Chapter 1

Equations and Inequalities

1.6 Equations and Inequalities Involving Absolute Value

1. $|2x| = 6$
$2x = 6$ or $2x = -6$
$x = 3$ or $x = -3$
The solution set is {–3, 3}.

2. $|3x| = 12$
$3x = 12$ or $3x = -12$
$x = 4$ or $x = -4$
The solution set is $\{-4, 4\}$.

3. $|2x+3| = 5$
$2x+3 = 5$ or $2x+3 = -5$
$2x = 2$ or $2x = -8$
$x = 1$ or $x = -4$
The solution set is {–4, 1}.

4. $|3x-1| = 2$
$3x-1 = 2$ or $3x-1 = -2$
$3x = 3$ or $3x = -1$
$x = 1$ or $x = -\frac{1}{3}$
The solution set is $\{-\frac{1}{3}, 1\}$.

5. $|1-4t|+8 = 13 \to |1-4t| = 5$
$1-4t = 5$ or $1-4t = -5$
$-4t = 4$ or $-4t = -6$
$t = -1$ or $t = \frac{3}{2}$
The solution set is $\left\{-1, \frac{3}{2}\right\}$.

6. $|1-2z|+6 = 9 \to |1-2z| = 3$
$1-2z = 3$ or $1-2z = -3$
$-2z = 2$ or $-2z = -4$
$z = -1$ or $z = 2$
The solution set is $\{-1, 2\}$.

7. $|-2x| = 8$
$-2x = 8$ or $-2x = -8$
$x = -4$ or $x = 4$
The solution set is {–4, 4}.

8. $|-x| = |1|$
$-x = 1$ or $-x = -1$
The solution set is {–1, 1}.

9. $|-2|x = 4$
$2x = 4$
$x = 2$
The solution set is {2}.

10. $|3|x = 9$
$3x = 9$
$x = 3$
The solution set is {3}.

11. $\frac{2}{3}|x| = 9$
$|x| = \frac{27}{2} \to x = \frac{27}{2}$ or $x = -\frac{27}{2}$
The solution set is $\left\{-\frac{27}{2}, \frac{27}{2}\right\}$.

12. $\frac{3}{4}|x| = 9$
$|x| = 12 \to x = 12$ or $x = -12$
The solution set is {–12, 12}.

13. $\left|\frac{x}{3}+\frac{2}{5}\right| = 2$

$\frac{x}{3}+\frac{2}{5} = 2$ or $\frac{x}{3}+\frac{2}{5} = -2$

$5x+6 = 30$ or $5x+6 = -30$

$5x = 24$ or $5x = -36$

$x = \frac{24}{5}$ or $x = \frac{-36}{5}$

The solution set is $\left\{\frac{-36}{5}, \frac{24}{5}\right\}$.

14. $\left|\frac{x}{2}-\frac{1}{3}\right| = 1$

$\frac{x}{2}-\frac{1}{3} = 1$ or $\frac{x}{2}-\frac{1}{3} = -1$

$3x-2 = 6$ or $3x-2 = -6$

$3x = 8$ or $3x = -4$

$x = \frac{8}{3}$ or $x = -\frac{4}{3}$

The solution set is $\left\{-\frac{4}{3}, \frac{8}{3}\right\}$.

15. $|u-2| = -\frac{1}{2}$

impossible, since absolute value always yields a non-negative number.

16. $|2-v| = -1$

impossible, since absolute value always yields a non-negative number.

17.

$4-|2x| = 3 \rightarrow -|2x| = -1$

$\rightarrow |2x| = 1$

$2x = 1$ or $2x = -1$

$x = \frac{1}{2}$ or $x = -\frac{1}{2}$

The solution set is $\left\{-\frac{1}{2}, \frac{1}{2}\right\}$.

18.

$5-\left|\frac{1}{2}x\right| = 3 \rightarrow -\left|\frac{1}{2}x\right| = -2$

$\rightarrow \left|\frac{1}{2}x\right| = 2$

$\frac{1}{2}x = 2$ or $\frac{1}{2}x = -2$

$x = 4$ or $x = -4$

The solution set is { - 4, 4}.

