Chapter 3

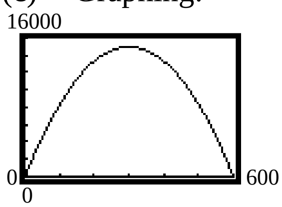
Functions and Their Graphs

3.6 Mathematical Models; Constructing Functions

1. If $V = \pi r^2 h$ and $h = 2r$, then $V(r) = \pi r^2(2r) = 2\pi r^3$.

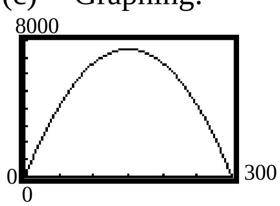
2. If $V = \frac{1}{3}\pi r^2 h$ and $h = 2r$, then $V(r) = \frac{1}{3}\pi r^2(2r) = \frac{2}{3}\pi r^3$.

3. (a) If $p = \frac{-1}{6}x + 100$ and $R = xp$, then $R(x) = x\left(\frac{-1}{6}x + 100\right) = \frac{-1}{6}x^2 + 100x$.
   (b) $R(200) = \frac{-1}{6}(200)^2 + 100(200) = \$13{,}333$
   (c) Graphing:



(d) 300; $15,000 (e) $p = \frac{-1}{6}(300) + 100 = -50 + 100 = \$50$

4. (a) If $p = \frac{-1}{3}x + 100$ and $R = xp$, then $R(x) = x\left(\frac{-1}{3}x + 100\right) = \frac{-1}{3}x^2 + 100x$.
   (b) $R(100) = \frac{-1}{3}(100)^2 + 100(100) = \$6{,}666.67$
   (c) Graphing:

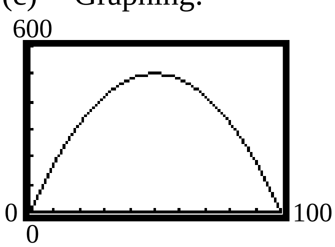


(d) 150; $7,500 (e) $p = \frac{-1}{3}(150) + 100 = -50 + 100 = \$50$

5. (a) If $x = -5p + 100$ and $R = xp$, then $p = \dfrac{100 - x}{5}$ and

$$R(x) = x\left(\frac{100 - x}{5}\right) = \frac{-1}{5}x^2 + 20x.$$

   (b) $R(15) = \frac{-1}{5}(15)^2 + 20(15) = \$255$
   (c) Graphing:



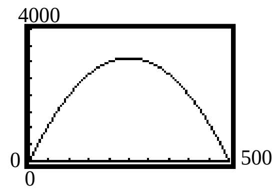
(d) 50; $500 (e) $p = \dfrac{100 - 50}{5} = \dfrac{50}{5} = \$10$

6. (a) If $x = -20p + 500$ and $R = xp$, then $p = \dfrac{500 - x}{20}$ and

$$R(x) = x\left(\frac{500 - x}{20}\right) = \frac{-1}{20}x^2 + 25x.$$

(b) $R(20) = \frac{-1}{20}(20)^2 + 25(20) = -20 + 500 = \$480$

(c) Graphing:



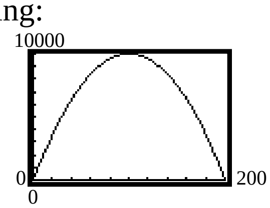
(d) 250; \$3,125

(e) $p = \dfrac{500 - 250}{20} = \dfrac{250}{20} = \$12.50$

7. (a) Let x be the width of the rectangle and let y be the length of the rectangle.
Then, the perimeter is: $P = 2y + 2x = 400$.
Solving for y: $y = \dfrac{400 - 2x}{2} = 200 - x$.
The area function is: $A(x) = y(x) = (200 - x)x = -x^2 + 200x$.

(b) The domain is: $\{x \mid 0 < x < 200\}$

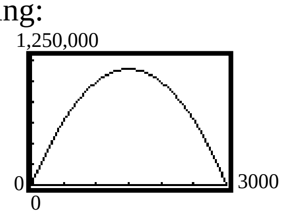
(c) Graphing:



The area is largest when $x = 100$ yards.

