Chapter 13

Sequences; Induction; The Binomial Theorem

13.4 Mathematical Induction

1. I: $n=1$: $2 \cdot 1 = 2$ and $1(1+1) = 2$

   II: If $2+4+6+\cdots+2k = k(k+1)$

   then $2+4+6+\cdots+2k+2(k+1)$

   $$= \left[2+4+6+\cdots+2k\right]+2(k+1) = k(k+1)+2(k+1)$$
   $$= (k+1)(k+2)$$

   Conditions I and II are satisfied; the statement is true.

2. I: $n=1$: $4 \cdot 1 - 3 = 1$ and $1(2 \cdot 1 - 1) = 1$

   II: If $1+5+9+\cdots+(4k-3) = k(2k-1)$

   then $1+5+9+\cdots+(4k-3)+(4(k+1)-3)$

   $$= \left[1+5+9+\cdots+(4k-3)\right]+4k+4-3 = k(2k-1)+4k+1$$
   $$= 2k^2-k+4k+1 = 2k^2+3k+1 = (k+1)(2k+1)$$

   Conditions I and II are satisfied; the statement is true.

3. I: $n=1$: $1+2 = 3$ and $\frac{1}{2} \cdot 1(1+5) = 3$

   II: If $3+4+5+\cdots+(k+2) = \frac{1}{2}k(k+5)$

   then $3+4+5+\cdots+(k+2)+[(k+1)+2]$

   $$= \left[3+4+5+\cdots+(k+2)\right]+(k+3) = \frac{1}{2}k(k+5)+(k+3)$$
   $$= \frac{1}{2}k^2+\frac{5}{2}k+k+3 = \frac{1}{2}k^2+\frac{7}{2}k+3 = \frac{1}{2}\left(k^2+7k+6\right)$$
   $$= \frac{1}{2}(k+1)(k+6)$$

   Conditions I and II are satisfied; the statement is true.

4. I: $n=1$: $2 \cdot 1+1 = 3$ and $1(1+2) = 3$

   II: If $3+5+7+\cdots+(2k+1) = k(k+2)$

   then $3+5+7+\cdots+(2k+1)+[2(k+1)+1]$

   $$= \left[3+5+7+\cdots+(2k+1)\right]+(2k+3) = k(k+2)+(2k+3)$$
   $$= k^2+2k+2k+3 = k^2+4k+3 = (k+1)(k+3)$$

   Conditions I and II are satisfied; the statement is true.

5. I: $n=1$: $3\;1-1=2$ and $\frac{1}{2}\;1(3\;1+1)=2$

II: If $2+5+8+\cdots+(3k-1)=\frac{1}{2}\;k(3k+1)$

then $2+5+8+\cdots+(3k-1)+[3(k+1)-1]$

$$=[2+5+8+\cdots+(3k-1)]+(3k+2)=\frac{1}{2}\;k(3k+1)+(3k+2)$$
$$=\frac{3}{2}k^2+\frac{1}{2}k+3k+2=\frac{3}{2}k^2+\frac{7}{2}k+2=\frac{1}{2}\;\left(3k^2+7k+4\right)$$
$$=\frac{1}{2}\;(k+1)(3k+4)$$

Conditions I and II are satisfied; the statement is true.

6. I: $n=1$: $3\;1-2=1$ and $\frac{1}{2}\;1(3\;1-1)=1$

II: If $1+4+7+\cdots+(3k-2)=\frac{1}{2}\;k(3k-1)$

then $1+4+7+\cdots+(3k-2)+[3(k+1)-2]$

$$=[1+4+7+\cdots+(3k-2)]+(3k+1)=\frac{1}{2}\;k(3k-1)+(3k+1)$$
$$=\frac{3}{2}k^2-\frac{1}{2}k+3k+1=\frac{3}{2}k^2+\frac{5}{2}k+1=\frac{1}{2}\;\left(3k^2+5k+2\right)$$
$$=\frac{1}{2}\;(k+1)(3k+2)$$

Conditions I and II are satisfied; the statement is true.

7. I: $n=1$: $2^{1-1}=1$ and $2^1-1=1$

II: If $1+2+2^2+\cdots+2^{k-1}=2^k-1$

then $1+2+2^2+\cdots+2^{k-1}+2^{k+1-1}$

$$=\left[1+2+2^2+\cdots+2^{k-1}\right]+2^k=2^k-1+2^k$$
$$=2\;\;2^k-1=2^{k+1}-1$$

Conditions I and II are satisfied; the statement is true.