19. $|x^2-9| = 0$

$x^2-9 = 0$

$x^2 = 9$

$x = \pm 3$

The solution set is $\{-3, 3\}$.

20. $|x^2-16| = 0$

$x^2-16 = 0$

$x^2 = 16$

$x = \pm 4$

The solution set is $\{-4, 4\}$.

21. $|x^2-2x| = 3$

$x^2-2x = 3$ or $x^2-2x = -3$

$x^2-2x-3 = 0$ or $x^2-2x+3 = 0$

$(x-3)(x+1) = 0$

or $x = \frac{2\pm\sqrt{4-12}}{2} = \frac{2\pm\sqrt{-8}}{2}$

$x = 3, x = -1$ or $\rightarrow$ no real solution

The solution set is $\{-1, 3\}$.

22. $|x^2+x| = 12$

$x^2+x = 12$ or $x^2+x = -12$

$x^2+x-12 = 0$ or $x^2+x+12 = 0$

$(x-3)(x+4) = 0$ or $x = \frac{-1\pm\sqrt{1-48}}{2} = \frac{1\pm\sqrt{-47}}{2}$

$x = 3, x = -4$ or no real solution

The solution set is $\{-4, 3\}$.

23. $\left|x^2 + x - 1\right| = 1$

$x^2 + x - 1 = 1$ or $x^2 + x - 1 = -1$

$x^2 + x - 2 = 0$ or $x^2 + x = 0$

$(x-1)(x+2) = 0$ or $x(x+1) = 0$

$x = 1, x = -2$ or $x = 0, x = -1$

The solution set is $\{-2, -1, 0, 1\}$.

24. $\left|x^2 + 3x - 2\right| = 2$

$x^2 + 3x - 2 = 2$ or $x^2 + 3x - 2 = -2$

$x^2 + 3x = 4$ or $x^2 + 3x = 0$

$x^2 + 3x - 4 = 0$ or $x(x+3) = 0$

$(x+4)(x-1) = 0$ or $x = 0, x = -3$

$x = -4, x = 1$

The solution set is $\{-4, -3, 0, 1\}$.

25. $|2x| < 8$

$-8 < 2x < 8$

$-4 < x < 4$

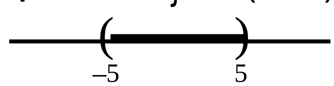
$\{x \mid -4 < x < 4\}$ or $(-4, 4)$

26. $|3x| < 15$

$-15 < 3x < 15$

$-5 < x < 5$

$\{x \mid -5 < x < 5\}$ or $(-5, 5)$



27. $|3x| > 12$

$3x < -12$ or $3x > 12$

$x < -4$ or $x > 4$

$\{x \mid x < -4 \text{ or } x > 4\}$ or

$(-\infty, -4)$ $(4, +\infty)$

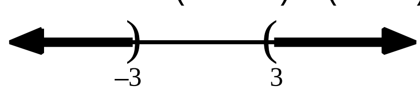
28. $|2x| > 6$

$2x < -6$ or $2x > 6$

$x < -3$ or $x > 3$

$\{x \mid x < -3 \text{ or } x > 3\}$ or

$(-\infty, -3)$ $(3, +\infty)$



29. $|x - 2| + 2 < 3$

$|x - 2| < 1$

$-1 < x - 2 < 1$

$1 < x < 3$

$\{x \mid 1 < x < 3\}$ or $(1, 3)$

30. $|x + 4| + 3 < 5$

$|x + 4| < 2$

$-2 < x + 4 < 2$

$-6 < x < -2$

$\{x \mid -6 < x < -2\}$ or $(-6, -2)$

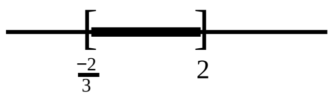
31. $|3t-2| \le 4$

$-4 \le 3t-2 \le 4$

$-2 \le 3t \le 6$

$\frac{-2}{3} \le t \le 2$

$\left\{ t \,\middle|\, \frac{-2}{3} \le t \le 2 \right\}$ or $\left[\frac{-2}{3}, 2\right]$