8. (a) Let x be the length of the side parallel to the river and let y be the width of the field.
Then, the amount of fencing used is: $P = x + 2y = 3000$.
Solving for y: $y = \dfrac{3000 - x}{2} = 1500 - \dfrac{1}{2}x$.
The area function is: $A(x) = y(x) = \left(1500 - \frac{1}{2}x\right)x = -\frac{1}{2}x^2 + 1500x$.

(b) Graphing:



The area is largest when $x = 1500$ feet.

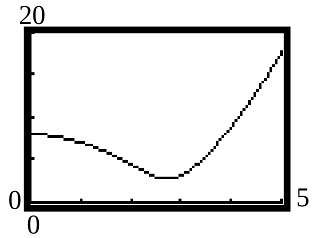
9. (a) The distance d from P to the origin is $d = \sqrt{x^2 + y^2}$. Since P is a point on the graph of $y = x^2 - 8$, we have:

$$d(x) = \sqrt{x^2 + (x^2 - 8)^2} = \sqrt{x^4 - 15x^2 + 64}$$

(b) $d(0) = \sqrt{0^4 - 15(0)^2 + 64} = \sqrt{64} = 8$

(c) $d(1) = \sqrt{(1)^4 - 15(1)^2 + 64} = \sqrt{1 - 15 + 64} = \sqrt{50} = 5\sqrt{2} \approx 7.07$

(d) Graphing:



(e) d is smallest when x is 2.74.

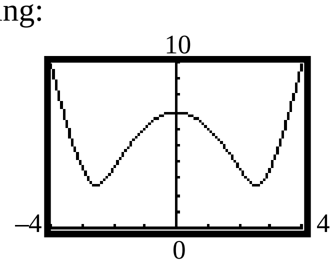
10. (a) The distance d from P to (0, –1) is $d = \sqrt{x^2 + (y+1)^2}$. Since P is a point on the graph of $y = x^2 - 8$, we have:

$$d(x) = \sqrt{x^2 + (x^2 - 8 + 1)^2} = \sqrt{x^2 + \left(x^2 - 7\right)^2} = \sqrt{x^4 - 13x^2 + 49}$$

(b) $d(0) = \sqrt{0^4 - 13(0)^2 + 49} = \sqrt{49} = 7$

(c) $d(-1) = \sqrt{(-1)^4 - 13(-1)^2 + 49} = \sqrt{1 - 13 + 49} = \sqrt{37} \approx 6.08$

(d) Graphing:

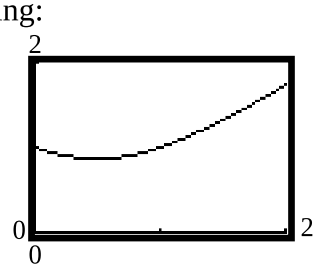


(e) d is smallest when x is 2.55 or –2.55.

11. (a) The distance d from P to the point (1, 0) is $d = \sqrt{(x-1)^2 + y^2}$. Since P is a point on the graph of $y = \sqrt{x}$, we have:

$$d(x) = \sqrt{(x-1)^2 + \left(\sqrt{x}\right)^2} = \sqrt{x^2 - x + 1}$$

(b) Graphing:

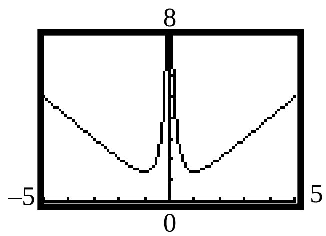


(c) d is smallest when x is 0.50.

12. (a) The distance d from P to the origin is $d = \sqrt{x^2 + y^2}$. Since P is a point on the graph of $y = \dfrac{1}{x}$, we have:

$$d(x) = \sqrt{x^2 + \left(\frac{1}{x}\right)^2} = \sqrt{x^2 + \frac{1}{x^2}} = \sqrt{\frac{x^4 + 1}{x^2}}$$

(b) Graphing:



(c) d is smallest when x is 1 or –1.

