Chapter 13

Sequences; Induction; The Binomial Theorem

13.5 The Binomial Theorem

1. $\binom{5}{3} = \dfrac{5!}{3!\,2!} = \dfrac{5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 2\cdot 1} = \dfrac{5\cdot 4}{2\cdot 1} = 10$

2. $\binom{7}{3} = \dfrac{7!}{3!\,4!} = \dfrac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 4\cdot 3\cdot 2\cdot 1} = \dfrac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 35$

3. $\binom{7}{5} = \dfrac{7!}{5!\,2!} = \dfrac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 2\cdot 1} = \dfrac{7\cdot 6}{2\cdot 1} = 21$

4. $\binom{9}{7} = \dfrac{9!}{7!\,2!} = \dfrac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 2\cdot 1} = \dfrac{9\cdot 8}{2\cdot 1} = 36$

5. $\binom{50}{49} = \dfrac{50!}{49!\,1!} = \dfrac{50\cdot 49!}{49!\cdot 1} = \dfrac{50}{1} = 50$

6. $\binom{100}{98} = \dfrac{100!}{98!\,2!} = \dfrac{100\cdot 99\cdot 98!}{98!\cdot 2\cdot 1} = \dfrac{100\cdot 99}{2\cdot 1} = 4950$

7. $\binom{1000}{1000} = \dfrac{1000!}{1000!\,0!} = \dfrac{1}{1} = 1$

8. $\binom{1000}{0} = \dfrac{1000!}{0!\,1000!} = \dfrac{1}{1} = 1$

9. $\binom{55}{23} = \dfrac{55!}{23!\,32!} = 1.866442159 \times 10^{15}$

10. $\binom{60}{20} = \dfrac{60!}{20!\,40!} = 4.191844506 \times 10^{15}$

11. $\binom{47}{25} = \dfrac{47!}{25!\,22!} = 1.483389769 \times 10^{13}$

12. $\binom{37}{19} = \dfrac{37!}{19!\,18!} = 1.76726319 \times 10^{10}$

13. $(x+1)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4 + \binom{5}{2}x^3 + \binom{5}{3}x^2 + \binom{5}{4}x^1 + \binom{5}{5}x^0$
$= x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

14. $(x-1)^5 = \binom{5}{0}x^5 + \binom{5}{1}(-1)x^4 + \binom{5}{2}(-1)^2x^3 + \binom{5}{3}(-1)^3x^2 + \binom{5}{4}(-1)^4x^1 + \binom{5}{5}(-1)^5x^0$
$= x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$

15. $(x-2)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5(-2) + \binom{6}{2}x^4(-2)^2 + \binom{6}{3}x^3(-2)^3 + \binom{6}{4}x^2(-2)^4$
$+ \binom{6}{5}x(-2)^5 + \binom{6}{6}x^0(-2)^6$
$= x^6 + 6x^5(-2) + 15x^4 \cdot 4 + 20x^3(-8) + 15x^2 \cdot 16 + 6x \cdot (-32) + 64$
$= x^6 - 12x^5 + 60x^4 - 160x^3 + 240x - 192x + 64$

16. $(x+3)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4(3) + \binom{5}{2}x^3(3)^2 + \binom{5}{3}x^2(3)^3 + \binom{5}{4}x^1(3)^4 + \binom{5}{5}x^0(3)^5$
$= x^5 + 5x^4(3) + 10x^3 \cdot 9 + 10x^2(27) + 5x \cdot 81 + 243$
$= x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243$

17. $(3x+1)^4 = \binom{4}{0}(3x)^4 + \binom{4}{1}(3x)^3 + \binom{4}{2}(3x)^2 + \binom{4}{3}(3x) + \binom{4}{4}$
$= 81x^4 + 4 \cdot 27x^3 + 6 \cdot 9x^2 + 4 \cdot 3x + 1 = 81x^4 + 108x^3 + 54x^2 + 12x + 1$

18. $(2x+3)^5 = \binom{5}{0}(2x)^5 + \binom{5}{1}(2x)^4 \cdot 3 + \binom{5}{2}(2x)^3 \cdot 3^2 + \binom{5}{3}(2x)^2 \cdot 3^3$
$+ \binom{5}{4} \cdot 2x \cdot 3^4 + \binom{5}{5} \cdot 3^5$
$= 32x^5 + 5 \cdot 16x^4 \cdot 3 + 10 \cdot 8x^3 \cdot 9 + 10 \cdot 4x^2 \cdot 27 + 5 \cdot 2x \cdot 81 + 243$
$= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$