8. I: $n=1$: $3^{1-1}=1$ and $\frac{1}{2}(3^1-1)=1$

II: If $1+3+3^2+\cdots+3^{k-1}=\frac{1}{2}\;(3^k-1)$

then $1+3+3^2+\cdots+3^{k-1}+3^{k+1-1}$

$$=\left[1+3+3^2+\cdots+3^{k-1}\right]+3^k=\frac{1}{2}\;(3^k-1)+3^k$$
$$=\frac{1}{2}\;3^k-\frac{1}{2}+3^k=\frac{3}{2}\;3^k-\frac{1}{2}=\frac{1}{2}\;\left(3\;3^k-1\right)=\frac{1}{2}\left(3^{k+1}-1\right)$$

Conditions I and II are satisfied; the statement is true.

9. I: $n=1$: $4^{1-1}=1$ and $\frac{1}{3}\left(4^1-1\right)=1$

II: If $1+4+4^2+\cdots+4^{k-1}=\frac{1}{3}\left(4^k-1\right)$

then $1+4+4^2+\cdots+4^{k-1}+4^{k+1-1}$

$$=\left[1+4+4^2+\cdots+4^{k-1}\right]+4^k=\frac{1}{3}\left(4^k-1\right)+4^k$$

$$=\frac{1}{3}\cdot 4^k-\frac{1}{3}+4^k=\frac{4}{3}\cdot 4^k-\frac{1}{3}=\frac{1}{3}\left(4\cdot 4^k-1\right)=\frac{1}{3}\left(4^{k+1}-1\right)$$

Conditions I and II are satisfied; the statement is true.

10. I: $n=1$: $5^{1-1}=1$ and $\frac{1}{4}\left(5^1-1\right)=1$

II: If $1+5+5^2+\cdots+5^{k-1}=\frac{1}{4}\left(5^k-1\right)$

then $1+5+5^2+\cdots+5^{k-1}+5^{k+1-1}$

$$=\left[1+5+5^2+\cdots+5^{k-1}\right]+5^k=\frac{1}{4}\left(5^k-1\right)+5^k$$

$$=\frac{1}{4}\cdot 5^k-\frac{1}{4}+5^k=\frac{5}{4}\cdot 5^k-\frac{1}{4}=\frac{1}{4}\left(5\cdot 5^k-1\right)=\frac{1}{4}\left(5^{k+1}-1\right)$$

Conditions I and II are satisfied; the statement is true.

11. I: $n=1$: $\frac{1}{1\cdot(1+1)}=\frac{1}{2}$ and $\frac{1}{1+1}=\frac{1}{2}$

II: If $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}$

then $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+1+1)}$

$$=\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}\right]+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}\cdot\frac{k+2}{k+2}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)(k+1)}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

Conditions I and II are satisfied; the statement is true.

12. I: $n=1$: $\frac{1}{(2\cdot 1-1)(2\cdot 1+1)}=\frac{1}{3}$ and $\frac{1}{2\cdot 1+1}=\frac{1}{3}$

II: If $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$

$$\text{then}\quad \frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$=\left[\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2k-1)(2k+1)}\right]+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}=\frac{k}{2k+1}\cdot\frac{2k+3}{2k+3}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}=\frac{(k+1)(2k+1)}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

Conditions I and II are satisfied; the statement is true.

13. I: $n=1$: $1^2=1$ and $\frac{1}{6}\cdot 1(1+1)(2\cdot 1+1)=1$

II: If $1^2+2^2+3^2+\cdots+k^2=\frac{1}{6}\, k(k+1)(2k+1)$

then $1^2+2^2+3^2+\cdots+k^2+(k+1)^2$

$$=\left[1^2+2^2+3^2+\cdots+k^2\right]+(k+1)^2=\frac{1}{6}k(k+1)(2k+1)+(k+1)^2$$

$$=(k+1)\left[\frac{1}{6}k(2k+1)+k+1\right]=(k+1)\left[\frac{1}{3}k^2+\frac{1}{6}k+k+1\right]$$

$$=(k+1)\left[\frac{1}{3}k^2+\frac{7}{6}k+1\right]=\frac{1}{6}(k+1)\left[2k^2+7k+6\right]$$

$$=\frac{1}{6}\,(k+1)(k+2)(2k+3)$$

Conditions I and II are satisfied; the statement is true.

