Chapter 4

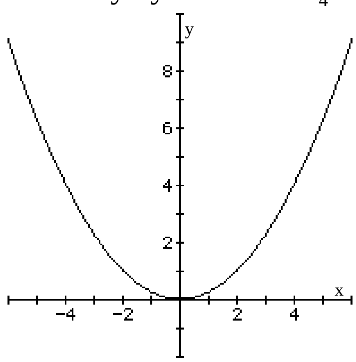
Polynomial and Rational Functions

4.1 Quadratic Functions and Models

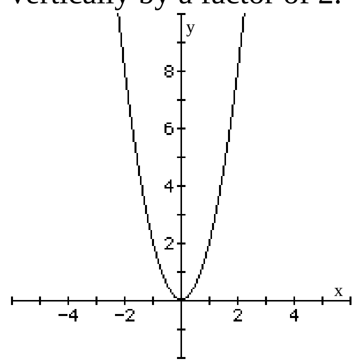
1. D 2. F 3. A 4. H 5. B 6. C

7. E 8. G

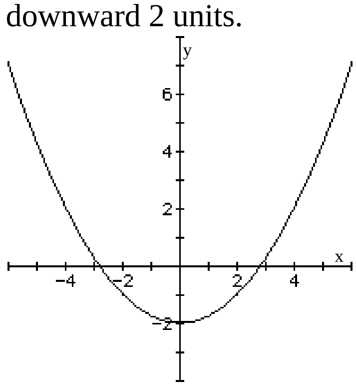
9. $f(x) = \frac{1}{4}x^2$
Using the function $y = x^2$, compress vertically by a factor of $\frac{1}{4}$.



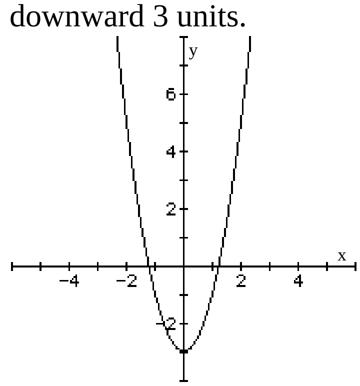
10. $f(x) = 2x^2$
Using the function $y = x^2$, stretch vertically by a factor of 2.



11. $f(x) = \frac{1}{4}x^2 - 2$
Using the function $y = x^2$, compress vertically by a factor of $\frac{1}{4}$, and shift downward 2 units.

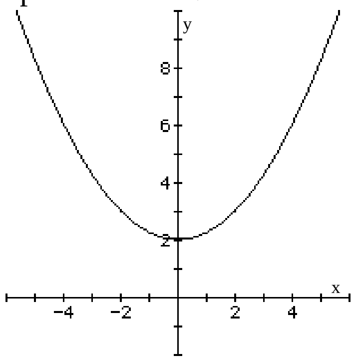


12. $f(x) = 2x^2 - 3$
Using the function $y = x^2$, stretch vertically by a factor of 2, and shift downward 3 units.



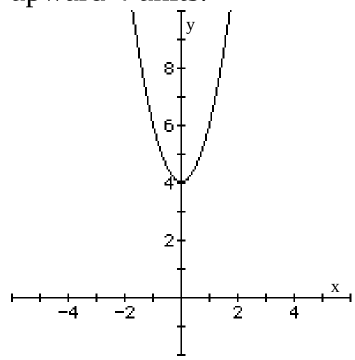
13. $f(x) = \frac{1}{4}x^2 + 2$

Using the function $y = x^2$, compress vertically by a factor of $\frac{1}{4}$, and shift upward 2 units.



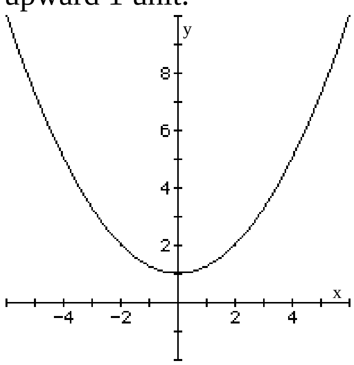
14. $f(x) = 2x^2 + 4$

Using the function $y = x^2$, stretch vertically by a factor of 2, and shift upward 4 units.



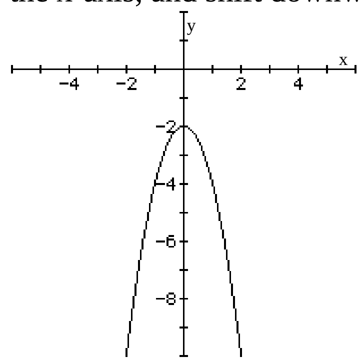
15. $f(x) = \frac{1}{4}x^2 + 1$

Using the function $y = x^2$, compress vertically by a factor of $\frac{1}{4}$, and shift upward 1 unit.

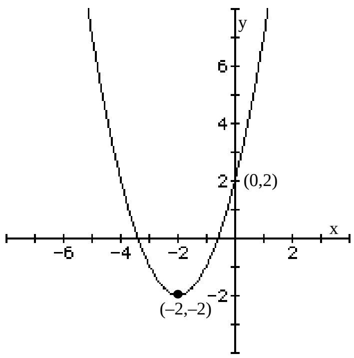


16. $f(x) = -2x^2 - 2$

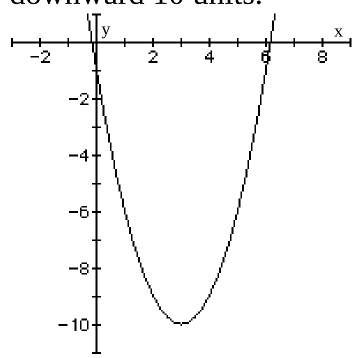
Using the function $y = x^2$, stretch vertically by a factor of 2, reflect about the x-axis, and shift downward 2 units.



