Chapter 13

Sequences; Induction; The Binomial Theorem

13.3 Geometric Sequences; Geometric Series

1. $r = \dfrac{3^{n+1}}{3^n} = 3^{n+1-n} = 3$

   $a_1 = 3^1 = 3,\ a_2 = 3^2 = 9,\ a_3 = 3^3 = 27,\ a_4 = 3^4 = 81$

2. $r = \dfrac{(-5)^{n+1}}{(-5)^n} = (-5)^{n+1-n} = -5$

   $a_1 = (-5)^1 = -5,\ a_2 = (-5)^2 = 25,\ a_3 = (-5)^3 = -125,\ a_4 = (-5)^4 = 625$

3. $r = \dfrac{-3\left(\frac{1}{2}\right)^{n+1}}{-3\left(\frac{1}{2}\right)^{n}} = \left(\dfrac{1}{2}\right)^{n+1-n} = \dfrac{1}{2}$

   $a_1 = -3\left(\frac{1}{2}\right)^1 = -\dfrac{3}{2},\ a_2 = -3\left(\frac{1}{2}\right)^2 = -\dfrac{3}{4},\ a_3 = -3\left(\frac{1}{2}\right)^3 = -\dfrac{3}{8},\ a_4 = -3\left(\frac{1}{2}\right)^4 = -\dfrac{3}{16}$

4. $r = \dfrac{\left(\frac{5}{2}\right)^{n+1}}{\left(\frac{5}{2}\right)^{n}} = \left(\dfrac{5}{2}\right)^{n+1-n} = \dfrac{5}{2}$

   $a_1 = \left(\frac{5}{2}\right)^1 = \dfrac{5}{2},\ a_2 = \left(\frac{5}{2}\right)^2 = \dfrac{25}{4},\ a_3 = \left(\frac{5}{2}\right)^3 = \dfrac{125}{8},\ a_4 = \left(\frac{5}{2}\right)^4 = \dfrac{625}{16}$

5. $r = \dfrac{\left(\frac{2^{n+1-1}}{4}\right)}{\left(\frac{2^{n-1}}{4}\right)} = \dfrac{2^n}{2^{n-1}} = 2^{n-(n-1)} = 2$

   $a_1 = \dfrac{2^{1-1}}{4} = \dfrac{2^0}{2^2} = 2^{-2} = \dfrac{1}{4},\ a_2 = \dfrac{2^{2-1}}{4} = \dfrac{2^1}{2^2} = 2^{-1} = \dfrac{1}{2},\ a_3 = \dfrac{2^{3-1}}{4} = \dfrac{2^2}{2^2} = 1,$

   $a_4 = \dfrac{2^{4-1}}{4} = \dfrac{2^3}{2^2} = 2$

6. $r = \dfrac{\left(\dfrac{3^{n+1}}{9}\right)}{\left(\dfrac{3^n}{9}\right)} = \dfrac{3^{n+1}}{3^n} = 3^{n+1-n} = 3$

$a_1 = \dfrac{3^1}{9} = \dfrac{1}{3}, \; a_2 = \dfrac{3^2}{9} = \dfrac{9}{9} = 1, \; a_3 = \dfrac{3^3}{9} = \dfrac{27}{9} = 3, \; a_4 = \dfrac{3^4}{9} = \dfrac{81}{9} = 9$

7. $r = \dfrac{2^{\left(\frac{n+1}{3}\right)}}{2^{\left(\frac{n}{3}\right)}} = 2^{\left(\frac{n+1}{3} - \frac{n}{3}\right)} = 2^{1/3}$

$a_1 = 2^{1/3}, \; a_2 = 2^{2/3}, \; a_3 = 2^{3/3} = 2, \; a_4 = 2^{4/3}$

8. $r = \dfrac{3^{2(n+1)}}{3^{2n}} = 3^{2n+2-2n} = 3^2 = 9$

$a_1 = 3^{2 \cdot 1} = 9, \; a_2 = 3^{2 \cdot 2} = 3^4 = 81, \; a_3 = 3^{2 \cdot 3} = 3^6 = 729, \; a_4 = 3^{2 \cdot 4} = 3^8 = 6561$