14. I: $n=1$: $1^3=1$ and $\frac{1}{4}\cdot 1^2(1+1)^2=1$

II: If $1^3+2^3+3^3+\cdots+k^3=\frac{1}{4}k^2(k+1)^2$

then $1^3+2^3+3^3+\cdots+k^3+(k+1)^3$

$$=\left[1^3+2^3+3^3+\cdots+k^3\right]+(k+1)^3=\frac{1}{4}k^2(k+1)^2+(k+1)^3$$

$$=(k+1)^2\left[\frac{1}{4}k^2+k+1\right]=\frac{1}{4}(k+1)^2\left[k^2+4k+4\right]$$

$$=\frac{1}{4}\,(k+1)^2(k+2)^2$$

Conditions I and II are satisfied; the statement is true.

15. I: $n=1$: $5-1=4$ and $\frac{1}{2}\cdot 1(9-1)=4$

II: If $4+3+2+\cdots+(5-k)=\frac{1}{2}k(9-k)$

then $4+3+2+\cdots+(5-k)+\left(5-(k+1)\right)$

$$=\left[4+3+2+\cdots+(5-k)\right]+(4-k)=\frac{1}{2}k(9-k)+(4-k)$$

$$=\frac{9}{2}k-\frac{1}{2}k^2+4-k=-\frac{1}{2}k^2+\frac{7}{2}k+4=-\frac{1}{2}\left[k^2-7k-8\right]$$

$$=-\frac{1}{2}(k+1)(k-8)=\frac{1}{2}(k+1)(8-k)=\frac{1}{2}(k+1)\left[9-(k+1)\right]$$

Conditions I and II are satisfied; the statement is true.

16. I: $n=1$: $-(1+1)=-2$ and $-\frac{1}{2}\cdot 1(1+3)=-2$

II: If $-2-3-4-\cdots-(k+1)=-\frac{1}{2}k(k+3)$

then $-2-3-4-\cdots-(k+1)-\left((k+1)+1\right)$

$$=\left[-2-3-4-\cdots-(k+1)\right]-(k+2)=-\frac{1}{2}k(k+3)-(k+2)$$

$$=-\frac{1}{2}k^2-\frac{3}{2}k-k-2=-\frac{1}{2}k^2-\frac{5}{2}k-2=-\frac{1}{2}\left[k^2+5k+4\right]$$

$$=-\frac{1}{2}(k+1)(k+4)$$

Conditions I and II are satisfied; the statement is true.

17. I: $n=1$: $1(1+1)=2$ and $\frac{1}{3}\cdot 1(1+1)(1+2)=2$

II: If $1\cdot 2+2\cdot 3+3\cdot 4+\cdots+k(k+1)=\frac{1}{3}k(k+1)(k+2)$

then $1\cdot 2+2\cdot 3+3\cdot 4+\cdots+k(k+1)+(k+1)(k+1+1)$

$$=\left[1\cdot 2+2\cdot 3+3\cdot 4+\cdots+k(k+1)\right]+(k+1)(k+2)$$

$$=\frac{1}{3}k(k+1)(k+2)+(k+1)(k+2)=(k+1)(k+2)\left[\frac{1}{3}k+1\right]$$

$$=\frac{1}{3}(k+1)(k+2)(k+3)$$

Conditions I and II are satisfied; the statement is true.

18. I: $n=1$: $(2\cdot 1-1)(2\cdot 1)=2$ and $\frac{1}{3}\cdot 1(1+1)(4\cdot 1-1)=2$

II: If $1\cdot 2+3\cdot 4+5\cdot 6+\cdots+(2k-1)(2k)=\frac{1}{3}k(k+1)(4k-1)$

then $1\cdot 2+3\cdot 4+5\cdot 6+\cdots+(2k-1)(2k)+(2(k+1)-1)(2(k+1))$

$$=\left[1\cdot 2+3\cdot 4+5\cdot 6+\cdots+(2k-1)(2k)\right]+(2k+1)(k+1)\cdot 2$$

$$=\frac{1}{3}k(k+1)(4k-1)+2(k+1)(2k+1)=(k+1)\left[\frac{1}{3}k(4k-1)+2(2k+1)\right]$$

$$=(k+1)\left[\frac{4}{3}k^2-\frac{1}{3}k+4k+2\right]=\frac{1}{3}(k+1)\left(4k^2-k+12k+6\right)$$

$$=\frac{1}{3}(k+1)\left(4k^2+11k+6\right)=\frac{1}{3}(k+1)(k+2)(4k+3)$$

Conditions I and II are satisfied; the statement is true.

