Chapter 13

Sequences; Induction; The Binomial Theorem

13.R Chapter Review

1. $a_1 = (-1)^1 \frac{1+3}{1+2} = -\frac{4}{3},\ a_2 = (-1)^2 \frac{2+3}{2+2} = \frac{5}{4},\ a_3 = (-1)^3 \frac{3+3}{3+2} = -\frac{6}{5},$
$a_4 = (-1)^4 \frac{4+3}{4+2} = \frac{7}{6},\ a_5 = (-1)^5 \frac{5+3}{5+2} = -\frac{8}{7}$

2. $a_1 = (-1)^{1+1}(2 \cdot 1 + 3) = 5,\ a_2 = (-1)^{2+1}(2 \cdot 2 + 3) = -7,\ a_3 = (-1)^{3+1}(2 \cdot 3 + 3) = 9,$
$a_4 = (-1)^{4+1}(2 \cdot 4 + 3) = -11,\ a_5 = (-1)^{5+1}(2 \cdot 5 + 3) = 13$

3. $a_1 = \frac{2^1}{1^2} = \frac{2}{1} = 2,\ a_2 = \frac{2^2}{2^2} = \frac{4}{4} = 1,\ a_3 = \frac{2^3}{3^2} = \frac{8}{9},\ a_4 = \frac{2^4}{4^2} = \frac{16}{16} = 1,\ a_5 = \frac{2^5}{5^2} = \frac{32}{25}$

4. $a_1 = \frac{e^1}{1} = e,\ a_2 = \frac{e^2}{2},\ a_3 = \frac{e^3}{3},\ a_4 = \frac{e^4}{4},\ a_5 = \frac{e^5}{5}$

5. $a_1 = 3,\ a_2 = \frac{2}{3} \cdot 3 = 2,\ a_3 = \frac{2}{3} \cdot 2 = \frac{4}{3},\ a_4 = \frac{2}{3} \cdot \frac{4}{3} = \frac{8}{9},\ a_5 = \frac{2}{3} \cdot \frac{8}{9} = \frac{16}{27}$

6. $a_1 = 4,\ a_2 = -\frac{1}{4} \cdot 4 = -1,\ a_3 = -\frac{1}{4} \cdot -1 = \frac{1}{4},\ a_4 = -\frac{1}{4} \cdot \frac{1}{4} = -\frac{1}{16},\ a_5 = -\frac{1}{4} \cdot -\frac{1}{16} = \frac{1}{64}$

7. $a_1 = 2,\ a_2 = 2 - 2 = 0,\ a_3 = 2 - 0 = 2,\ a_4 = 2 - 2 = 0,\ a_5 = 2 - 0 = 2$

8. $a_1 = -3,\ a_2 = 4 + (-3) = 1,\ a_3 = 4 + 1 = 5,\ a_4 = 4 + 5 = 9,\ a_5 = 4 + 9 = 13$

9. $\{n+5\}$ Arithmetic
$d = (n+1+5) - (n+5) = n + 6 - n - 5 = 1$
$S_n = \frac{n}{2}[6 + n + 5] = \frac{n}{2}(n+11)$

10. $\{4n+3\}$ Arithmetic
$d = (4(n+1)+3) - (4n+3) = 4n + 4 + 3 - 4n - 3 = 4$
$S_n = \frac{n}{2}[7 + 4n + 3] = \frac{n}{2}(4n+10) = 2n^2 + 5n$

11. $\left\{2n^3\right\}$ Examine the terms of the sequence: 2, 16, 54, 128, 250, ...
There is no common difference; there is no common ratio; neither.

12. $\left\{2n^2-1\right\}$ Examine the terms of the sequence: 1, 7, 17, 31, 49, ...
There is no common difference; there is no common ratio; neither.