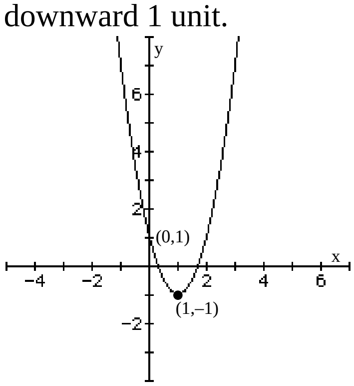
17. $f(x) = x^2 + 4x + 2$
Completing the square:
$f(x) = (x^2 + 4x + 4) + 2 - 4$
$\quad = (x + 2)^2 - 2$
Using the function $y = x^2$, shift the graph to the left 2 units, and shift downward 2 units.



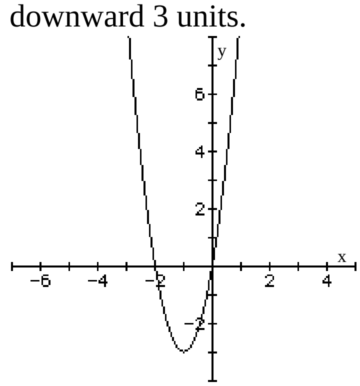
18. $f(x) = x^2 - 6x - 1$
Completing the square:
$f(x) = (x^2 - 6x + 9) - 1 - 9$
$\quad = (x - 3)^2 - 10$
Using the function $y = x^2$, shift the graph to the right 3 units, and shift downward 10 units.



19. $f(x) = 2x^2 - 4x + 1$
Completing the square:
$f(x) = 2(x^2 - 2x + 1) + 1 - 2$
$\quad = 2(x - 1)^2 - 1$
Using the function $y = x^2$, shift the graph to the right 1 unit, stretch the graph vertically by a factor of 2, and shift downward 1 unit.



20. $f(x) = 3x^2 + 6x$
Completing the square:
$f(x) = 3(x^2 + 2x + 1) - 3$
$\quad = 3(x + 1)^2 - 3$
Using the function $y = x^2$, shift the graph to the left 1 unit, stretch the graph vertically by a factor of 3, and shift downward 3 units.

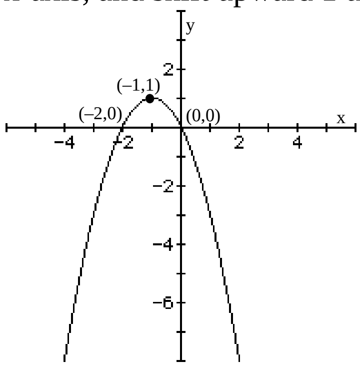


21. $f(x) = -x^2 - 2x$

Completing the square:

$$f(x) = -(x^2 + 2x + 1) + 1$$
$$= -(x + 1)^2 + 1$$

Using the function $y = x^2$, shift the graph to the left 1 unit, reflect the graph on the x-axis, and shift upward 1 unit.

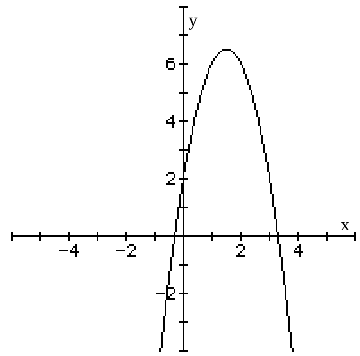


22. $f(x) = -2x^2 + 6x + 2$

Completing the square:

$$f(x) = -2\left(x^2 - 3x + \tfrac{9}{4}\right) + 2 + \tfrac{9}{2}$$
$$= -2\left(x - \tfrac{3}{2}\right)^2 + \tfrac{13}{2}$$

Using the function $y = x^2$, shift the graph to the right $\frac{3}{2}$ units, stretch vertically by a factor of 2, reflect the graph on the x-axis, and shift upward $\frac{13}{2}$ units.

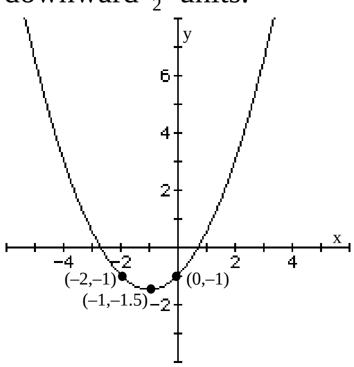


23. $f(x) = \frac{1}{2}x^2 + x - 1$

Completing the square:

$$f(x) = \tfrac{1}{2}(x^2 + 2x + 1) - 1 - \tfrac{1}{2}$$
$$= \tfrac{1}{2}(x + 1)^2 - \tfrac{3}{2}$$

Using the function $y = x^2$, shift the graph to the left 1 unit, compress the graph vertically by a factor of $\frac{1}{2}$, and shift downward $\frac{3}{2}$ units.

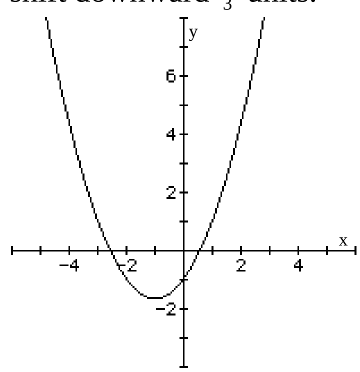


24. $f(x) = \frac{2}{3}x^2 + \frac{4}{3}x - 1$

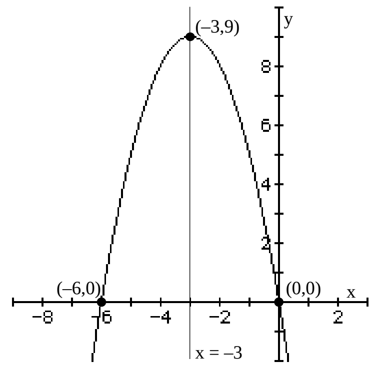
Completing the square:

$$f(x) = \tfrac{2}{3}(x^2 + 2x + 1) - 1 - \tfrac{2}{3}$$
$$= \tfrac{2}{3}(x + 1)^2 - \tfrac{5}{3}$$