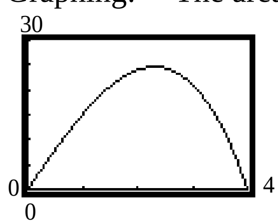
13. By definition, a triangle has area $A = \frac{1}{2}bh$, $b =$ base, $h =$ height. Because a vertex of the triangle is at the origin, we know that $b = x$ and $h = y$. Expressing the area of the triangle as a function of x, we have: $A(x) = \frac{1}{2}xy = \frac{1}{2}x\left(x^3\right) = \frac{1}{2}x^4$.

14. By definition, a triangle has area $A = \frac{1}{2}bh$, $b =$ base, $h =$ height. Because a vertex of the triangle is at the origin, we know that $b = x$ and $h = y$. Expressing the area of the triangle as a function of x, we have: $A(x) = \frac{1}{2}xy = \frac{1}{2}x\left(9 - x^2\right) = \frac{9}{2}x - \frac{1}{2}x^3$.

15. (a) $A(x) = xy = x\left(16 - x^2\right) = -x^3 + 16x$

(b) Domain: $\{x \mid 0 < x < 4\}$

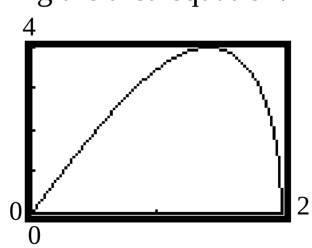
(c) Graphing: The area is largest when x is approximately 2.31.



16. (a) $A(x) = 2xy = 2x\sqrt{4 - x^2}$

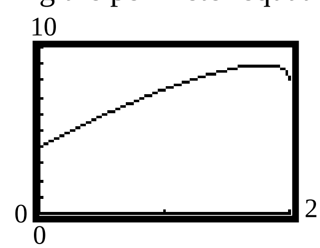
(b) $p(x) = 2(2x) + 2(y) = 4x + 2\sqrt{4 - x^2}$

(c) Graphing the area equation:



The area is largest when x is approximately 1.41.

(d) Graphing the perimeter equation:

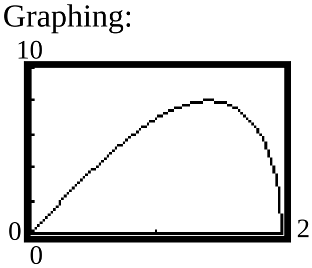


The perimeter is largest when x is approximately 1.79.

17. (a) $A(x) = (2x)(2y) = 4x\left(4 - x^2\right)^{\frac{1}{2}}$

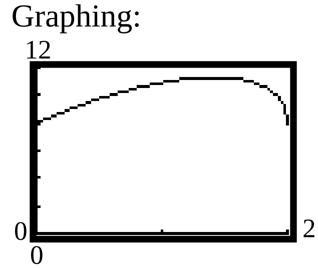
(b) $p(x) = 2(2x) + 2(2y) = 4x + 4\left(4 - x^2\right)^{\frac{1}{2}}$

(c) Graphing:



The area is largest when x is approximately 1.41.

(d) Graphing:



The perimeter is largest when x is approximately 1.41.

18. (a) $A(r) = (2r)(2r) = 4r^2$ (b) $p(r) = 4(2r) = 8r$

19. (a) $C =$ circumference, $A =$ area, $r =$ radius, $x =$ side of square

$C = 2\pi r = 10 - 4x \quad \rightarrow \quad r = \dfrac{5-2x}{\pi}$

$A(x) = x^2 + \pi r^2 = x^2 + \pi\left(\dfrac{5-2x}{\pi}\right)^2 = x^2 + \dfrac{25-20x+4x^2}{\pi}$

(b) Since the lengths must be positive, we have:

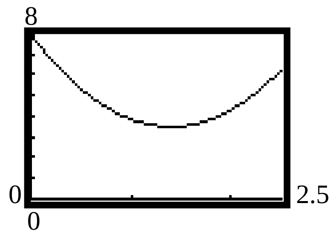
$10 - 4x > 0$ and $x > 0$

$-4x > -10$

$x < 2.5$ and $x > 0$

Domain: $\{x \mid 0 < x < 2.5\}$

(c) Graphing: The area is smallest when x is approximately 1.40 meters.