13. $\left\{2^{3n}\right\}$ Geometric $\quad r=\dfrac{2^{3(n+1)}}{2^{3n}}=\dfrac{2^{3n+3}}{2^{3n}}=2^{3n+3-3n}=2^3=8$

$$S_n=8\left(\frac{1-8^n}{1-8}\right)=8\left(\frac{1-8^n}{-7}\right)=\frac{8}{7}\left(8^n-1\right)$$

14. $\left\{3^{2n}\right\}$ Geometric $\quad r=\dfrac{3^{2(n+1)}}{3^{2n}}=\dfrac{3^{2n+2}}{3^{2n}}=3^{2n+2-2n}=3^2=9$

$$S_n=9\left(\frac{1-9^n}{1-9}\right)=9\left(\frac{1-9^n}{-8}\right)=\frac{9}{8}\left(9^n-1\right)$$

15. 0, 4, 8, 12, ... Arithmetic $\quad d=4-0=4$

$$S_n=\frac{n}{2}\big(2(0)+(n-1)4\big)=\frac{n}{2}\big(4(n-1)\big)=2n(n-1)$$

16. 1, –3, –7, –11, ... Arithmetic $d=-3-1=-4$

$$S_n=\frac{n}{2}\big(2(1)+(n-1)(-4)\big)=\frac{n}{2}\left(2-4n+4\right)=\frac{n}{2}(6-4n)=3n-2n^2$$

17. $3,\dfrac{3}{2},\dfrac{3}{4},\dfrac{3}{5},\dfrac{3}{16},\ldots$ Geometric $\quad r=\dfrac{\left(\frac{3}{2}\right)}{3}=\dfrac{3}{2}\ \dfrac{1}{3}=\dfrac{1}{2}$

$$S_n=3\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right)=3\left(\frac{1-\left(\frac{1}{2}\right)^n}{\left(\frac{1}{2}\right)}\right)=6\left(1-\left(\frac{1}{2}\right)^n\right)$$

18. $5,-\dfrac{5}{3},\dfrac{5}{9},-\dfrac{5}{27},\dfrac{5}{81},\ldots$ Geometric $\quad r=\dfrac{\left(-\frac{5}{3}\right)}{5}=-\dfrac{5}{3}\ \dfrac{1}{5}=-\dfrac{1}{3}$

$$S_n=5\left(\frac{1-\left(-\frac{1}{3}\right)^n}{1-\left(-\frac{1}{3}\right)}\right)=5\left(\frac{1-\left(-\frac{1}{3}\right)^n}{\left(\frac{4}{3}\right)}\right)=\frac{15}{4}\left(1-\left(-\frac{1}{3}\right)^n\right)$$

19. Neither. There is no common difference or common ratio.

20. $\dfrac{3}{2},\dfrac{5}{4},\dfrac{7}{6},\dfrac{9}{8},\dfrac{11}{10},\ldots$ Neither. There is no common difference or common ratio.

21. $\sum_{k=1}^{5}(k^2+12) = 13+16+21+28+37 = 115$

22. $\sum_{k=1}^{3}(k+2)^2 = 9+16+25 = 50$

23. $\sum_{k=1}^{10}(3k-9) = \sum_{k=1}^{10}3k - \sum_{k=1}^{10}9 = 3\sum_{k=1}^{10}k - \sum_{k=1}^{10}9 = 3\left(\frac{10(10+1)}{2}\right) - 10(9) = 165-90 = 75$

24. $\sum_{k=1}^{9}(-2k+8) = \sum_{k=1}^{9}-2k + \sum_{k=1}^{9}8 = -2\sum_{k=1}^{9}k + \sum_{k=1}^{9}8 = -2\left(\frac{9(1+9)}{2}\right) + 9(8) = -90+72 = -18$

25. $\sum_{k=1}^{7}\left(\frac{1}{3}\right)^k = \frac{1}{3}\left(\frac{1-\left(\frac{1}{3}\right)^7}{1-\frac{1}{3}}\right) = \frac{1}{3}\left(\frac{1-\left(\frac{1}{3}\right)^7}{\frac{2}{3}}\right) = \frac{1}{2}\left(1-\frac{1}{2187}\right) = \frac{1}{2}\;\frac{2186}{2187} = \frac{1093}{2187}$