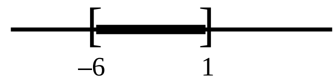
32. $|2u+5| \le 7$

$-7 \le 2u+5 \le 7$

$-12 \le 2u \le 2$

$-6 \le u \le 1$

$\{u \mid -6 \le u \le 1\}$ or $[-6, 1]$



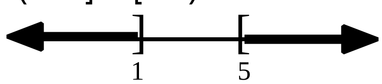
33. $|x-3| \ge 2$

$x-3 \le -2$ or $x-3 \ge 2$

$x \le 1$ or $x \ge 5$

$\{x \mid x \le 1 \text{ or } x \ge 5\}$

or $(-\infty, 1] \quad [5, \infty)$



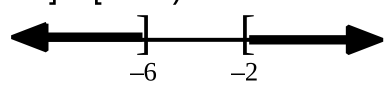
34. $|x+4| \ge 2$

$x+4 \le -2$ or $x+4 \ge 2$

$x \le -6$ or $x \ge -2$

$\{x \mid x \le -6 \text{ or } x \ge -2\}$

or $(-\infty, -6] \quad [-2, \infty)$



35. $|1-4x| - 7 < -2$

$|1-4x| < 5$

$-5 < 1-4x < 5$

$-6 < -4x < 4$

$-1 < x < \frac{3}{2}$

$\left\{ x \,\middle|\, -1 < x < \frac{3}{2} \right\}$ or $\left(-1, \frac{3}{2}\right)$

36. $|1-2x| - 4 < -1$

$|1-2x| < 3$

$-3 < 1-2x < 3$

$-4 < -2x < 2$

$-1 < x < 2$

$\{x \mid -1 < x < 2\}$ or $(-1, 2)$

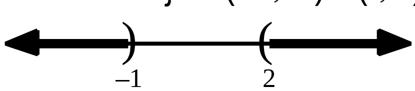
37. $|1-2x| > 3$

$1-2x < -3$ or $1-2x > 3$

$-2x < -4$ or $-2x > 2$

$x > 2$ or $x < -1$

$\{x \mid x < -1 \text{ or } x > 2\}$ or $(-\infty, -1) \quad (2, \infty)$



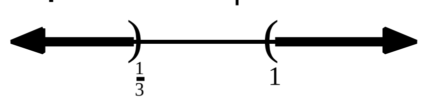
38. $|2-3x| > 1$

$2-3x < -1$ or $2-3x > 1$

$-3x < -3$ or $-3x > -1$

$x > 1$ or $x < \frac{1}{3}$

$\left\{ x \,\middle|\, x < \frac{1}{3} \text{ or } x > 1 \right\}$ or $\left(-\infty, \frac{1}{3}\right) \quad (1, \infty)$



39. $|-4x|+|-5|\le 1$

$|4x|+5\le 1$

$|4x|\le -4$

but this is impossible since absolute value always yields a non-negative number.

40. $|-x|-|4|\le 2$

$|x|-4\le 2$

$|x|\le 6$

$-6\le x\le 6$

$\{x|-6\le x\le 6\}$ or $[-6,6]$

41. $|-2x|>|-3|$

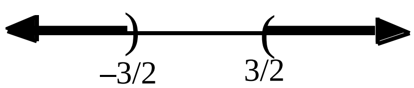
$|2x|>3$

$2x<-3$ or $2x>3$

$x<-\frac{3}{2}$ or $x>\frac{3}{2}$

$\left\{x \middle| x<-\frac{3}{2} \text{ or } x>\frac{3}{2}\right\}$

or $\left(-\infty,-\frac{3}{2}\right)$ $\left(\frac{3}{2},+\infty\right)$



42. $|-x-2|\ge 1$

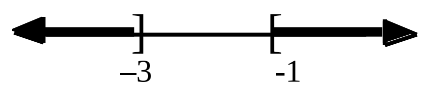
$-x-2\le -1$ or $-x-2\ge 1$

$-x\le 1$ or $-x\ge 3$

$x\ge -1$ or $x\le -3$

$\{x|x\le -3 \text{ or } x\ge -1\}$ or

$(-\infty,-3]$ $[-1,+\infty)$



43. $-|2x-1|\ge -3$

$|2x-1|\le 3$

$-3\le 2x-1\le 3$

$-2\le 2x\le 4$

$-1\le x\le 2$

$\{x|-1\le x\le 2\}$ or $[-1,2]$

44. $-|1-2x|\ge -3$

$|1-2x|\le 3$

$-3\le 1-2x\le 3$

$-4\le -2x\le 2$

$-1\le x\le 2$

$\{x|-1\le x\le 2\}$ or $[-1,2]$

45.