19. I: $n=1$: $1^2+1=2$ is divisible by 2
II: If k^2+k is divisible by 2
then $(k+1)^2+(k+1)=k^2+2k+1+k+1=(k^2+k)+(2k+2)$
Since k^2+k is divisible by 2 and $2k+2$ is divisible by 2, then $(k+1)^2+(k+1)$ is divisible by 2.
Conditions I and II are satisfied; the statement is true.

20. I: $n=1$: $1^3+2\cdot 1=3$ is divisible by 3
II: If k^3+2k is divisible by 3
then $(k+1)^3+2(k+1)=k^3+3k^2+3k+1+2k+2=(k^3+2k)+(3k^2+3k+3)$
Since k^3+2k is divisible by 3 and $3k^2+3k+3$ is divisible by 3, then $(k+1)^3+2(k+1)$ is divisible by 3.
Conditions I and II are satisfied; the statement is true.

21. I: $n=1$: $1^2-1+2=2$ is divisible by 2
II: If k^2-k+2 is divisible by 2
then $(k+1)^2-(k+1)+2=k^2+2k+1-k-1+2=(k^2-k+2)+(2k)$
Since k^2-k+2 is divisible by 2 and $2k$ is divisible by 2, then $(k+1)^2-(k+1)+2$ is divisible by 2.
Conditions I and II are satisfied; the statement is true.

22. I: $n=1$: $1(1+1)(1+2)=6$ is divisible by 6
II: If $k(k+1)(k+2)$ is divisible by 6
then $(k+1)(k+1+1)(k+1+2)=(k+1)(k+2)(k+3)$
$=k(k+1)(k+2)+3(k+1)(k+2)$. $k(k+1)(k+2)$ is divisible by 6 and either $k+1$ or $k+2$ is even. Thus, $3(k+1)(k+2)$ is divisible by 6.
Conditions I and II are satisfied; the statement is true.

23. I: $n=1$: If $x>1$ then $x^1=x>1$.
II: Assume, for any natural number k, that if $x>1$, then $x^k>1$.

Show that if $x^k > 1$, then $x^{k+1} > 1$

$$x^{k+1} = x^k \; x > 1 \; x = x > 1$$

$$\uparrow$$

$$(x^k > 1)$$

Conditions I and II are satisfied; the statement is true.

24. I: $n = 1$: If $0 < x < 1$ then $0 < x^1 < 1$.

II: Assume, for any natural number k, that if $0 < x < 1$, then $0 < x^k < 1$.

Show that if $0 < x < 1$, then $0 < x^{k+1} < 1$:

$$0 < x^{k+1} = x^k \; x < 1 \; x = x < 1$$

Thus, $0 < x^{k+1} < 1$.

Conditions I and II are satisfied; the statement is true.

25. I: $n = 1$: $a - b$ is a factor of $a^1 - b^1 = a - b$.

II: If $a - b$ is a factor of $a^k - b^k$

Show that $a - b$ is a factor of $a^{k+1} - b^{k+1} = a \; a^k - b \; b^k$

$$= a \; a^k - a \; b^k + a \; b^k - b \; b^k = a\left(a^k - b^k\right) + b^k(a - b)$$

Since $a - b$ is a factor of $a^k - b^k$ and $a - b$ is a factor of $a - b$, then $a - b$ is a factor of $a^{k+1} - b^{k+1}$.

Conditions I and II are satisfied; the statement is true.

26. I: $n = 1$: $a + b$ is a factor of $a^{2\;1+1} + b^{2\;1+1} = a^3 + b^3$.

II: If $a + b$ is a factor of $a^{2k+1} + b^{2k+1}$

Show that $a + b$ is a factor of $a^{2(k+1)+1} + b^{2(k+1)+1} = a^{2k+3} + b^{2k+3}$

$$= a^2 \; a^{2k+1} + a^2 \; b^{2k+1} - a^2 \; b^{2k+1} + b^{2k+3}$$

$$= a^2\left(a^{2k+1} + b^{2k+1}\right) - b^{2k+1}(a^2 - b^2)$$

Since $a + b$ is a factor of $a^{2k+1} + b^{2k+1}$ and $a + b$ is a factor of $a^2 - b^2$, then $a + b$ is a factor of $a^{2k+3} + b^{2k+3}$.