26. $\sum_{k=1}^{10}(-2)^k = -2\left(\frac{1-(-2)^{10}}{1-(-2)}\right) = -2\left(\frac{1-1024}{3}\right) = -\frac{2}{3}(-1023) = 682$

27. Arithmetic $a_1 = 3,\; d = 4,\; a_n = a+(n-1)d$
$a_9 = 3+(9-1)4 = 3+8(4) = 3+32 = 35$

28. Arithmetic $a_1 = 1,\; d = -2,\; a_n = a+(n-1)d$
$a_8 = 1+(8-1)(-2) = 1+7(-2) = 1-14 = -13$

29. Geometric $a = 1,\; r = \frac{1}{10},\; n = 11$ $a_{11} = 1\left(\frac{1}{10}\right)^{11-1} = \left(\frac{1}{10}\right)^{10} = \frac{1}{10{,}000{,}000{,}000}$

30. Geometric $a = 1,\; r = 2,\; n = 11$ $a_{11} = 1\,(2)^{11-1} = (2)^{10} = 1024$

31. Arithmetic $a_1 = \sqrt{2},\; d = \sqrt{2},\; n = 9,\; a_n = a+(n-1)d$
$a_9 = \sqrt{2}+(9-1)\sqrt{2} = \sqrt{2}+8\sqrt{2} = 9\sqrt{2}$

32. Geometric $a_1 = \sqrt{2},\; d = \sqrt{2},\; n = 9,\; a_n = a\,r^{n-1}$
$a_9 = \sqrt{2}\left(\sqrt{2}\right)^{9-1} = \sqrt{2}\left(\sqrt{2}\right)^{8} = \sqrt{2}\;16 = 16\sqrt{2}$

33. $a_7 = a+6d = 31$ $a_{20} = a+19d = 96$
Solve the system of equations:

$$\begin{aligned} 31-6d+19d &= 96 \\ 13d &= 65 \\ d &= 5 \\ a &= 31-6(5) = 31-30 = 1 \end{aligned}$$

General formula: $\{5n-4\}$

34. $a_8 = a + 7d = -20 \qquad a_{17} = a + 16d = -47$

Solve the system of equations:

$$\begin{aligned} -20 - 7d + 16d &= -47 \\ 9d &= -27 \\ d &= -3 \\ a &= -20 - 7(-3) = -20 + 21 = 1 \end{aligned}$$

General formula: $\{-3n + 4\}$

35. $a_{10} = a + 9d = 0 \qquad a_{18} = a + 17d = 8$

Solve the system of equations:

$$\begin{aligned} -9d + 17d &= 8 \\ 8d &= 8 \\ d &= 1 \\ a &= -9(1) = -9 \end{aligned}$$

General formula: $\{n - 10\}$

36. $a_{12} = a + 11d = 30 \qquad a_{22} = a + 21d = 50$

Solve the system of equations:

$$\begin{aligned} 30 - 11d + 21d &= 50 \\ 10d &= 20 \\ d &= 2 \\ a &= 30 - 11(2) = 30 - 22 = 8 \end{aligned}$$

General formula: $\{2n + 6\}$

37.

$a = 3,\ r = \frac{1}{3}$ Since $|r| < 1$, $S_n = \frac{a}{1-r} = \frac{3}{\left(1 - \frac{1}{3}\right)} = \frac{3}{\left(\frac{2}{3}\right)} = \frac{9}{2}$

38.

$a = 2,\ r = \frac{1}{2}$ Since $|r| < 1$, $S_n = \frac{a}{1-r} = \frac{2}{\left(1 - \frac{1}{2}\right)} = \frac{2}{\left(\frac{1}{2}\right)} = 4$

39.

$a = 2,\ r = -\frac{1}{2}$ Since $|r| < 1$, $S_n = \frac{a}{1-r} = \frac{2}{\left(1 - \left(-\frac{1}{2}\right)\right)} = \frac{2}{\left(\frac{3}{2}\right)} = \frac{4}{3}$

40.

$a = 6,\ r = -\frac{2}{3}$ Since $|r| < 1$, $S_n = \frac{a}{1-r} = \frac{6}{\left(1 - \left(-\frac{2}{3}\right)\right)} = \frac{6}{\left(\frac{5}{3}\right)} = \frac{18}{5}$

41.