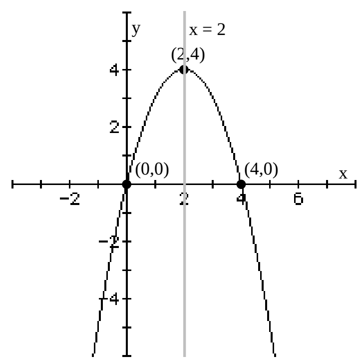
Using the function $y = x^2$, shift the graph to the left 1 unit, compress the graph vertically by a factor of $\frac{2}{3}$, and shift downward $\frac{5}{3}$ units.



25. $f(x) = -x^2 - 6x$
$a = -1,\ b = -6,\ c = 0.$ Since $a = -1 < 0,$ the graph opens down.
The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-(-6)}{2(-1)} = \frac{6}{-2} = -3.$
The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f(-3) = -(-3)^2 - 6(-3) = -9 + 18 = 9.$
Thus, the vertex is (–3, 9).
The axis of symmetry is the line $x = -3.$
The discriminant is:
$b^2 - 4ac = (-6)^2 - 4(-1)(0) = 36 > 0,$
so the graph has two x-intercepts.
The x-intercepts are found by solving:
$-x^2 - 6x = 0$
$-x(x + 6) = 0$
$x = 0$ or $x = -6$
The x-intercepts are –6 and 0.
The y-intercept is $f(0) = 0.$



26. $f(x) = -x^2 + 4x$
$a = -1,\ b = 4,\ c = 0.$ Since $a = -1 < 0,$ the graph opens down.
The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-4}{2(-1)} = \frac{-4}{-2} = 2.$
The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f(2) = -(2)^2 + 4(2) = -4 + 8 = 4.$
Thus, the vertex is (2, 4).
The axis of symmetry is the line $x = 2.$
The discriminant is:
$b^2 - 4ac = 4^2 - 4(-1)(0) = 16 > 0,$
so the graph has two x-intercepts.
The x-intercepts are found by solving:
$-x^2 + 4x = 0$
$-x(x - 4) = 0$
$x = 0$ or $x = 4$
The x-intercepts are 0 and 4.
The y-intercept is $f(0) = 0.$



27. $f(x) = 2x^2 - 8x$
$a = 2,\ b = -8,\ c = 0.$ Since $a = 2 > 0,$ the graph opens up.
The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-(-8)}{2(2)} = \frac{8}{4} = 2.$
The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f(2) = 2(2)^2 - 8(2) = 8 - 16 = -8.$

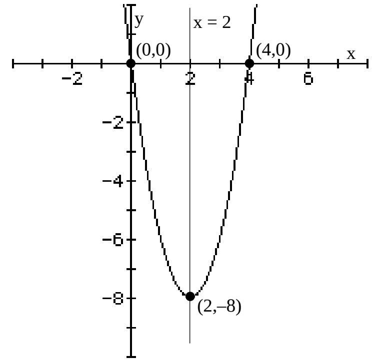
Thus, the vertex is (2, –8).
The axis of symmetry is the line $x = 2$.
The discriminant is:

$$b^2 - 4ac = (-8)^2 - 4(2)(0) = 64 > 0,$$

so the graph has two x-intercepts.
The x-intercepts are found by solving:

$$2x^2 - 8x = 0$$
$$2x(x - 4) = 0$$
$$x = 0 \quad \text{or} \quad x = 4$$

The x-intercepts are 0 and 4.
The y-intercept is $f(0) = 0$.



28. $f(x) = 3x^2 + 18x$

$a = 3$, $b = 18$, $c = 0$. Since $a = 3 > 0$, the graph opens up.

The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-18}{2(3)} = \frac{-18}{6} = -3$.

The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f(-3) = 3(-3)^2 + 18(-3) = 27 - 54 = -27$.

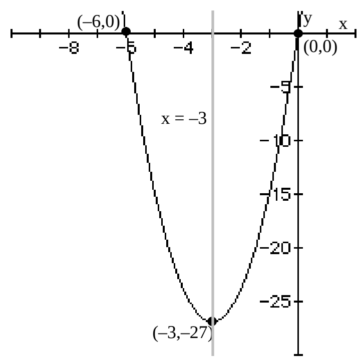
Thus, the vertex is (–3, –27).
The axis of symmetry is the line $x = -3$.
The discriminant is:

$$b^2 - 4ac = (18)^2 - 4(3)(0) = 324 > 0,$$

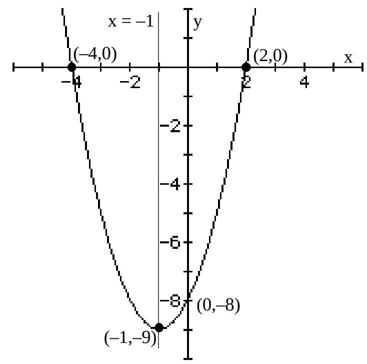
so the graph has two x-intercepts.
The x-intercepts are found by solving:

$$3x^2 + 18x = 0$$
$$3x(x + 6) = 0$$
$$x = 0 \quad \text{or} \quad x = -6$$

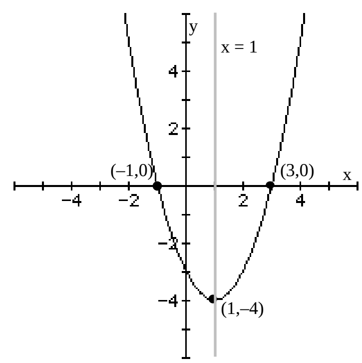
The x-intercepts are 0 and –6.
The y-intercept is $f(0) = 0$.