19. $\left(x^2+y^2\right)^5 = \binom{5}{0}\left(x^2\right)^5\left(y^2\right)^0 + \binom{5}{1}\left(x^2\right)^4\left(y^2\right) + \binom{5}{2}\left(x^2\right)^3\left(y^2\right)^2 + \binom{5}{3}\left(x^2\right)^2\left(y^2\right)^3$
$+ \binom{5}{4}x^2\left(y^2\right)^4 + \binom{5}{5}\left(y^2\right)^5$
$= x^{10} + 5x^8y^2 + 10x^6y^4 + 10x^4y^6 + 5x^2y^8 + y^{10}$

20. $\left(x^2-y^2\right)^6 = \binom{6}{0}\left(x^2\right)^6\left(-y^2\right)^0 + \binom{6}{1}\left(x^2\right)^5\left(-y^2\right) + \binom{6}{2}\left(x^2\right)^4\left(-y^2\right)^2 + \binom{6}{3}\left(x^2\right)^3\left(-y^2\right)^3$
$\binom{6}{4}\left(x^2\right)^2\left(-y^2\right)^4 + \binom{6}{5}x^2\left(-y^2\right)^5 + \binom{6}{6}\left(-y^2\right)^6$
$= x^{12} - 6x^{10}y^2 + 15x^8y^4 - 20x^6y^6 + 15x^4y^8 - 6x^2y^{10} + y^{12}$

21. $\left(\sqrt{x}+\sqrt{2}\right)^6 = \binom{6}{0}\left(\sqrt{x}\right)^6\left(\sqrt{2}\right)^0 + \binom{6}{1}\left(\sqrt{x}\right)^5\left(\sqrt{2}\right)^1 + \binom{6}{2}\left(\sqrt{x}\right)^4\left(\sqrt{2}\right)^2 + \binom{6}{3}\left(\sqrt{x}\right)^3\left(\sqrt{2}\right)^3$

$$\binom{6}{4}\left(\sqrt{x}\right)^2\left(\sqrt{2}\right)^4 + \binom{6}{5}\left(\sqrt{x}\right)\left(\sqrt{2}\right)^5 + \binom{6}{6}\left(\sqrt{x}\right)^0\left(\sqrt{2}\right)^6$$

$$= x^3 + 6\sqrt{2}x^{5/2} + 15\,2x^2 + 20\,2\sqrt{2}x^{3/2} + 15\,4x + 6\,4\sqrt{2}x^{1/2} + 8$$

$$= x^3 + 6\sqrt{2}x^{5/2} + 30x^2 + 40\sqrt{2}x^{3/2} + 60x + 24\sqrt{2}x^{1/2} + 8$$

22. $\left(\sqrt{x}-\sqrt{3}\right)^4 = \binom{4}{0}\left(\sqrt{x}\right)^4\left(-\sqrt{3}\right)^0 + \binom{4}{1}\left(\sqrt{x}\right)^3\left(-\sqrt{3}\right)^1 + \binom{4}{2}\left(\sqrt{x}\right)^2\left(-\sqrt{3}\right)^2$

$$+\binom{4}{3}\left(\sqrt{x}\right)\left(-\sqrt{3}\right)^3 + \binom{4}{4}\left(\sqrt{x}\right)^0\left(-\sqrt{3}\right)^4$$

$$= x^2 - 4\sqrt{3}x^{3/2} + 6\,3x - 4\,3\sqrt{3}x^{1/2} + 9 = x^2 - 4\sqrt{3}x^{3/2} + 18x - 12\sqrt{3}x^{1/2} + 9$$

23. $(ax+by)^5 = \binom{5}{0}(ax)^5 + \binom{5}{1}(ax)^4\,by + \binom{5}{2}(ax)^3(by)^2 + \binom{5}{3}(ax)^2(by)^3$

$$+\binom{5}{4}ax(by)^4 + \binom{5}{5}(by)^5$$

$$= a^5x^5 + 5a^4x^4by + 10a^3x^3b^2y^2 + 10a^2x^2b^3y^3 + 5axb^4y^4 + b^5y^5$$

24. $(ax-by)^4 = \binom{4}{0}(ax)^4 + \binom{4}{1}(ax)^3(-by) + \binom{4}{2}(ax)^2(-by)^2 + \binom{4}{3}(ax)(-by)^3$

$$+\binom{4}{4}(-by)^4$$

$$= a^4x^4 - 4a^3x^3by + 6a^2x^2b^2y^2 - 4axb^3y^3 + b^4y^4$$

25. $n=10,\ j=4,\ x=x,\ a=3$

$$\binom{10}{4}x^6\,3^4 = \frac{10!}{4!\,6!}81x^6 = \frac{10\,9\,8\,7}{4\,3\,2\,1}\,81x^6 = 17{,}010x^6$$

The coefficient of x^6 is $17{,}010$.