$a = 4,\ r = \frac{1}{2}$ Since $|r| < 1$, $S_n = \frac{a}{1-r} = \frac{4}{\left(1-\frac{1}{2}\right)} = \frac{4}{\left(\frac{1}{2}\right)} = 8$

42.

$a = 3,\ r = -\frac{3}{4}$ Since $|r| < 1$, $S_n = \frac{a}{1-r} = \frac{3}{\left(1-\left(-\frac{3}{4}\right)\right)} = \frac{3}{\left(\frac{7}{4}\right)} = \frac{12}{7}$

43. I: $n = 1$: $3 \cdot 1 = 3$ and $\frac{3 \cdot 1}{2}(1+1) = 3$

II: If $3 + 6 + 9 + \cdots + 3k = \frac{3k}{2}(k+1)$

then $3 + 6 + 9 + \cdots + 3k + 3(k+1)$

$= \left[3 + 6 + 9 + \cdots + 3k\right] + 3(k+1) = \frac{3k}{2}(k+1) + 3(k+1)$

$= (k+1)\left(\frac{3k}{2} + 3\right) = \frac{3}{2}(k+1)(k+2)$

Conditions I and II are satisfied; the statement is true.

44. I: $n = 1$: $4 \cdot 1 - 2 = 2$ and $2(1)^2 = 2$

II: If $2 + 6 + 10 + \cdots + (4k-2) = 2k^2$

then $2 + 6 + 10 + \cdots + (4k-2) + (4(k+1)-2)$

$= \left[2 + 6 + 10 + \cdots + (4k-2)\right] + 4k + 2 = 2k^2 + 4k + 2$

$= 2\left(k^2 + 2k + 1\right) = 2(k+1)^2$

Conditions I and II are satisfied; the statement is true.

45. I: $n = 1$: $2 \cdot 3^{1-1} = 2$ and $3^1 - 1 = 2$

II: If $2 + 6 + 18 + \cdots + 2 \cdot 3^{k-1} = 3^k - 1$

then $2 + 6 + 18 + \cdots + 2 \cdot 3^{k-1} + 2 \cdot 3^{k+1-1}$

$= \left[2 + 6 + 18 + \cdots + 2 \cdot 3^{k-1}\right] + 2 \cdot 3^k = 3^k - 1 + 2 \cdot 3^k$

$= 3 \cdot 3^k - 1 = 3^{k+1} - 1$

Conditions I and II are satisfied; the statement is true.

46. I: $n = 1$: $3 \cdot 2^{1-1} = 3$ and $3\left(2^1 - 1\right) = 3$

II: If $3 + 6 + 12 + \cdots + 3 \cdot 2^{k-1} = 3\left(2^k - 1\right)$

then $3 + 6 + 12 + \cdots + 3 \cdot 2^{k-1} + 3 \cdot 2^{k+1-1}$

$= \left[3 + 6 + 12 + \cdots + 3 \cdot 2^{k-1}\right] + 3 \cdot 2^k = 3\left(2^k - 1\right) + 3 \cdot 2^k$

$= 3 \cdot \left(2^k - 1 + 2^k\right) = 3\left(2 \cdot 2^k - 1\right) = 3\left(2^{k+1} - 1\right)$

Conditions I and II are satisfied; the statement is true.