9. $r = \dfrac{\left(\dfrac{3^{n+1-1}}{2^{n+1}}\right)}{\left(\dfrac{3^{n-1}}{2^n}\right)} = \dfrac{3^n}{3^{n-1}} \cdot \dfrac{2^n}{2^{n+1}} = 3^{n-(n-1)} \cdot 2^{n-(n+1)} = 3 \cdot 2^{-1} = \dfrac{3}{2}$

$a_1 = \dfrac{3^{1-1}}{2^1} = \dfrac{3^0}{2} = \dfrac{1}{2}, \; a_2 = \dfrac{3^{2-1}}{2^2} = \dfrac{3^1}{2^2} = \dfrac{3}{4}, \; a_3 = \dfrac{3^{3-1}}{2^3} = \dfrac{3^2}{2^3} = \dfrac{9}{8},$

$a_4 = \dfrac{3^{4-1}}{2^4} = \dfrac{3^3}{2^4} = \dfrac{27}{16}$

10. $r = \dfrac{\left(\dfrac{2^{n+1}}{3^{n+1-1}}\right)}{\left(\dfrac{2^n}{3^{n-1}}\right)} = \dfrac{3^{n-1}}{3^n} \cdot \dfrac{2^{n+1}}{2^n} = 3^{n-1-n} \cdot 2^{n+1-n} = 3^{-1} \cdot 2 = \dfrac{2}{3}$

$a_1 = \dfrac{2^1}{3^{1-1}} = \dfrac{2}{3^0} = \dfrac{2}{1} = 2, \; a_2 = \dfrac{2^2}{3^{2-1}} = \dfrac{4}{3}, \; a_3 = \dfrac{2^3}{3^{3-1}} = \dfrac{8}{3^2} = \dfrac{8}{9}, a_4 = \dfrac{2^4}{3^{4-1}} = \dfrac{16}{3^3} = \dfrac{16}{27}$

11. $\{n+2\}$ Arithmetic
$d = (n+1+2) - (n+2) = n+3-n-2 = 1$

12. $\{2n-5\}$ Arithmetic
$d = 2(n+1) - 5 - (2n-5) = 2n+2-5-2n+5 = 2$

13. $\{4n^2\}$ Examine the terms of the sequence: 4, 16, 36, 64, 100, ...
There is no common difference; there is no common ratio; neither.

14. $\{5n^2+1\}$ Examine the terms of the sequence: 6, 21, 46, 81, 126, ...
There is no common difference; there is no common ratio; neither.

15. $\left\{3-\frac{2}{3}n\right\}$ Arithmetic

$$d=\left(3-\frac{2}{3}(n+1)\right)-\left(3-\frac{2}{3}n\right)=3-\frac{2}{3}n-\frac{2}{3}-3+\frac{2}{3}n=-\frac{2}{3}$$

16. $\left\{8-\frac{3}{4}n\right\}$ Arithmetic

$$d=\left(8-\frac{3}{4}(n+1)\right)-\left(8-\frac{3}{4}n\right)=8-\frac{3}{4}n-\frac{3}{4}-8+\frac{3}{4}n=-\frac{3}{4}$$

17. 1, 3, 6, 10, ... Neither
There is no common difference or common ratio.

18. 2, 4, 6, 8, ... Arithmetic
The common difference is 2.

19. $\left\{\left(\frac{2}{3}\right)^n\right\}$ Geometric

$$r=\frac{\left(\frac{2}{3}\right)^{n+1}}{\left(\frac{2}{3}\right)^n}=\left(\frac{2}{3}\right)^{n+1-n}=\frac{2}{3}$$

20. $\left\{\left(\frac{5}{4}\right)^n\right\}$ Geometric

$$r=\frac{\left(\frac{5}{4}\right)^{n+1}}{\left(\frac{5}{4}\right)^n}=\left(\frac{5}{4}\right)^{n+1-n}=\frac{5}{4}$$

21. –1, –2, –4, –8, ... Geometric $r=\frac{-2}{-1}=\frac{-4}{-2}=\frac{-8}{-4}=2$

22. 1,1,2,3,5,8, ... Neither There is no common difference; there is no common ratio.