20. (a) $C =$ circumference, $A =$ area, $r =$ radius, $x =$ side of equilateral triangle

$C = 2\pi r = 10 - 3x \quad \rightarrow \quad r = \dfrac{10-3x}{2\pi}$

height of the equilateral triangle is $\dfrac{\sqrt{3}}{2}x$

$A(x) = \dfrac{1}{2}x\left(\dfrac{\sqrt{3}}{2}x\right) + \pi r^2 = \dfrac{\sqrt{3}}{4}x^2 + \pi\left(\dfrac{10-3x}{2\pi}\right)^2 = \dfrac{\sqrt{3}}{4}x^2 + \dfrac{100-60x+9x^2}{4\pi}$

(b) Since the lengths must be positive, we have:

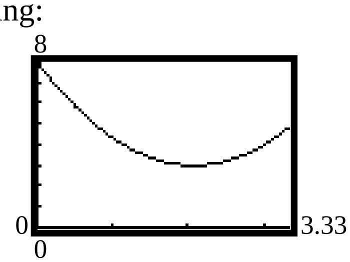
$10 - 3x > 0$ and $x > 0$

$-3x > -10$

$x < \dfrac{10}{3}$ and $x > 0$

Domain: $\left\{x \,\middle|\, 0 < x < \dfrac{10}{3}\right\}$

(c) Graphing:



The area is smallest when x is approximately 2.08 meters.

21. (a) Since the wire of length x is bent into a circle, the circumference is x. Therefore, $C(x) = x$.

(b) Since $C = x = 2\pi r$, $r = \dfrac{x}{2\pi}$.

$$A(x) = \pi r^2 = \pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}.$$

22. (a) Since the wire of length x is bent into a square, the perimeter is x. Therefore, $P(x) = x$.

(b) Since $P = x = 4s$, $s = \dfrac{x}{4}$. $\qquad A(x) = s^2 = \left(\dfrac{x}{4}\right)^2 = \dfrac{x^2}{16}$.

23. (a) A = area; r = radius; diameter $= 2r$

$A(r) = (2r)(r) = 2r^2$

(b) p = perimeter

$p(r) = 2(2r) + 2r = 6r$

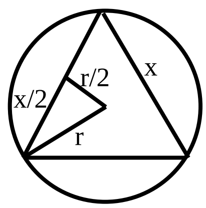
24. A = area, r = radius; x = length of a side of the triangle

$$r^2 = \left(\frac{r}{2}\right)^2 + \left(\frac{x}{2}\right)^2$$

$$r^2 - \frac{r^2}{4} = \frac{x^2}{4}$$

$$3r^2 = x^2$$

$$r^2 = \frac{x^2}{3}$$



$$C(x) = 2\pi r = 2\pi\left(\frac{x}{\sqrt{3}}\right) = \frac{2\pi\sqrt{3}}{3}x$$

25. Area of the equilateral triangle $= \dfrac{1}{2}x\left(\dfrac{\sqrt{3}}{2}x\right) = \dfrac{\sqrt{3}}{4}x^2$

Area of $\frac{1}{3}$ of the equilateral triangle $= \dfrac{1}{2}x\sqrt{r^2 - \left(\dfrac{x}{2}\right)^2} = \dfrac{1}{2}x\sqrt{r^2 - \dfrac{x^2}{4}} = \dfrac{1}{3}\left(\dfrac{\sqrt{3}}{4}x^2\right)$

Solving for r^2:

$$\frac{1}{2}x\sqrt{r^2 - \frac{x^2}{4}} = \frac{1}{3}\left(\frac{\sqrt{3}}{4}x^2\right)$$

$$\sqrt{r^2 - \frac{x^2}{4}} = \frac{2}{x}\left(\frac{\sqrt{3}}{12}x^2\right)$$

$$\sqrt{r^2 - \frac{x^2}{4}} = \frac{\sqrt{3}}{6}x$$

$$r^2 - \frac{x^2}{4} = \frac{3}{36}x^2$$

$$r^2 = \frac{x^2}{3}$$

Area inside the circle, but outside the triangle:

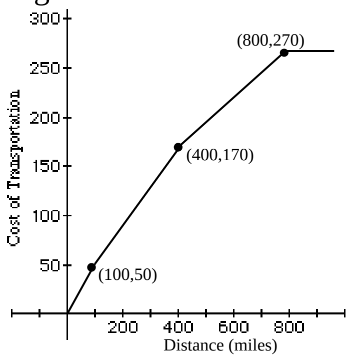
$$A(x) = \pi r^2 - \frac{\sqrt{3}}{4}x^2 = \pi\frac{x^2}{3} - \frac{\sqrt{3}}{4}x^2 = \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)x^2$$

26. (a) Let x represent the number of miles and C be the cost of transportation.

$$C(x)=\begin{cases}0.50x & \text{if } 0\le x\le 100\\ 0.50(100)+0.40(x-100) & \text{if } 100< x\le 400\\ 0.50(100)+0.40(300)+0.25(x-400) & \text{if } 400< x\le 800\\ 0.50(100)+0.40(300)+0.25(400)+0(x-800) & \text{if } 800< x\le 960\end{cases}$$

$$C(x)=\begin{cases}0.50x & \text{if } 0\le x\le 100\\ 10+0.40x & \text{if } 100< x\le 400\\ 70+0.25x & \text{if } 400< x\le 800\\ 270 & \text{if } 800< x\le 960\end{cases}$$

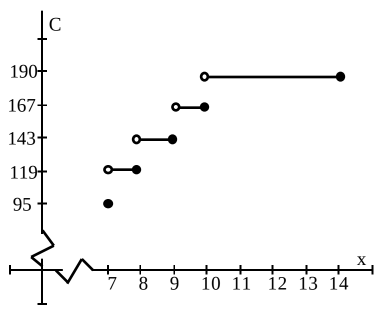
Graphing:



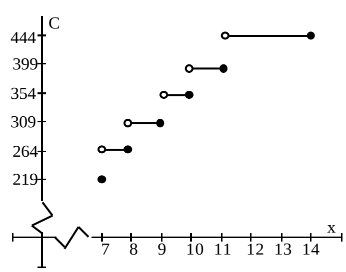
(b) For hauls between 100 and 400 miles the cost is: $C(x)=10+0.40x$.

(c) For hauls between 400 and 800 miles the cost is: $C(x)=70+0.25x$.

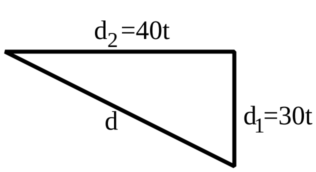
27.
$$C=\begin{cases}95 & \text{if } x=7\\ 119 & \text{if } 7<x\le 8\\ 143 & \text{if } 8<x\le 9\\ 167 & \text{if } 9<x\le 10\\ 190 & \text{if } 10<x\le 14\end{cases}$$



28.
$$C=\begin{cases}219 & \text{if } x=7\\ 264 & \text{if } 7<x\le 8\\ 309 & \text{if } 8<x\le 9\\ 354 & \text{if } 9<x\le 10\\ 399 & \text{if } 10<x\le 11\\ 438 & \text{if } 11<x\le 14\end{cases}$$

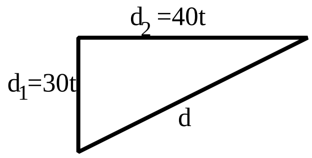


29.



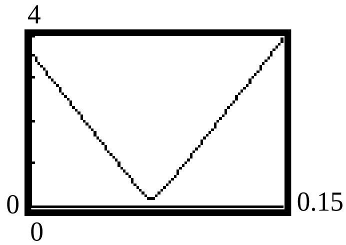
$$d^2=d_1^{\,2}+d_2^{\,2}$$
$$d^2=(30t)^2+(40t)^2$$
$$d(t)=\sqrt{900t^2+1600t^2}$$
$$d(t)=\sqrt{2500t^2}=50t$$

30. (a)



$$d^2 = d_1^{\ 2} + d_2^{\ 2}$$
$$d^2 = (2-30t)^2 + (3-40t)^2$$
$$d(t) = \sqrt{(2-30t)^2 + (3-40t)^2}$$

(b) Graphing:



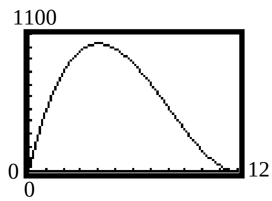
The distance is smallest at $t = 0.072$ hours.