29. $f(x) = x^2 + 2x - 8$
$a = 1,\ b = 2,\ c = -8.$ Since $a = 1 > 0,$ the graph opens up.
The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-2}{2(1)} = \frac{-2}{2} = -1$.
The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f(-1) = (-1)^2 + 2(-1) - 8 = 1 - 2 - 8 = -9$.
Thus, the vertex is (–1, –9).
The axis of symmetry is the line $x = -1$.
The discriminant is:
$b^2 - 4ac = 2^2 - 4(1)(-8) = 4 + 32 = 36 > 0$,
so the graph has two x-intercepts.
The x-intercepts are found by solving:
$x^2 + 2x - 8 = 0$
$(x + 4)(x - 2) = 0$
$x = -4$ or $x = 2$
The x-intercepts are –4 and 2.
The y-intercept is $f(0) = -8$.



30. $f(x) = x^2 - 2x - 3$
$a = 1,\ b = -2,\ c = -3.$ Since $a = 1 > 0,$ the graph opens up.
The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-(-2)}{2(1)} = \frac{2}{2} = 1$.
The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f(1) = 1^2 - 2(1) - 3 = 1 - 2 - 3 = -4$.
Thus, the vertex is (1, –4).
The axis of symmetry is the line $x = 1$.
The discriminant is:
$b^2 - 4ac = (-2)^2 - 4(1)(-3) = 4 + 12 = 16 > 0$,
so the graph has two x-intercepts.
The x-intercepts are found by solving:
$x^2 - 2x - 3 = 0$
$(x + 1)(x - 3) = 0$
$x = -1$ or $x = 3$
The x-intercepts are –1 and 3.
The y-intercept is $f(0) = -3$.



31. $f(x) = x^2 + 2x + 1$
$a = 1,\ b = 2,\ c = 1.$ Since $a = 1 > 0,$ the graph opens up.
The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-2}{2(1)} = \frac{-2}{2} = -1$.
The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f(-1) = (-1)^2 + 2(-1) + 1 = 1 - 2 + 1 = 0$.

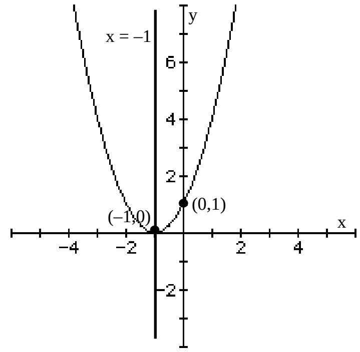
Thus, the vertex is (–1, 0).
The axis of symmetry is the line $x = -1$.
The discriminant is:

$$b^2 - 4ac = 2^2 - 4(1)(1) = 4 - 4 = 0,$$

so the graph has one x-intercept.
The x-intercept is found by solving:

$$x^2 + 2x + 1 = 0$$
$$(x+1)^2 = 0$$
$$x = -1$$

The x-intercept is –1.
The y-intercept is $f(0) = 1$.



32. $f(x) = x^2 + 6x + 9$
$a = 1,\ b = 6,\ c = 9$. Since $a = 1 > 0,$ the graph opens up.
The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-6}{2(1)} = \frac{-6}{2} = -3$. The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f(-3) = (-3)^2 + 6(-3) + 9 = 9 - 18 + 9 = 0$.

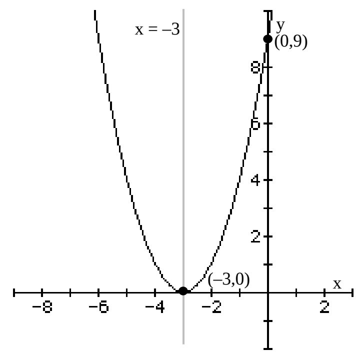
Thus, the vertex is (–3, 0).
The axis of symmetry is the line $x = -3$.
The discriminant is:

$$b^2 - 4ac = 6^2 - 4(1)(9) = 36 - 36 = 0,$$

so the graph has one x-intercept.
The x-intercept is found by solving:

$$x^2 + 6x + 9 = 0$$
$$(x+3)^2 = 0$$
$$x = -3$$

The x-intercept is –3.
The y-intercept is $f(0) = 9$.



33. $f(x) = 2x^2 - x + 2$

$a = 2,\ b = -1,\ c = 2.$ Since $a = 2 > 0,$ the graph opens up.

The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-(-1)}{2(2)} = \frac{1}{4}.$

The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f\left(\frac{1}{4}\right) = 2\left(\frac{1}{4}\right)^2 - \frac{1}{4} + 2 = \frac{1}{8} - \frac{1}{4} + 2 = \frac{15}{8}.$

Thus, the vertex is $\left(\frac{1}{4}, \frac{15}{8}\right).$

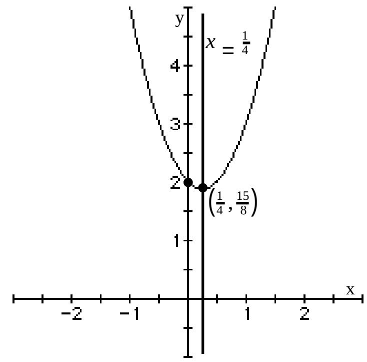
The axis of symmetry is the line $x = \frac{1}{4}.$

The discriminant is:

$$b^2 - 4ac = (-1)^2 - 4(2)(2) = 1 - 16 = -15,$$

so the graph has no x-intercepts.