23. $\left\{3^{n/2}\right\}$ Geometric

$$r=\frac{3^{\left(\frac{n+1}{2}\right)}}{3^{\left(\frac{n}{2}\right)}}=3^{\left(\frac{n+1}{2}-\frac{n}{2}\right)}=3^{1/2}$$

24. $\left\{(-1)^n\right\}$ Geometric

$$r=\frac{(-1)^{n+1}}{(-1)^n}=(-1)^{n+1-n}=-1$$

25. $a_5=2\cdot 3^{5-1}=2\cdot 3^4=2\cdot 81=162$ $\quad a_n=2\cdot 3^{n-1}$

26. $a_5=-2\cdot 4^{5-1}=-2\cdot 4^4=-2\cdot 256=-512$ $\quad a_n=-2\cdot 4^{n-1}$

27. $a_5 = 5(-1)^{5-1} = 5(-1)^4 = 5\cdot 1 = 5$ $\quad a_n = 5\cdot(-1)^{n-1}$

28. $a_5 = 6(-2)^{5-1} = 6(-2)^4 = 6\cdot 16 = 96$ $\quad a_n = 6\cdot(-2)^{n-1}$

29. $a_5 = 0\cdot\left(\frac{1}{2}\right)^{5-1} = 0\cdot\left(\frac{1}{2}\right)^{4} = 0$ $\quad a_n = 0\cdot\left(\frac{1}{2}\right)^{n-1} = 0$

30. $a_5 = 1\cdot\left(-\frac{1}{3}\right)^{5-1} = 1\cdot\left(-\frac{1}{3}\right)^{4} = \frac{1}{81}$ $\quad a_n = 1\cdot\left(-\frac{1}{3}\right)^{n-1} = \left(-\frac{1}{3}\right)^{n-1}$

31. $a_5 = \sqrt{2}\cdot\left(\sqrt{2}\right)^{5-1} = \sqrt{2}\cdot\left(\sqrt{2}\right)^{4} = \sqrt{2}\cdot 4 = 4\sqrt{2}$ $\quad a_n = \sqrt{2}\cdot\left(\sqrt{2}\right)^{n-1} = \left(\sqrt{2}\right)^{n}$

32. $a_5 = 0\cdot\left(\frac{1}{\pi}\right)^{5-1} = 0\cdot\left(\frac{1}{\pi}\right)^{4} = 0$ $\quad a_n = 0\cdot\left(\frac{1}{\pi}\right)^{n-1} = 0$

33. $a = 1,\ r = \frac{1}{2},\ n = 7$ $\quad a_7 = 1\cdot\left(\frac{1}{2}\right)^{7-1} = \left(\frac{1}{2}\right)^{6} = \frac{1}{64}$

34. $a = 1,\ r = 3,\ n = 8$ $\quad a_8 = 1\cdot 3^{8-1} = 3^7 = 2187$

35. $a = 1,\ r = -1,\ n = 9$ $\quad a_7 = 1\cdot(-1)^{9-1} = (-1)^8 = 1$

36. $a = -1,\ r = -2,\ n = 10$ $\quad a_{10} = -1\cdot(-2)^{10-1} = -1\cdot(-2)^9 = -1(-512) = 512$

37. $a = 0.4,\ r = 0.1,\ n = 8$ $\quad a_7 = 0.4\cdot(0.1)^{8-1} = 0.4(0.1)^7 = 0.00000004$

38. $a = 0.1,\ r = 10,\ n = 7$ $\quad a_7 = 0.1\cdot 10^{7-1} = 0.1(10)^6 = 100{,}000$

39. $a = \frac{1}{4},\ r = 2$ $\quad S_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{1}{4}\left(\frac{1-2^n}{1-2}\right) = -\frac{1}{4}\left(1-2^n\right)$