31. (a) length = $24-2x$
width = $24-2x$
height = x
$V(x) = x(24-2x)(24-2x) = x(24-2x)^2$

(b) $V(3) = 3(24-2(3))^2 = 3(18)^2 = 3(324) = 972$ cu.in.

(c) $V(10) = 3(24-2(10))^2 = 3(4)^2 = 3(16) = 48$ cu.in.

(d)



The volume is largest when $x = 4$ inches.

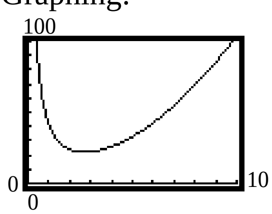
32. (a) Let A = amount of material, x = side of base, h = height, V = volume

$$V = x^2h = 10 \quad \rightarrow \quad h = \frac{10}{x^2}$$

$$A = x^2 + 4xh = x^2 + 4x\left(\frac{10}{x^2}\right) = x^2 + \frac{40}{x}$$

(b) $A = 1^2 + \frac{40}{1} = 1 + 40 = 41 \text{ ft}^2$ (c) $A = 2^2 + \frac{40}{2} = 4 + 20 = 24 \text{ ft}^2$

(d) Graphing:



The amount of material is least when $x = 2.71$ ft.

33. r = radius of cylinder, h = height of cylinder, V = volume of cylinder

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2$$

$$r^2 + \frac{h^2}{4} = R^2$$

$$r^2 = R^2 - \frac{h^2}{4}$$

$$r^2 = \frac{4R^2 - h^2}{4}$$

$$V = \pi r^2 h \longrightarrow V(h) = \pi\left(\frac{4R^2 - h^2}{4}\right)h = \frac{\pi}{4}\left(4R^2h - h^3\right)$$

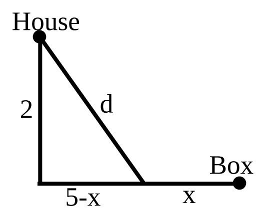
34. $r =$ radius of cylinder, $h =$ height of cylinder, $V =$ volume of cylinder

By similar triangles: $\frac{H}{R} = \frac{H-h}{r}$

$$H\,r = R(H-h)$$
$$H\,r = RH - Rh$$
$$Rh = RH - H\,r$$
$$h = \frac{RH - H\,r}{R} = H - \frac{H\,r}{R}$$

$$V = \pi r^2 h = \pi r^2\left(H - \frac{H\,r}{R}\right) = H\pi r^2\left(1 - \frac{r}{R}\right)$$

35. (a) The total cost of installing the cable along the road is $10x$. If cable is installed x miles along the road, there are $5 - x$ miles left from the road to the house and where the cable ends.



$$d = \sqrt{(5-x)^2 + 2^2} = \sqrt{25 - 10x + x^2 + 4}$$
$$= \sqrt{x^2 - 10x + 29}$$

The total cost of installing the cable is:

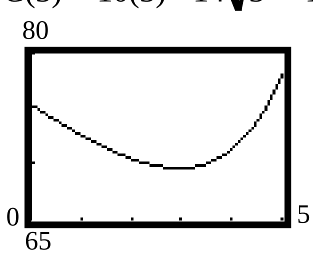
$$C(x) = 10x + 14\sqrt{x^2 - 10x + 29}$$

Domain: $\{x \mid 0 < x < 5\}$

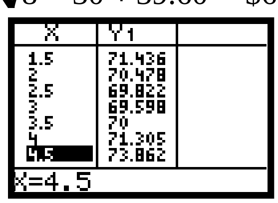
(b) $C(1) = 10(1) + 14\sqrt{1^2 - 10(1) + 29} = 10 + 14\sqrt{20} \approx 10 + 62.61 = \$72.61$

(c) $C(3) = 10(3) + 14\sqrt{3^2 - 10(3) + 29} = 30 + 14\sqrt{8} \approx 30 + 39.60 = \$69.60$

(d)



(e)



The table indicates that $x = 3$ results in the least cost.