The y-intercept is $f(0) = 2.$



34. $f(x) = 4x^2 - 2x + 1$

$a = 4,\ b = -2,\ c = 1.$ Since $a = 4 > 0,$ the graph opens up.

The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-(-2)}{2(4)} = \frac{2}{8} = \frac{1}{4}.$ The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f\left(\frac{1}{4}\right) = 4\left(\frac{1}{4}\right)^2 - 2\left(\frac{1}{4}\right) + 1 = \frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4}.$

Thus, the vertex is $\left(\frac{1}{4}, \frac{3}{4}\right).$

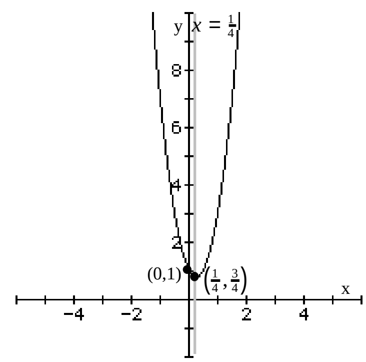
The axis of symmetry is the line $x = \frac{1}{4}.$

The discriminant is:

$$b^2 - 4ac = (-2)^2 - 4(4)(1) = 4 - 16 = -12,$$

so the graph has no x-intercepts.

The y-intercept is $f(0) = 1.$



35. $f(x) = -2x^2 + 2x - 3$

$a = -2,\ b = 2,\ c = -3.$ Since $a = -2 < 0,$ the graph opens down.

The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-(2)}{2(-2)} = \frac{-2}{-4} = \frac{1}{2}.$ The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f\left(\frac{1}{2}\right) = -2\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 3 = \frac{-1}{2} + 1 - 3 = \frac{-5}{2}.$

Thus, the vertex is $\left(\frac{1}{2}, \frac{-5}{2}\right)$.

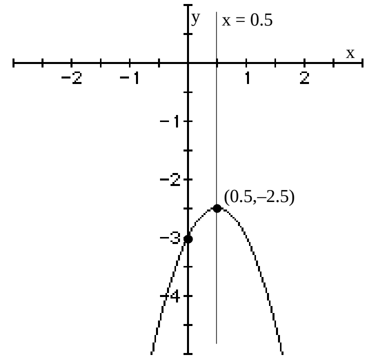
The axis of symmetry is the line $x = \frac{1}{2}$.

The discriminant is:

$$b^2 - 4ac = 2^2 - 4(-2)(-3) = 4 - 24 = -20,$$

so the graph has no x-intercepts.

The y-intercept is $f(0) = -3$.



36. $f(x) = -3x^2 + 3x - 2$

$a = -3,\ b = 3,\ c = -2.$ Since $a = -3 < 0,$ the graph opens down.

The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-3}{2(-3)} = \frac{-3}{-6} = \frac{1}{2}$. The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f\left(\frac{1}{2}\right) = -3\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right) - 2 = -\frac{3}{4} + \frac{3}{2} - 2 = \frac{-5}{4}$.

Thus, the vertex is $\left(\frac{1}{2}, \frac{-5}{4}\right)$.

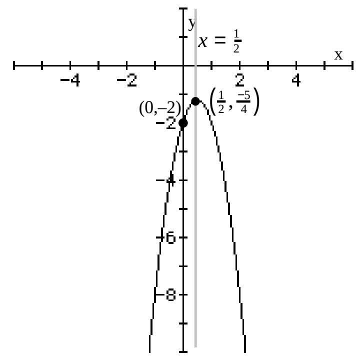
The axis of symmetry is the line $x = \frac{1}{2}$.

The discriminant is:

$$b^2 - 4ac = 3^2 - 4(-3)(-2) = 9 - 24 = -15,$$

so the graph has no x-intercepts.

The y-intercept is $f(0) = -2$.



37. $f(x) = 3x^2 + 6x + 2$

$a = 3,\ b = 6,\ c = 2.$ Since $a = 3 > 0,$ the graph opens up.

The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-6}{2(3)} = \frac{-6}{6} = -1$. The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f(-1) = 3(-1)^2 + 6(-1) + 2 = 3 - 6 + 2 = -1$.

Thus, the vertex is (–1, –1).

The axis of symmetry is the line $x = -1$.

The discriminant is:

$$b^2 - 4ac = 6^2 - 4(3)(2) = 36 - 24 = 12,$$

so the graph has two x-intercepts.

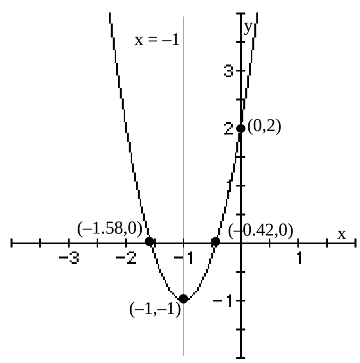
The x-intercepts are found by solving:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{12}}{2(3)}$$

$$= \frac{-6 \pm 2\sqrt{3}}{6} = \frac{-3 \pm \sqrt{3}}{3} = \frac{-3 \pm 1.732}{3}$$

The x-intercepts are approximately –0.42 and –1.58.

The y-intercept is $f(0) = 2$.



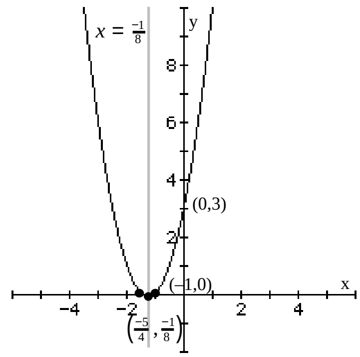
38. $f(x) = 2x^2 + 5x + 3$

$a = 2,\ b = 5,\ c = 3.$ Since $a = 2 > 0,$ the graph opens up.