40. $a = \frac{3}{9} = \frac{1}{3},\ r = 3$ $\quad S_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{1}{3}\left(\frac{1-3^n}{1-3}\right) = \frac{1}{3}\left(\frac{1-3^n}{-2}\right) = -\frac{1}{6}\left(1-3^n\right)$

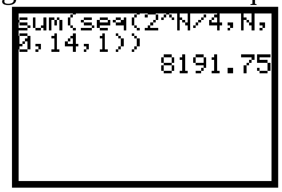
41. $a = \frac{2}{3},\ r = \frac{2}{3}$ $\quad S_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{2}{3}\left(\frac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}}\right) = \frac{2}{3}\left(\frac{1-\left(\frac{2}{3}\right)^n}{\frac{1}{3}}\right) = 2\left(1-\left(\frac{2}{3}\right)^n\right)$

42. $a = 4,\ r = 3$ $\quad S_n = a\left(\frac{1-r^n}{1-r}\right) = 4\left(\frac{1-3^n}{1-3}\right) = 4\left(\frac{1-3^n}{-2}\right) = -2\left(1-3^n\right)$

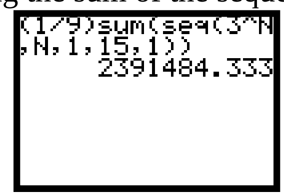
43. $a = -1,\ r = 2$ $\quad S_n = a\left(\frac{1-r^n}{1-r}\right) = -1\left(\frac{1-2^n}{1-2}\right) = 1-2^n$

44. $a=2,\ r=\dfrac{3}{5}\qquad S_n=a\left(\dfrac{1-r^n}{1-r}\right)=2\left(\dfrac{1-\left(\frac{3}{5}\right)^n}{1-\frac{3}{5}}\right)=2\left(\dfrac{1-\left(\frac{3}{5}\right)^n}{\left(\frac{2}{5}\right)}\right)=5\left(1-\left(\frac{3}{5}\right)^n\right)$

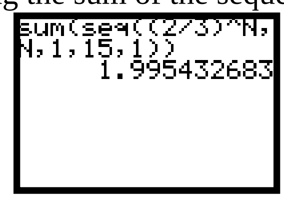
45. Using the sum of the sequence feature:



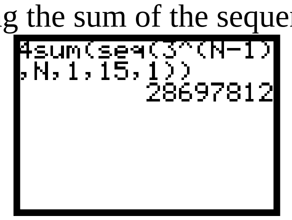
46. Using the sum of the sequence feature:



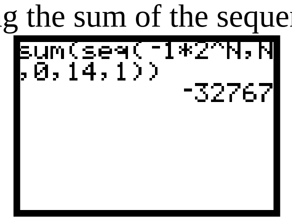
47. Using the sum of the sequence feature:



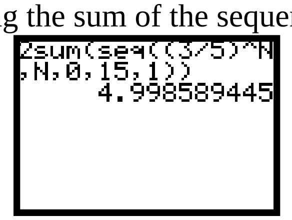
48. Using the sum of the sequence feature:



49. Using the sum of the sequence feature:



50. Using the sum of the sequence feature:



51.

$a=1,\ r=\frac{1}{3}$ Since $|r|<1$, $S_n=\frac{a}{1-r}=\frac{1}{\left(1-\frac{1}{3}\right)}=\frac{1}{\left(\frac{2}{3}\right)}=\frac{3}{2}$

52.

$a=2,\ r=\frac{2}{3}$ Since $|r|<1$, $S_n=\frac{a}{1-r}=\frac{2}{\left(1-\frac{2}{3}\right)}=\frac{2}{\left(\frac{1}{3}\right)}=6$

53.

$a=8,\ r=\frac{1}{2}$ Since $|r|<1$, $S_n=\frac{a}{1-r}=\frac{8}{\left(1-\frac{1}{2}\right)}=\frac{8}{\left(\frac{1}{2}\right)}=16$

54.

$a=6,\ r=\frac{1}{3}$ Since $|r|<1$, $S_n=\frac{a}{1-r}=\frac{6}{\left(1-\frac{1}{3}\right)}=\frac{6}{\left(\frac{2}{3}\right)}=9$

55.