$|x-1|<3 \rightarrow -3<x-1<3$

$\rightarrow -2<x<4$

$\rightarrow 2<x+4<8$

$\rightarrow a=2, b=8$

46

$|x+2|<5 \rightarrow -5<x+2<3$

$\rightarrow -7<x<-1$

$\rightarrow -9<x-2<-3$

$\rightarrow a=-9, b=-3$

47.

$|x+4| \le 2 \rightarrow -2 \le x+4 \le 2$

$\rightarrow -6 \le x \le -2$

$\rightarrow -12 \le 2x \le -4$

$\rightarrow -15 \le 2x-3 \le -7$

$\rightarrow a = -15, b = -7$

48.

$|x-3| \le 1 \rightarrow -1 \le x-3 \le 1$

$\rightarrow 2 \le x \le 4$

$\rightarrow 6 \le 3x \le 12$

$\rightarrow 7 \le 3x+1 \le 13$

$\rightarrow a = 7, b = 13$

49.

$|x-2| \le 7 \rightarrow -7 \le x-2 \le 7$

$\rightarrow -5 \le x \le 9$

$\rightarrow -15 \le x-10 \le -1$

$\rightarrow -\frac{1}{15} \ge \frac{1}{x-10} \ge -1$

$\rightarrow -1 \le \frac{1}{x-10} \le -\frac{1}{15}$

$\rightarrow a = -1, b = -\frac{1}{15}$

50.

$|x+1| \le 3 \rightarrow -3 \le x+1 \le 3$

$\rightarrow -4 \le x \le 2$

$\rightarrow 1 \le x+5 \le 7$

$\rightarrow 1 \ge \frac{1}{x+5} \ge \frac{1}{7}$

$\rightarrow \frac{1}{7} \le \frac{1}{x+5} \le 1$

$\rightarrow a = \frac{1}{7}, b = 1$

51. If $b \ne 0$, prove $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.

Case 1: $\frac{a}{b} \ge 0 \rightarrow a \ge 0$ and $b > 0$ or $a \le 0$ and $b < 0$.

if $a \ge 0$ and $b > 0$ then $|a| = a$ and $|b| = b$.

so, $\frac{a}{b} \ge 0 \rightarrow \left|\frac{a}{b}\right| = \frac{a}{b} = \frac{|a|}{|b|}$.

if $a \le 0$ and $b < 0$ then $|a| = -a$ and $|b| = -b$.

so, $\frac{a}{b} \ge 0 \rightarrow \left|\frac{a}{b}\right| = \frac{a}{b} = \frac{-|a|}{-|b|} = \frac{|a|}{|b|}$.

Case 2: $\frac{a}{b} < 0 \rightarrow a > 0$ and $b < 0$ or $a < 0$ and $b > 0$.

if $a > 0$ and $b < 0$ then $|a| = a$ and $|b| = -b$.

now, $\frac{a}{b} < 0 \rightarrow \left|\frac{a}{b}\right| = -\left(\frac{a}{b}\right) = -\left(\frac{a}{-b}\right) = \frac{a}{b}$.

if $a < 0$ and $b > 0$ then $|a| = -a$ and $|b| = b$.

now, $\frac{a}{b} < 0 \rightarrow \left|\frac{a}{b}\right| = -\left(\frac{a}{b}\right) = -\left(\frac{-a}{b}\right) = \frac{a}{b}$.

52. Show that $a \le |a|$.

We know that $0 \le |a|$. So if a < 0, then we have $a < 0 \le |a| \rightarrow a \le |a|$

Now, if $a \ge 0$, then $|a| = a$. So $a \le |a|$.