The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-5}{2(2)} = \frac{-5}{4}$. The y-coordinate of the vertex is

$$f\left(\frac{-b}{2a}\right) = f\left(\frac{-5}{4}\right) = 2\left(\frac{-5}{4}\right)^2 + 5\left(\frac{-5}{4}\right) + 3 = \frac{25}{8} - \frac{25}{4} + 3 = -\frac{1}{8}.$$

Thus, the vertex is $\left(\frac{-5}{4}, \frac{-1}{8}\right)$.



The axis of symmetry is the line $x = -\frac{5}{4}$.

The discriminant is:

$$b^2 - 4ac = 5^2 - 4(2)(3) = 25 - 24 = 1,$$

so the graph has two x-intercepts.

The x-intercepts are found by solving:

$$(2x + 3)(x + 1) = 0$$

$$x = -\frac{3}{2} \text{ or } x = -1$$

The x-intercepts are $\frac{-3}{2}$ and –1.

The y-intercept is $f(0) = 3$.

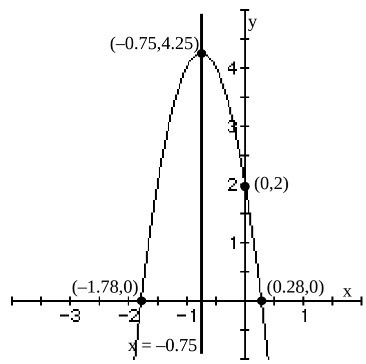
39. $f(x) = -4x^2 - 6x + 2$

$a = -4,\ b = -6,\ c = 2.$ Since $a = -4 < 0,$ the graph opens down.

The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-(-6)}{2(-4)} = \frac{6}{-8} = \frac{-3}{4}$. The y-coordinate of the vertex is

$$f\left(\frac{-b}{2a}\right) = f\left(\frac{-3}{4}\right) = -4\left(\frac{-3}{4}\right)^2 - 6\left(\frac{-3}{4}\right) + 2 = \frac{-9}{4} + \frac{9}{2} + 2 = \frac{17}{4}.$$

Thus, the vertex is $\left(\frac{-3}{4}, \frac{17}{4}\right)$.



The axis of symmetry is the line $x = \frac{-3}{4}$.

The discriminant is:

$$b^2 - 4ac = (-6)^2 - 4(-4)(2) = 36 + 32 = 68,$$

so the graph has two x-intercepts.

The x-intercepts are found by solving:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{68}}{2(-4)}$$

$$= \frac{6 \pm 2\sqrt{17}}{-8} = \frac{-3 \pm \sqrt{17}}{4} = \frac{-3 \pm 4.123}{4}$$

The x-intercepts are approximately –1.78 and 0.28.

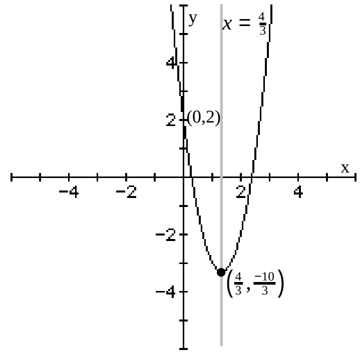
The y-intercept is $f(0) = 2$.

40. $f(x) = 3x^2 - 8x + 2$

$a = 3,\ b = -8,\ c = 2.$ Since $a = 3 > 0,$ the graph opens up.

The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-(-8)}{2(3)} = \frac{8}{6} = \frac{4}{3}$. The y-coordinate of the vertex is

$$f\left(\frac{-b}{2a}\right) = f\left(\frac{4}{3}\right) = 3\left(\frac{4}{3}\right)^2 - 8\left(\frac{4}{3}\right) + 2 = \frac{16}{3} - \frac{32}{3} + 2 = -\frac{10}{3}.$$



Thus, the vertex is $\left(\frac{4}{3}, -\frac{10}{3}\right)$.

The axis of symmetry is the line $x = \frac{4}{3}$.

The discriminant is:

$$b^2 - 4ac = (-8)^2 - 4(3)(2) = 64 - 24 = 40,$$

so the graph has two x-intercepts.

The x-intercepts are found by solving:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-8) \pm \sqrt{40}}{2(3)}$$

$$= \frac{8 \pm 2\sqrt{10}}{6} = \frac{4 \pm \sqrt{10}}{3} = \frac{4 \pm 3.162}{3}$$

The x-intercepts are approximately 0.28 and 2.39.

The y-intercept is $f(0) = 2$.

41. $f(x) = 2x^2 + 12x$, $a = 2, b = 12, c = 0$. Since $a = 2 > 0$, the graph opens up, so the vertex is a minimum point. The minimum occurs at $x = \frac{-b}{2a} = \frac{-12}{2(2)} = \frac{-12}{4} = -3$. The minimum value is

$$f\left(\frac{-b}{2a}\right) = f(-3) = 2(-3)^2 + 12(-3) = 18 - 36 = -18.$$

42. $f(x) = -2x^2 + 12x$, $a = -2, b = 12, c = 0$. Since $a = -2 < 0$, the graph opens down, so the vertex is a maximum point. The maximum occurs at $x = \frac{-b}{2a} = \frac{-12}{2(-2)} = \frac{-12}{-4} = 3$. The maximum value is

$$f\left(\frac{-b}{2a}\right) = f(3) = -2(3)^2 + 12(3) = -18 + 36 = 18.$$

43. $f(x) = 2x^2 + 12x - 3$, $a = 2, b = 12, c = -3$. Since $a = 2 > 0$, the graph opens up, so the vertex is a minimum point. The minimum occurs at $x = \frac{-b}{2a} = \frac{-12}{2(2)} = \frac{-12}{4} = -3$. The minimum value is

$$f\left(\frac{-b}{2a}\right) = f(-3) = 2(-3)^2 + 12(-3) - 3 = 18 - 36 - 3 = -21.$$

44. $f(x) = 4x^2 - 8x + 3$, $a = 4, b = -8, c = 3$. Since $a = 4 > 0$, the graph opens up, so the vertex is a minimum point. The minimum occurs at $x = \frac{-b}{2a} = \frac{-(-8)}{2(4)} = \frac{8}{8} = 1$.