$a=2,\ r=-\frac{1}{4}$ Since $|r|<1$, $S_n=\frac{a}{1-r}=\frac{2}{\left(1-\left(-\frac{1}{4}\right)\right)}=\frac{2}{\left(\frac{5}{4}\right)}=\frac{8}{5}$

56.

$a=1,\ r=-\frac{3}{4}$ Since $|r|<1$, $S_n=\frac{a}{1-r}=\frac{1}{\left(1-\left(-\frac{3}{4}\right)\right)}=\frac{1}{\left(\frac{7}{4}\right)}=\frac{4}{7}$

57.

$a=5,\ r=\frac{1}{4}$ Since $|r|<1$, $S_n=\frac{a}{1-r}=\frac{5}{\left(1-\frac{1}{4}\right)}=\frac{5}{\left(\frac{3}{4}\right)}=\frac{20}{3}$

58.

$a=8,\ r=\frac{1}{3}$ Since $|r|<1$, $S_n=\frac{a}{1-r}=\frac{8}{\left(1-\frac{1}{3}\right)}=\frac{8}{\left(\frac{2}{3}\right)}=12$

59.

$$a = 6,\ r = -\frac{2}{3} \qquad \text{Since } |r| < 1,\ S_n = \frac{a}{1-r} = \frac{6}{\left(1 - \left(-\frac{2}{3}\right)\right)} = \frac{6}{\left(\frac{5}{3}\right)} = \frac{18}{5}$$

60.

$$a = 4,\ r = -\frac{1}{2} \qquad \text{Since } |r| < 1,\ S_n = \frac{a}{1-r} = \frac{4}{\left(1 - \left(-\frac{1}{2}\right)\right)} = \frac{4}{\left(\frac{3}{2}\right)} = \frac{8}{3}$$

61. Find the common ratio of the terms and solve the system of equations:

$$\frac{x+2}{x} = r$$

$$\frac{x+3}{x+2} = r$$

$$\frac{x+2}{x} = \frac{x+3}{x+2} \rightarrow x^2 + 4x + 4 = x^2 + 3x \rightarrow x = -4$$

62. Find the common ratio of the terms and solve the system of equations:

$$\frac{x}{x-1} = r$$

$$\frac{x+2}{x} = r$$

$$\frac{x+2}{x} = \frac{x}{x-1} \rightarrow x^2 + x - 2 = x^2 \rightarrow x = 2$$

63. This is a geometric series with $a = \$18,000,\quad r = 1.05,\quad n = 5$. Find the 5th term:

$$a_5 = 18000(1.05)^{5-1} = 18000(1.05)^4 = \$21,879.11$$

64. This is a geometric series with $a = \$15,000,\quad r = 0.85,\quad n = 6$. Find the 6th term:

$$a_6 = 15000(0.85)^{6-1} = 15000(0.85)^5 = \$6655.58$$

65. (a) Find the 10th term of the geometric sequence:

$$a = 2,\ r = 0.9,\ n = 10 \qquad a_{10} = 2(0.9)^{10-1} = 2(0.9)^9 = 0.775 \text{ feet}$$

(b) Find n when $a_n < 1$:

$$2(0.9)^{n-1} < 1 \rightarrow 0.9^{n-1} < 0.5$$

$$(n-1)\log 0.9 < \log 0.5 \rightarrow n - 1 > \frac{\log 0.5}{\log 0.9} \rightarrow n > \frac{\log 0.5}{\log 0.9} + 1 = 7.58$$

On the 8th swing the arc is less than 1 foot.