(f) Using MINIMUM, the graph indicates that $x = 2.96$ results in the least cost.

36. (a) The time on the boat is given by $\frac{d}{3}$. The time on land is given by $\frac{12-x}{5}$.

$$d=\sqrt{x^2+2^2}=\sqrt{x^2+4}$$

The total time for the trip is:

$$T(x)=\frac{12-x}{5}+\frac{d}{3}=\frac{12-x}{5}+\frac{\sqrt{x^2+4}}{3}$$

(b) Domain: $\{x \mid 0 \le x \le 12\}$

(c) $T(4)=\frac{12-4}{5}+\frac{\sqrt{4^2+4}}{3}=\frac{8}{5}+\frac{\sqrt{20}}{3}\approx 3.09$ hours

(d) $T(8)=\frac{12-8}{5}+\frac{\sqrt{8^2+4}}{3}=\frac{4}{5}+\frac{\sqrt{68}}{3}\approx 3.55$ hours

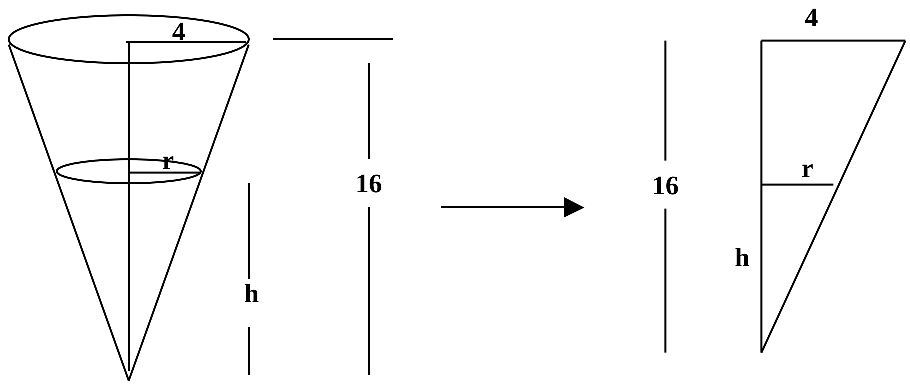
37. For schedule X:

$$f(x)=\begin{cases} 0.15x & \text{if } 0 < x \le 25{,}750 \\ 3{,}862.50+0.28(x-25{,}750) & \text{if } 25{,}750 < x \le 62{,}450 \\ 14{,}138.50+0.31(x-62{,}450) & \text{if } 62{,}450 < x \le 130{,}250 \\ 35{,}156.50+0.36(x-130{,}250) & \text{if } 130{,}250 < x \le 283{,}150 \\ 90{,}200.50+0.396(x-283{,}150) & \text{if } x > 283{,}150 \end{cases}$$

For Schedule Y-1:

$$f(x)=\begin{cases} 0.15x & \text{if } 0 < x \le 43{,}050 \\ 6{,}457.50+0.28(x-43{,}050) & \text{if } 43{,}050 < x \le 104{,}050 \\ 23{,}537.50+0.31(x-104{,}050) & \text{if } 104{,}050 < x \le 158{,}550 \\ 40{,}432.50+0.36(x-158{,}550) & \text{if } 158{,}550 < x \le 283{,}150 \\ 85{,}288.50+0.396(x-283{,}150) & \text{if } x > 283{,}150 \end{cases}$$

38. Consider the diagram shown below.



We can extract a pair of similar triangles from the diagram.
Since the smaller triangle is similar to the larger triangle, we have the proportion

$$\frac{r}{h}=\frac{4}{16} \rightarrow \frac{r}{h}=\frac{1}{4} \rightarrow r=\frac{1}{4}h$$

Substituting into the volume formula for the conical portion of water,

$$V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h=\frac{1}{48}\pi h^3,$$

so we have the volume function $V(h)=\frac{1}{48}\pi h^3$.