47. I: $n=1$: $(3\cdot 1-2)^2=1$ and $\frac{1}{2}\cdot 1(6\cdot 1^2-3\cdot 1-1)=1$

II: If $1^2+4^2+7^2+\cdots+(3k-2)^2=\frac{1}{2}\cdot k\left(6k^2-3k-1\right)$

then $1^2+4^2+7^2+\cdots+(3k-2)^2+\left(3(k+1)-2\right)^2$

$$=\left[1^2+4^2+7^2+\cdots+(3k-2)^2\right]+(3k+1)^2$$
$$=\frac{1}{2}\cdot k\left(6k^2-3k-1\right)+(3k+1)^2$$
$$=\frac{1}{2}\cdot\left[6k^3-3k^2-k+18k^2+12k+2\right]=\frac{1}{2}\cdot\left[6k^3+15k^2+11k+2\right]$$
$$=\frac{1}{2}\cdot(k+1)\left[6k^2+9k+2\right]=\frac{1}{2}\cdot(k+1)\left[6k^2+12k+6-3k-3-1\right]$$
$$=\frac{1}{2}\cdot(k+1)\left[6(k^2+2k+1)-3(k+1)-1\right]$$
$$=\frac{1}{2}\cdot(k+1)\left[6(k+1)^2-3(k+1)-1\right]$$

Conditions I and II are satisfied; the statement is true.

48. I: $n=1$: $1(1+2)=3$ and $\frac{1}{6}\cdot 1(1+1)(2\cdot 1+7)=3$

II: If $1\cdot 3+2\cdot 4+3\cdot 5+\cdots+k(k+2)=\frac{k}{6}(k+1)(2k+7)$

then $1\cdot 3+2\cdot 4+3\cdot 5+\cdots+k(k+2)+(k+1)(k+1+2)$

$$=\left[1\cdot 3+2\cdot 4+3\cdot 5+\cdots+k(k+2)\right]+(k+1)(k+3)$$
$$=\frac{k}{6}(k+1)(2k+7)+(k+1)(k+3)=\frac{(k+1)}{6}\left(2k^2+7k+6k+18\right)$$
$$=\frac{(k+1)}{6}\left(2k^2+13k+18\right)=\frac{(k+1)}{6}(k+2)(2k+9)$$

Conditions I and II are satisfied; the statement is true.

49. $$(x+2)^5=\binom{5}{0}x^5+\binom{5}{1}x^4\cdot 2+\binom{5}{2}x^3\cdot 2^2+\binom{5}{3}x^2\cdot 2^3+\binom{5}{4}x^1\cdot 2^4+\binom{5}{5}\cdot 2^5$$
$$=x^5+5\cdot 2x^4+10\cdot 4x^3+10\cdot 8x^2+5\cdot 16x+1\cdot 32$$
$$=x^5+10x^4+40x^3+80x^2+80x+32$$

50. $$(x-3)^4=\binom{4}{0}x^4+\binom{4}{1}x^3(-3)+\binom{4}{2}x^2(-3)^2+\binom{4}{3}x(-3)^3+\binom{4}{4}x^0(-3)^4$$
$$=x^4+4(-3)x^3+6\cdot 9x^2+4(-27)x+81$$
$$=x^4-12x^3+54x^2-108x+81$$

51. $$(2x+3)^5=\binom{5}{0}(2x)^5+\binom{5}{1}(2x)^4\cdot 3+\binom{5}{2}(2x)^3\cdot 3^2+\binom{5}{3}(2x)^2\cdot 3^3$$
$$+\binom{5}{4}(2x)^1\cdot 3^4+\binom{5}{5}\cdot 3^5$$

$= 32x^5 + 5 \cdot 16x^4 \cdot 3 + 10 \cdot 8x^3 \cdot 9 + 10 \cdot 4x^2 \cdot 27 + 5 \cdot 2x \cdot 81 + 1 \cdot 243$
$= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$

52. $(3x-4)^4 = \binom{4}{0}(3x)^4 + \binom{4}{1}(3x)^3(-4) + \binom{4}{2}(3x)^2(-4)^2 + \binom{4}{3}(3x)(-4)^3 + \binom{4}{4}(-4)^4$
$= 81x^4 + 4 \cdot 27x^3(-4) + 6 \cdot 9x^2 \cdot 16 + 4 \cdot 3x(-64) + 1 \cdot 256$
$= 81x^4 - 432x^3 + 864x^2 - 768x + 256$

53. $n = 9,\ j = 2,\ x = x,\ a = 2$
$\binom{9}{2}x^7 \cdot 2^2 = \frac{9!}{2!\,7!} \cdot 4x^7 = \frac{9 \cdot 8}{2 \cdot 1} \cdot 4x^7 = 144x^7$
The coefficient of x^7 is 144.

