx \frac{-49 \pm 63.25}{2}$$

$$n \approx 7.13 \text{ or } n \approx -56.13$$

It takes about 7.13 years to have an aggregate salary of \$280,000.

55. Answers will vary.