Conditions I and II are satisfied; the statement is true.

27. $n = 1$: $1^2 - 1 + 41 = 41$ is a prime number.

$n = 41$: $41^2 - 41 + 41 = 41^2$ is not a prime number.

28. II: If $2 + 4 + 6 + \cdots + 2k = k^2 + k + 2$

then $2 + 4 + 6 + \cdots + 2k + 2(k+1)$

$$= \left[2 + 4 + 6 + \cdots + 2k\right] + 2k + 2 = k^2 + k + 2 + 2k + 2$$

$$= (k^2 + 2k + 1) + (k + 1) + 2 = (k+1)^2 + (k+1) + 2$$

I: $n = 1$: $2 \; 1 = 2$ and $1^2 + 1 + 2 = 4 \neq 2$

29. I: $n = 1$: $a r^{1-1} = a$ and $a\left(\frac{1 - r^1}{1 - r}\right) = a$

II: If $a + ar + ar^2 + \cdots + ar^{k-1} = a\left(\frac{1 - r^k}{1 - r}\right)$

then $a + ar + ar^2 + \cdots + ar^{k-1} + ar^{k+1-1}$

$$= \left[a + ar + ar^2 + \cdots + ar^{k-1}\right] + ar^k = a\left(\frac{1-r^k}{1-r}\right) + ar^k$$

$$= \frac{a(1-r^k) + ar^k(1-r)}{1-r} = \frac{a - ar^k + ar^k - ar^{k+1}}{1-r} = a\left(\frac{1-r^{k+1}}{1-r}\right)$$

Conditions I and II are satisfied; the statement is true.

30. I: $n = 1$: $a + (1-1)d = a$ and $1a + d\dfrac{1(1-1)}{2} = a$

II: If $a + (a+d) + (a+2d) + \cdots + [a + (k-1)d] = ka + d\dfrac{k(k-1)}{2}$

then $a + (a+d) + (a+2d) + \cdots + [a + (k-1)d] + (a + kd)$

$$= \left[a + (a+d) + (a+2d) + \cdots + [a + (k-1)d]\right] + (a + kd)$$

$$= ka + d\frac{k(k-1)}{2} + (a + kd) = (k+1)a + d\left[\frac{k(k-1)}{2} + k\right]$$

$$= (k+1)a + d\left[\frac{k^2 - k + 2k}{2}\right] = (k+1)a + d\left[\frac{k^2 + k}{2}\right]$$

$$= (k+1)a + d\left[\frac{(k+1)k}{2}\right]$$

Conditions I and II are satisfied; the statement is true.

31. I: $n = 4$: The number of diagonals of a quadrilateral is $\frac{1}{2}\,4(4-3) = 2$

II: Assume that for any integer k the number of diagonals of a convex polygon with k sides (k vertices) is $\frac{1}{2}\,k(k-3)$. A convex polygon with $k+1$ sides ($k+1$ vertices) consists of a convex polygon with k sides (k vertices) plus a triangle for a total of $k+1$ vertices. The number of diagonals of this convex polygon consists of the original ones plus $k-1$ additional ones,

namely, $\frac{1}{2}\,k(k-3) + (k-1) = \frac{1}{2}k^2 - \frac{3}{2}k + k - 1 = \frac{1}{2}k^2 - \frac{1}{2}k - 1$

$$= \frac{1}{2}\left(k^2 - k - 2\right) = \frac{1}{2}\,(k+1)(k-2)$$

Conditions I and II are satisfied; the statement is true.

32. I: $n = 3$: $(3-2)\,180° = 180°$ which is the sum of the angles of a triangle.

II: Assume that for any integer k the sum of the angles of a convex polygon with k sides is $(k-2)\,180°$. A convex polygon with $k+1$ sides consists of a convex polygon with k sides plus a triangle. Thus the sum of the angles is $(k-2)\,180° + 180° = (k-1)\,180°$.

Conditions I and II are satisfied; the statement is true.

33. Answers will vary.