The minimum value is $f\left(\frac{-b}{2a}\right) = f(1) = 4\ 1^2 - 8(1) + 3 = 4 - 8 + 3 = -1$.

45. $f(x) = -x^2 + 10x - 4$
$a = -1,\ b = 10,\ c = -4.$ Since $a = -1 < 0,$ the graph opens down, so the vertex is a maximum point. The maximum occurs at $x = \frac{-b}{2a} = \frac{-10}{2(-1)} = \frac{-10}{-2} = 5$. The maximum value is $f\left(\frac{-b}{2a}\right) = f(5) = -(5)^2 + 10(5) - 4 = -25 + 50 - 4 = 21$.

46. $f(x) = -2x^2 + 8x + 3$
$a = -2,\ b = 8,\ c = 3.$ Since $a = -2 < 0,$ the graph opens down, so the vertex is a maximum point. The maximum occurs at $x = \frac{-b}{2a} = \frac{-8}{2(-2)} = \frac{-8}{-4} = 2$. The maximum value is $f\left(\frac{-b}{2a}\right) = f(2) = -2(2)^2 + 8(2) + 3 = -8 + 16 + 3 = 11$.

47. $f(x) = -3x^2 + 12x + 1$
$a = -3,\ b = 12,\ c = 1.$ Since $a = -3 < 0,$ the graph opens down, so the vertex is a maximum point. The maximum occurs at $x = \frac{-b}{2a} = \frac{-12}{2(-3)} = \frac{-12}{-6} = 2$. The maximum value is $f\left(\frac{-b}{2a}\right) = f(2) = -3(2)^2 + 12(2) + 1 = -12 + 24 + 1 = 13$.

48. $f(x) = 4x^2 - 4x$
$a = 4,\ b = -4,\ c = 0.$ Since $a = 4 > 0,$ the graph opens up, so the vertex is a minimum point. The minimum occurs at $x = \frac{-b}{2a} = \frac{-(-4)}{2(4)} = \frac{4}{8} = \frac{1}{2}$. The minimum value is
$f\left(\frac{-b}{2a}\right) = f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) = 1 - 2 = -1$.

49. (a) $f(x) = 1(x - (-3))(x - 1) = 1(x + 3)(x - 1) = 1(x^2 + 2x - 3) = x^2 + 2x - 3$
$f(x) = 2(x - (-3))(x - 1) = 2(x + 3)(x - 1) = 2(x^2 + 2x - 3) = 2x^2 + 4x - 6$
$f(x) = -2(x - (-3))(x - 1) = -2(x + 3)(x - 1)$
$= -2(x^2 + 2x - 3) = -2x^2 - 4x + 6$
$f(x) = 5(x - (-3))(x - 1) = 5(x + 3)(x - 1) = 5(x^2 + 2x - 3) = 5x^2 + 10x - 15$

(b) The value of a multiplies the value of the y-intercept by the value of a. The values of the x-intercepts are not changed.
(c) The axis of symmetry is unaffected by the value of a.
(d) The y-coordinate of the vertex is multiplied by the value of a.
(e) The x-coordinate of the vertex is the midpoint of the x-intercepts.

50. (a) $f(x)=1(x-(-5))(x-3)=1(x+5)(x-3)=1\left(x^2+2x-15\right)=x^2+2x-15$
$f(x)=2(x-(-5))(x-3)=2(x+5)(x-3)=2\left(x^2+2x-15\right)=2x^2+4x-30$
$f(x)=-2(x-(-5))(x-3)=-2(x+5)(x-3)=-2\left(x^2+2x-15\right)=-2x^2-4x+30$
$f(x)=5(x-(-5))(x-3)=5(x+5)(x-3)=5\left(x^2+2x-15\right)=5x^2+10x-75$

(b) The value of a multiplies the value of the y-intercept by the value of a. The values of the x-intercepts are not changed.

(c) The axis of symmetry is unaffected by the value of a.

(d) The y-coordinate of the vertex is multiplied by the value of a.

(e) The x-coordinate of the vertex is the midpoint of the x-intercepts.

51. $R(p)=-4p^2+4000p$
$a=-4,\ b=4000,\ c=0.$ Since $a=-4<0,$ the graph is a parabola that opens down, so the vertex is a maximum point. The maximum occurs at
$p=\frac{-b}{2a}=\frac{-4000}{2(-4)}=\frac{-4000}{-8}=500$.
$R(500)=-4(500)^2+4000(500)=-1000000+2000000=1{,}000{,}000$
Thus, the unit price should be \$500 for maximum revenue. The maximum revenue is \$1,000,000.