(c) Find the sum of the first 15 swings:

$$S_{15} = 2\left(\frac{1-(0.9)^{15}}{1-0.9}\right) = 2\left(\frac{1-0.9^{15}}{0.1}\right) = 20\left(1-0.9^{15}\right) = 15.88 \text{ feet}$$

(d) Find the infinite sum of the geometric series:

$$S = \frac{2}{1-0.9} = \frac{2}{0.1} = 20 \text{ feet}$$

66. (a) Find the 3rd term of the geometric sequence:

$a = 24,\ r = 0.8,\ n = 3 \qquad a_3 = 24(0.8)^{3-1} = 24(0.8)^2 = 15.36$ feet

(b) The height after the nth bounce is: $a_n = 24(0.8)^{n-1}$

(c) Find n when $a_n < 0.5$:

$$24(0.8)^{n-1} < 0.5 \to 0.8^{n-1} < 0.020833 \to (n-1)\log 0.8 < \log 0.020833$$

$$n - 1 > \frac{\log 0.020833}{\log 0.8} \to n > \frac{\log 0.020833}{\log 0.8} + 1 = 18.35$$

On the 19th bounce the height is less than 0.5 feet.

(d) Find the infinite sum of the geometric series:

$S = \dfrac{24}{1-0.8} = \dfrac{24}{0.2} = 120$ feet on the upward bounce.

For the downward motion of the ball:

$S = \dfrac{30}{1-0.8} = \dfrac{30}{0.2} = 150$ feet

The total distance the ball travels is 120 + 150 = 270 feet.

67. This is an ordinary annuity with $P = \$100$ and $n = (12)(30) = 360$ payment periods.

The interest rate per period is $\dfrac{.12}{12} = .01$. Thus,

$$A = 100\left[\frac{\left(1 + \frac{.12}{12}\right)^{360} - 1}{\left(\frac{.12}{12}\right)}\right] = \$349496.41$$

68. This is an ordinary annuity with $P = \$400$ and $n = (12)(3) = 36$ payment periods.

The interest rate per period is $\dfrac{.10}{12} = .008\overline{3}$. Thus,

$$A = 400\left[\frac{\left(1 + \frac{.10}{12}\right)^{36} - 1}{\left(\frac{.10}{12}\right)}\right] = \$16712.73$$

69. This is an ordinary annuity with $P = \$500$ and $n = (4)(20) = 80$ payment periods.

The interest rate per period is $\dfrac{.08}{4} = .02$. Thus,

$$A = 500\left[\frac{\left(1 + \frac{.08}{4}\right)^{80} - 1}{\left(\frac{.08}{4}\right)}\right] = \$96885.98$$

70. This is an ordinary annuity with $P = \$1000$ and $n = (2)(15) = 30$ payment periods. The interest rate per period is $\frac{.10}{2} = .05$. Thus,

$$A = 1000\left[\frac{\left(1+\frac{.10}{2}\right)^{30} - 1}{\left(\frac{.10}{2}\right)}\right] = \$66438.85$$

71. This is an ordinary annuity with $A = \$50000$ and $n = (12)(10) = 120$ payment periods. The interest rate per period is $\frac{.06}{12} = .005$. Thus,

$$50000 = P\left[\frac{\left(1+\frac{.06}{12}\right)^{120} - 1}{\left(\frac{.06}{12}\right)}\right] \rightarrow P = 50000\left[\frac{\left(\frac{.06}{12}\right)}{\left(1+\frac{.06}{12}\right)^{120} - 1}\right] = \$305.10$$

72. This is an ordinary annuity with $A = \$150000$ and $n = (12)(18) = 216$ payment periods. The interest rate per period is $\frac{.08}{12} = .00\overline{6}$. Thus,

$$150000 = P\left[\frac{\left(1+\frac{.08}{12}\right)^{216} - 1}{\left(\frac{.08}{12}\right)}\right] \rightarrow P = 150000\left[\frac{\left(\frac{.08}{12}\right)}{\left(1+\frac{.08}{12}\right)^{216} - 1}\right] = \$312.44$$

73. Both options are geometric sequences:

Option A: $a = \$20{,}000$; $r = 1.06$; $n = 5$

$$a_5 = 20{,}000(1.06)^{5-1} = 20{,}000(1.06)^4 = \$25{,}250$$

$$S_5 = 20000\left(\frac{1-1.06^5}{1-1.06}\right) = \$112{,}742$$

Option B: $a = \$22{,}000$; $r = 1.03$; $n = 5$

$$a_5 = 22{,}000(1.03)^{5-1} = 22{,}000(1.03)^4 = \$24{,}761$$

$$S_5 = 22000\left(\frac{1-1.03^5}{1-1.03}\right) = \$116{,}801$$

Option A provides more money in the 5th year, while Option B provides the greatest total amount of money over the 5 year period.