26. $n=10,\ j=7,\ x=x,\ a=-3$

$$\binom{10}{7}x^3\,(-3)^7 = \frac{10!}{7!\,3!}(-2187x^3) = \frac{10\,9\,8}{3\,2\,1}(-2187x^3) = -262{,}440x^3$$

The coefficient of x^3 is $-262{,}440$.

27. $n=12,\ j=5,\ x=2x,\ a=-1$

$$\binom{12}{5}(2x)^7\,(-1)^5 = \frac{12!}{5!\,7!}128x^7(-1) = \frac{12\,11\,10\,9\,8}{5\,4\,3\,2\,1}(-128)x^7 = -101{,}376x^7$$

The coefficient of x^7 is $-101{,}376$.

28. $n = 12,\ j = 9,\ x = 2x,\ a = 1$

$$\binom{12}{9}(2x)^3\cdot(1)^9 = \frac{12!}{9!\,3!}\cdot 8x^3(1) = \frac{12\cdot 11\cdot 10}{3\cdot 2\cdot 1}\cdot 8x^3 = 1760x^3$$

The coefficient of x^3 is 1760.

29. $n = 9,\ j = 2,\ x = 2x,\ a = 3$

$$\binom{9}{2}(2x)^7\cdot 3^2 = \frac{9!}{2!\,7!}\cdot 128x^7(9) = \frac{9\cdot 8}{2\cdot 1}\cdot 128x^7\cdot 9 = 41{,}472x^7$$

The coefficient of x^7 is $41,472$.

30. $n = 9,\ j = 7,\ x = 2x,\ a = -3$

$$\binom{9}{7}(2x)^2\cdot(-3)^7 = \frac{9!}{7!\,2!}\cdot 4x^2(-2187) = \frac{9\cdot 8}{2\cdot 1}\cdot 4x^2\cdot(-2187) = -314{,}928x^2$$

The coefficient of x^2 is $-314{,}928$.

31. $n = 7,\ j = 4,\ x = x,\ a = 3$

$$\binom{7}{4}x^3\cdot 3^4 = \frac{7!}{4!\,3!}\cdot 81x^3 = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1}\cdot 81x^3 = 2835x^3$$

32. $n = 7,\ j = 2,\ x = x,\ a = -3$

$$\binom{7}{2}x^5\cdot(-3)^2 = \frac{7!}{2!\,5!}\cdot 9x^5 = \frac{7\cdot 6}{2\cdot 1}\cdot 9x^5 = 189x^5$$

33. $n = 9,\ j = 2,\ x = 3x,\ a = -2$

$$\binom{9}{2}(3x)^7\cdot(-2)^2 = \frac{9!}{2!\,7!}\cdot 2187x^7\cdot 4 = \frac{9\cdot 8}{2\cdot 1}\cdot 8748x^7 = 314{,}928x^7$$

34. $n = 8,\ j = 5,\ x = 3x,\ a = 2$

$$\binom{8}{5}(3x)^3\cdot(2)^5 = \frac{8!}{5!\,3!}\cdot 27x^3\cdot 32 = \frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1}\cdot 864x^3 = 48{,}384x^3$$

35. The constant term in $\binom{12}{j}\left(x^2\right)^{12-j}\left(\frac{1}{x}\right)^j$ occurs when:

$2(12 - j) = j \;\rightarrow\; 24 - 2j = j \;\rightarrow\; 3j = 24 \;\rightarrow\; j = 8$.

Evaluate the 9th term:

$$\binom{12}{8}\left(x^2\right)^4\cdot\left(\frac{1}{x}\right)^8 = \frac{12!}{8!\,4!}x^8\cdot\frac{1}{x^8} = \frac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2\cdot 1}x^0 = 495$$

36. The constant term in $\binom{9}{j}(x)^{9-j}\left(-\frac{1}{x^2}\right)^j$ occurs when:

$9 - j = 2j \;\rightarrow\; 9 = 3j \;\rightarrow\; j = 3$.

Evaluate the 4th term:

$$\binom{9}{3}(x)^6\cdot\left(-\frac{1}{x^2}\right)^3 = \frac{9!}{3!\,6!}x^6\cdot\left(-\frac{1}{x^6}\right) = -\frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1}x^0 = -84$$

37. The x^4 term in $\binom{10}{j}(x)^{10-j}\left(\frac{-2}{\sqrt{x}}\right)^j$ occurs when:

$$10-j-\tfrac{1}{2}j=4 \;\rightarrow\; -\tfrac{3}{2}j=-6 \;\rightarrow\; j=4.$$

Evaluate the 5th term:

$$\binom{10}{4}(x)^6\left(\frac{-2}{\sqrt{x}}\right)^4=\frac{10!}{6!\,4!}x^6\cdot\frac{16}{x^2}=\frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1}\cdot 16x^4=3360x^4$$

The coefficient is 3360.