53. $|a+b|^2 = |a+b|\ |a+b|$

Case 1: $a+b \ge 0 \rightarrow |a+b| = a+b$

so $|a+b|\ |a+b| = (a+b)(a+b) = a^2 + 2ab + b^2$

$\le |a|^2 + 2|a|\ |b| + |b|^2$ by problem 52

$= (|a| + |b|)^2$

$(|a+b|)^2 \le (|a|+|b|)^2 \rightarrow |a+b| \le |a| + |b|$

Case 2 $a+b < 0 \rightarrow |a+b| = -(a+b)$

so $|a+b|\ |a+b| = (-(a+b))(-(a+b))$

$= (a+b)(a+b) = a^2 + 2ab + b^2$

$\le |a|^2 + 2|a|\ |b| + |b|^2$ by problem 52

$= (|a| + |b|)^2$

$(|a+b|)^2 \le (|a|+|b|)^2 \rightarrow |a+b| \le |a| + |b|$

54. To prove $|a-b| \ge a-b$, consider the following:

$|a| = |(a-b)+b| \le |a-b| + |b|$ by the Triangle Inequality

so $|a| \le |a-b| + |b| \rightarrow |a| - |b| \le |a-b|$

therefore $|a-b| \ge |a| - |b|$.

55. x differs from 3 by less than $\frac{1}{2}$

$$|x-3| < \frac{1}{2}$$

$$\frac{-1}{2} < x-3 < \frac{7}{2}$$

$$\frac{5}{2} < x < \frac{7}{2}$$

$$\left\{ x \,\middle|\, \frac{3}{2} < x < \frac{5}{2} \right\}$$

56 x differs from -1 by less than 1

$$|x-(-4)| < 1$$

$$|x+4| < 1$$

$$-1 < x+4 < 1$$

$$-5 < x < -3$$

$$\{x \mid -5 < x < -3\}$$

57. x differs from -3 by more than 2

$$|x-(-3)|>2$$
$$x+3<-2 \text{ or } x+3>2$$
$$x<-5 \quad \text{or} \quad x>-1$$
$$\{x \mid x<-5 \text{ or } x>-1\}$$

58. x differs from 2 by more than 3

$$|x-2|>3$$
$$x-2<-3 \text{ or } x-2>3$$
$$x<-1 \quad \text{or} \quad x>5$$
$$\{x \mid x<-1 \text{ or } x>5\}$$

59. A temperature x that differs from 98.6°F by at least 1.5°

$$|x-98.6°|\geq 1.5°$$
$$x-98.6° \leq -1.5° \quad \text{or} \quad x-98.6° \geq 1.5°$$
$$x \leq 97.1° \quad \text{or} \quad x \geq 100.1°$$

The temperatures that are considered unhealthy are those that are less than 97.1˚F or greater than 100.1˚F, inclusive.

60. A voltage x that differs from 115 volts by at most 5 volts

$$|x-115| \leq 5$$
$$-5 \leq x-115 \leq 5$$
$$110 \leq x \leq 120$$

The actual voltage is between 110 and 120 volts, inclusive.

61. given that $a > 0$

$$x^2<a \rightarrow x^2-a<0$$
$$\left(x+\sqrt{a}\right)\left(x-\sqrt{a}\right)<0$$

if $x<-\sqrt{a}$, then $x+\sqrt{a}<0$ and $x-\sqrt{a}<-2\sqrt{a}<0$

therefore $\left(x+\sqrt{a}\right)\left(x-\sqrt{a}\right)>0$

if $-\sqrt{a}<x<\sqrt{a}$, then $0<x+\sqrt{a}<2\sqrt{a}$ and $-2\sqrt{a}<x-\sqrt{a}<0$

therefore $\left(x+\sqrt{a}\right)\left(x-\sqrt{a}\right)<0$

if $x>\sqrt{a}$, then $x+\sqrt{a}>2\sqrt{a}>0$ and $x-\sqrt{a}>0$

therefore $\left(x+\sqrt{a}\right)\left(x-\sqrt{a}\right)>0$

So the solution set for $x^2<a$ is $\{\text{real numbers } x \mid -\sqrt{a}<x<\sqrt{a}\}$