54. $n = 8,\ j = 5,\ x = x,\ a = -3$
$\binom{8}{5}x^3(-3)^5 = \frac{8!}{5!\,3!}(-243)x^3 = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}(-243)x^3 = -13{,}608x^3$
The coefficient of x^3 is $-13{,}608$.

55. $n = 7,\ j = 5,\ x = 2x,\ a = 1$
$\binom{7}{5}(2x)^2 \cdot 1^5 = \frac{7!}{5!\,2!} \cdot 4x^2(1) = \frac{7 \cdot 6}{2 \cdot 1} \cdot 4x^2 = 84x^2$
The coefficient of x^2 is 84.

56. $n = 8,\ j = 2,\ x = 2x,\ a = 1$
$\binom{8}{2}(2x)^6 \cdot 1^2 = \frac{8!}{2!\,6!} \cdot 64x^6(1) = \frac{8 \cdot 7}{2 \cdot 1} \cdot 64x^6 = 1792x^6$
The coefficient of x^6 is 1792.

57. This is an arithmetic sequence with $a = 80,\ d = -3,\ n = 25$
(a) $a_{25} = 80 + (25-1)(-3) = 80 - 72 = 8$ bricks
(b) $S_{25} = \frac{25}{2}(80 + 8) = 25(44) = 1100$ bricks
1100 bricks are needed to build the steps.

58. This is an arithmetic sequence with $a = 30,\ d = -1,\ a_n = 15$

$15 = 30 + (n-1)(-1) \rightarrow -15 = -n + 1 \rightarrow -16 = -n \rightarrow n = 16$
$S_{16} = \frac{16}{2}(30 + 15) = 8(45) = 360$ tiles
360 tiles are required to make the trapezoid.

59. This is an ordinary annuity with $P = \$200$ and $n = (12)(20) = 240$ payment periods.

The interest rate per period is $\frac{.10}{12} = .008\overline{3}$. Thus,

$$A = 200\left[\frac{\left(1 + \frac{.10}{12}\right)^{240} - 1}{\left(\frac{.10}{12}\right)}\right] = \$151873.77$$

60. This is an ordinary annuity with $P = \$500$ and $n = (4)(30) = 120$ payment periods.

The interest rate per period is $\frac{.08}{4} = .02$. Thus

$$A = 500\left[\frac{\left(1 + \frac{.08}{4}\right)^{120} - 1}{\left(\frac{.08}{4}\right)}\right] = \$244129.08$$

61. This is a geometric sequence with $a = 20,\ r = \frac{3}{4}$.

(a) After striking the ground the third time, the height is $20\left(\frac{3}{4}\right)^3 = \frac{135}{16} \approx 8.44$ feet.

(b) After striking the ground the n^{th} time, the height is $20\left(\frac{3}{4}\right)^n$ feet.

(c) If the height is less than 6 inches or 0.5 feet, then:

$$0.5 = 20\left(\frac{3}{4}\right)^n \to 0.025 = \left(\frac{3}{4}\right)^n \to \log 0.025 = n \log\left(\frac{3}{4}\right) \to n = \frac{\log 0.025}{\log\left(\frac{3}{4}\right)} = 12.82$$

The height is less than 6 inches after the 13th strike.

(d) Since this is a geometric sequence with $|r| < 1$, the distance is the sum of the two infinite geometric series - the distances going down plus the distances going up.

Distance going down: $S_{down} = \frac{20}{\left(1 - \frac{3}{4}\right)} = \frac{20}{\left(\frac{1}{4}\right)} = 80$ feet.

Distance going up: $S_{up} = \frac{15}{\left(1 - \frac{3}{4}\right)} = \frac{15}{\left(\frac{1}{4}\right)} = 60$ feet.

The total distance traveled is 140 feet.

62. This is a geometric sequence with $a = 20{,}000,\ r = 1.04,\quad n = 5$.

Find the fifth term of the sequence:

$a_5 = 20000(1.04)^{5-1} = 20000(1.04)^4 = \$23{,}397.17$

The salary in the fifth year will be \$23,397.17.