38. The x^2 term in $\binom{8}{j}\left(\sqrt{x}\right)^{8-j}\left(\frac{3}{\sqrt{x}}\right)^j$ occurs when:

$$\tfrac{1}{2}(8-j)-\tfrac{1}{2}j=2 \;\rightarrow\; 4-\tfrac{1}{2}j-\tfrac{1}{2}j=2 \;\rightarrow\; -j=-2 \;\rightarrow\; j=2.$$

Evaluate the 3rd term:

$$\binom{8}{2}\left(\sqrt{x}\right)^6\left(\frac{3}{\sqrt{x}}\right)^2=\frac{8!}{6!\,2!}x^3\cdot\frac{9}{x}=\frac{8\cdot 7}{2\cdot 1}\cdot 9x^2=252x^2$$

The coefficient is 252.

39. $$(1.001)^5=\left(1+10^{-3}\right)^5=\binom{5}{0}1^5+\binom{5}{1}1^4\cdot 10^{-3}+\binom{5}{2}1^3\cdot\left(10^{-3}\right)^2+\binom{5}{3}1^2\cdot\left(10^{-3}\right)^3+\ldots$$
$$=1+5(0.001)+10(0.000001)+10(0.000000001)+\ldots$$
$$=1+0.005+0.000010+0.000000010+\ldots$$
$$=1.00501 \quad \text{(correct to 5 decimal places)}$$

40. $$(0.998)^6=\left(1-0.002\right)^6=\binom{6}{0}1^6+\binom{6}{1}1^5\cdot(-0.002)+\binom{6}{2}1^4\cdot\left(-0.002\right)^2+\binom{6}{3}1^3\cdot\left(-0.002\right)^3+\ldots$$
$$=1+6(-0.002)+15(0.000004)+20(-0.000000008)+\ldots$$
$$=1-0.012+0.000060-0.000000160+\ldots$$
$$=0.98806 \quad \text{(correct to 5 decimal places)}$$

41. $$\binom{n}{n-1}=\frac{n!}{(n-1)!\,(n-(n-1))!}=\frac{n!}{(n-1)!\,(1)!}=n$$

$$\binom{n}{n}=\frac{n!}{n!\,(n-n)!}=\frac{n!}{n!\,0!}=\frac{n!}{n!\cdot 1}=\frac{n!}{n!}=1$$

42. $$\binom{n}{j}=\frac{n!}{j!\,(n-j)!}=\frac{n!}{(n-j)!\,(n-(n-j))!}=\frac{n!}{(n-j)!\,j!}=\binom{n}{n-j}$$

43. Show that $\binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{n}=2^n$

$$2^n=(1+1)^n$$
$$=\binom{n}{0}1^n+\binom{n}{1}1^{n-1}\,1+\binom{n}{2}1^{n-2}\,1^2+\dots+\binom{n}{n}1^{n-n}\,1^n$$
$$=\binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{n}$$

44. Show that $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\dots+(-1)^n\binom{n}{n}=0$

$$0=(1-1)^n$$
$$=\binom{n}{0}1^n+\binom{n}{1}1^{n-1}\,(-1)+\binom{n}{2}1^{n-2}\,(-1)^2+\dots+\binom{n}{n}1^{n-n}\,(-1)^n$$
$$=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\dots+(-1)^n\binom{n}{n}$$

45. $\binom{5}{0}\left(\frac{1}{4}\right)^5+\binom{5}{1}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)+\binom{5}{2}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2+\binom{5}{3}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3$

$$+\binom{5}{4}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^4+\binom{5}{5}\left(\frac{3}{4}\right)^5=\left(\frac{1}{4}+\frac{3}{4}\right)^5=(1)^5=1$$

46. $12!=479{,}001{,}600=4.790016\times10^8$

$20!=2.432902008\times10^{18}$

$25!=1.551121004\times10^{25}$

$$12!\approx\sqrt{2\cdot12\pi}\left(\frac{12}{e}\right)^{12}\left(1+\frac{1}{12\cdot12-1}\right)\approx\sqrt{24\pi}(54782414.52)(1.006993007)$$
$$\approx 479013972.4$$
$$20!\approx\sqrt{2\cdot20\pi}\left(\frac{20}{e}\right)^{20}\left(1+\frac{1}{12\cdot20-1}\right)\approx\sqrt{40\pi}(2.161276221\times10^{17})(1.0041841)$$
$$\approx 2.43292403\times10^{18}$$
$$25!\approx\sqrt{2\cdot25\pi}\left(\frac{25}{e}\right)^{25}\left(1+\frac{1}{12\cdot25-1}\right)\approx\sqrt{50\pi}(1.233497203\times10^{24})(1.003344482)$$
$$\approx 1.551129917\times10^{25}$$