74. Find the sum of each sequence:
A: Arithmetic series with: $a = \$1000, \quad d = -1, \quad n = 1000$
Find the sum of the arithmetic series:
$$S_{1000} = \frac{1000}{2}\left(1000+1\right) = 500(1001) = \$500{,}500$$
B: This is a geometric sequence with $a = 1, \quad r = 2, \quad n = 19$.
Find the sum of the geometric series:
$$S_{19} = 1\left(\frac{1-2^{19}}{1-2}\right) = \frac{1-2^{19}}{-1} = 2^{19} - 1 = \$524{,}287$$
B results in more money.

75. Option 1: Total Salary = $\$2{,}000{,}000(7) + \$100{,}000(7) = \$14{,}700{,}000$
Option 2: Geometric series with: $a = \$2{,}000{,}000, \; r = 1.045, \quad n = 7$
Find the sum of the geometric series:
$$S = 2{,}000{,}000\left(\frac{1-1.045^7}{1-1.045}\right) = \$16{,}038{,}304$$
Option 3: Arithmetic series with: $a = \$2{,}000{,}000, \; d = \$95{,}000, \quad n = 7$
Find the sum of the arithmetic series:
$$S_7 = \frac{7}{2}\left(2(2{,}000{,}000) + (7-1)(95{,}000)\right) = \$15{,}995{,}000$$
Option 2 provides the most money; Option 1 provides the least money.

76. Given: $a = 1000, \; r = 0.9$
Find n when $a_n < 0.01$:
$$1000(0.9)^{n-1} < 0.01 \rightarrow 0.9^{n-1} < 0.00001$$
$$(n-1)\log 0.9 < \log 0.00001 \rightarrow n-1 > \frac{\log 0.00001}{\log 0.9} \rightarrow n > \frac{\log 0.00001}{\log 0.9} + 1 = 110.27$$
On the 111th day or December 20, 1998, the amount will be less than \$0.01.
Find the sum of the geometric series:
$$S_{110} = a\left(\frac{1-r^n}{1-r}\right) = 1000\left(\frac{1-0.9^{110}}{1-0.9}\right) = 1000\left(\frac{1-0.9^{110}}{0.1}\right) = \$9999.91$$

77. This is a geometric sequence with $a = 1, \quad r = 2, \quad n = 64$.
Find the sum of the geometric series:
$$S_{64} = 1\left(\frac{1-2^{64}}{1-2}\right) = \frac{1-2^{64}}{-1} = 2^{64} - 1 = 1.845 \times 10^{19} \text{ grains}$$

78. This is an infinite geometric series with $a = \frac{1}{4}, \quad r = \frac{1}{4}$.
Find the sum of the infinite geometric series:
$$S = \frac{\left(\frac{1}{4}\right)}{\left(1-\frac{1}{4}\right)} = \frac{\left(\frac{1}{4}\right)}{\left(\frac{3}{4}\right)} = \frac{1}{3}$$
$\frac{1}{3}$ of the square is eventually shaded.

79. The common ratio, $r = 0.90 < 1$. The sum is: $S = \frac{1}{1-0.9} = \frac{1}{0.10} = 10$.
The multiplier is 10.

80. The common ratio, $r = 0.95 < 1$. The sum is: $S = \frac{1}{1-0.95} = \frac{1}{0.05} = 20$.
The multiplier is 20.

81. This is an infinite geometric series with $a = 4$, and $r = \frac{1.03}{1.09}$.
Find the sum: $\text{Price} = \frac{4}{\left(1 - \frac{1.03}{1.09}\right)} = \$72.67$.

82. This is an infinite geometric series with $a = 2.5$, and $r = \frac{1.04}{1.11}$.
Find the sum: $\text{Price} = \frac{2.5}{\left(1 - \frac{1.04}{1.11}\right)} = \$39.64$.

83. – 85. Answers will vary.