62. given that $a > 0$

$$x^2 > a \rightarrow x^2 - a > 0$$
$$\left(x + \sqrt{a}\right)\left(x - \sqrt{a}\right) > 0$$

if $x < -\sqrt{a}$, then $x + \sqrt{a} < 0$ and $x - \sqrt{a} < -2\sqrt{a} < 0$
therefore $\left(x + \sqrt{a}\right)\left(x - \sqrt{a}\right) > 0$

if $-\sqrt{a} < x < \sqrt{a}$, then $0 < x + \sqrt{a} < 2\sqrt{a}$ and $-2\sqrt{a} < x - \sqrt{a} < 0$
therefore $\left(x + \sqrt{a}\right)\left(x - \sqrt{a}\right) < 0$

if $x > \sqrt{a}$, then $x + \sqrt{a} > 2\sqrt{a} > 0$ and $x - \sqrt{a} > 0$
therefore $\left(x + \sqrt{a}\right)\left(x - \sqrt{a}\right) > 0$

So the solution set for $x^2 > a$ is $\left\{\text{real numbers } x \middle| x < -\sqrt{a} \text{ or } x > \sqrt{a}\right\}$

63. $\left\{\text{real numbers } x \middle| -1 < x < 1\right\}$

64. $\left\{\text{real numbers } x \middle| -2 < x < 2\right\}$

65. $\left\{\text{real numbers } x \middle| x \le -3 \text{ or } x \ge 3\right\}$

66. $\left\{\text{real numbers } x \middle| x \le -1 \text{ or } x \ge 1\right\}$

67. $\left\{\text{real numbers } x \middle| -4 \le x \le 4\right\}$

68. $\left\{\text{real numbers } x \middle| -3 \le x \le 3\right\}$

69. $\left\{\text{real numbers } x \middle| x < -2 \text{ or } x > 2\right\}$

70. $\left\{\text{real numbers } x \middle| x < -4 \text{ or } x > 4\right\}$

71. $\left|3x - |2x+1|\right| = 4$

$$\rightarrow 3x - |2x+1| = 4 \quad \text{or} \quad 3x - |2x+1| = -4$$
$$3x - |2x+1| = 4 \rightarrow 3x - 4 = |2x+1|$$
$$\rightarrow 2x + 1 = 3x - 4 \quad \text{or} \quad 2x + 1 = -(3x - 4)$$
$$2x + 1 = 3x - 4 \rightarrow 5 = x$$
$$2x + 1 = -(3x - 4) \rightarrow 2x + 1 = -3x + 4 \rightarrow 5x = 3 \rightarrow x = \frac{3}{5}$$

$3x - |2x+1| = -4 \rightarrow 3x + 4 = |2x+1|$

$\rightarrow 2x+1 = 3x+4$ or $2x+1 = -(3x+4)$

$2x+1 = 3x+4 \rightarrow -3 = x$

$2x+1 = -(3x+4) \rightarrow 2x+1 = -3x-4 \rightarrow 5x = -5 \rightarrow x = -1$

however, the only values that check in the original equation are $x = 5$ and $x = -1$.

72. $\left| x + |3x - 2| \right| = 2$

$\rightarrow x + |3x-2| = 2$ or $x + |3x-2| = -2$

$x + |3x-2| = 2 \rightarrow |3x-2| = 2 - x$

$\rightarrow 3x - 2 = 2 - x$ or $3x - 2 = -(2-x)$

$3x - 2 = 2 - x \rightarrow 4x = 4 \rightarrow x = 1$

$3x - 2 = -(2-x) \rightarrow 3x - 2 = -2x + 1 \rightarrow 5x = -3 \rightarrow x = -\frac{3}{5}$

$x + |3x-2| = -2 \rightarrow |3x-2| = -2 - x$

$\rightarrow 3x - 2 = -2 - x$ or $3x - 2 = -(-2-x)$

$3x - 2 = -2 - x \rightarrow 4x = 0 \rightarrow x = 0$

$3x - 2 = -(-2-x) \rightarrow 3x - 2 = 2 + x \rightarrow 2x = 4 \rightarrow x = 2$

however, the only values that check in the original equation are $x = 0$ and $x = 1$.

73. The absolute value of a real number is always greater than or equal to zero.

74. The absolute value of a real number is always greater than or equal to zero, and zero is greater than – 0.5.

75. if $x > 0$, then $|x| = x$, therefore $|x| > 0$.

if $x < 0$, then $|x| = -x$. So $x < 0 \rightarrow -x > 0 \rightarrow |x| > 0$.