52. $R(p)=-\frac{1}{2}p^2+1900p$
$a=-\frac{1}{2},\ b=1900,\ c=0.$ Since $a=-\frac{1}{2}<0,$ the graph is a parabola that opens down, so the vertex is a maximum point. The maximum occurs at
$p=\frac{-b}{2a}=\frac{-1900}{2\left(-\frac{1}{2}\right)}=\frac{-1900}{-1}=1900$.
$R(1900)=-\frac{1}{2}(1900)^2+1900(1900)=-1805000+3610000=1{,}805{,}000$.
Thus, the unit price should be \$1900 for maximum revenue. The maximum revenue is \$1,805,000

53. (a) $R(x)=x\left(\frac{-1}{6}x+100\right)=\frac{-1}{6}x^2+100x$

(b) $R(200)=\frac{-1}{6}(200)^2+100(200)=\frac{-20000}{3}+20000=\frac{40000}{3}\approx \$13{,}333$

(c) $x=\frac{-b}{2a}=\frac{-100}{2\left(\frac{-1}{6}\right)}=\frac{-100}{\frac{-1}{3}}=\frac{300}{1}=300$

$R(300)=\frac{-1}{6}(300)^2+100(300)=-15000+30000=\$15{,}000$

(d) $p=\frac{-1}{6}(300)+100=-50+100=\$50$

54. (a) $R(x)=x\left(\frac{-1}{3}x+100\right)=\frac{-1}{3}x^2+100x$

(b) $R(100)=\frac{-1}{3}(100)^2+100(100)=\frac{-10000}{3}+10000=\frac{20000}{3}\approx \$6{,}666.6$

(c) $x=\frac{-b}{2a}=\frac{-100}{2\left(\frac{-1}{3}\right)}=\frac{-100}{\frac{-2}{3}}=\frac{300}{2}=150$

$R(150)=\frac{-1}{3}(150)^2+100(150)=-7500+15000=\$7{,}500$

(d) $p=\frac{-1}{3}(150)+100=-50+100=\$50$

55. (a) If $x = -5p + 100$, then $p = \dfrac{100 - x}{5}$. $R(x) = x\left(\dfrac{100 - x}{5}\right) = \frac{-1}{5}x^2 + 20x$

(b) $R(15) = \dfrac{-1}{5}(15)^2 + 20(15) = -45 + 300 = \$255$

(c) $x = \dfrac{-b}{2a} = \dfrac{-20}{2\left(\frac{-1}{5}\right)} = \dfrac{-20}{\frac{-2}{5}} = \dfrac{100}{2} = 50$

$R(50) = \dfrac{-1}{5}(50)^2 + 20(50) = -500 + 1000 = \$500$

(d) $p = \dfrac{100 - 50}{5} = \dfrac{50}{5} = \$10$

56. (a) If $x = -20p + 500$, then $p = \dfrac{500 - x}{20}$. $R(x) = x\left(\dfrac{500 - x}{20}\right) = \frac{-1}{20}x^2 + 25x$

(b) $R(20) = \dfrac{-1}{20}(20)^2 + 25(20) = -20 + 500 = \$480$

(c) $x = \dfrac{-b}{2a} = \dfrac{-25}{2\left(\frac{-1}{20}\right)} = \dfrac{-25}{\frac{-1}{10}} = \dfrac{250}{1} = 250$

$R(250) = \dfrac{-1}{20}(250)^2 + 25(250) = -3125 + 6250 = \$3125$

(d) $p = \dfrac{500 - 250}{20} = \dfrac{250}{20} = \$12.50$

57. (a) Let x = width and y = length of the rectangular area.

$P = 2x + 2y = 400$

$y = \dfrac{400 - 2x}{2} = 200 - x$

Then $A(x) = (200 - x)x = 200x - x^2 = -x^2 + 200x$

(b) $x = \dfrac{-b}{2a} = \dfrac{-200}{2(-1)} = \dfrac{-200}{-2} = 100$ yards

(c) $A(100) = -100^2 + 200(100) = -10000 + 20000 = 10{,}000$ sq yds.

58. (a) Let x = length and y = width of the rectangular field.

$P = 2x + 2y = 3000$

$y = \dfrac{3000 - 2x}{2} = 1500 - x$

Then $A(x) = (1500 - x)x = 1500x - x^2 = -x^2 + 1500x$

(b) $x = \dfrac{-b}{2a} = \dfrac{-1500}{2(-1)} = \dfrac{-1500}{-2} = 750$ feet

(c) $A(750) = -750^2 + 1500(750) = -562500 + 1125000 = 562{,}500$ sq ft.

59. Let x = width and y = length of the rectangular area.

$2x + y = 4000 \quad \rightarrow \quad y = 4000 - 2x$

Then $A(x) = (4000 - 2x)x = 4000x - 2x^2 = -2x^2 + 4000x$

$x = \dfrac{-b}{2a} = \dfrac{-4000}{2(-2)} = \dfrac{-4000}{-4} = 1000$

$A(1000) = -2(1000)^2 + 4000(1000) = -2000000 + 4000000 = 2{,}000{,}000$

The largest area that can be enclosed is 2,000,000 square meters.

60. Let x = width and y = length of the rectangular area.

$2x + y = 2000 \quad \rightarrow \quad y = 2000 - 2x$

Then $A(x) = (2000 - 2x)x = 2000x - 2x^2 = -2x^2 + 2000x$

$x = \frac{-b}{2a} = \frac{-2000}{2(-2)} = \frac{-2000}{-4} = 500$

$A(500) = -2(500)^2 + 2000(500) = -500000 + 1000000 = 500,000$

The largest area that can be enclosed is 500,000 square meters.

61. (a) $a = \frac{-32}{2500}$, $b = 1$, $c = 200$. The maximum height occurs when

$x = \frac{-b}{2a} = \frac{-1}{2\left(\frac{-32}{2500}\right)} = \frac{2500}{64} = 39.0625$ feet from base of the cliff.

(b) The maximum height is

$h(39.0625) = \frac{-32(39.0625)^2}{2500} + 39.0625 + 200 = 219.53$ feet

(c) Solving when $h(x) = 0$:

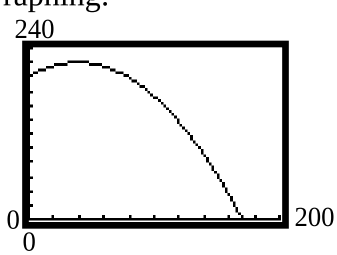
$\frac{-32}{2500}x^2 + x + 200 = 0$

$$x = \frac{-1 \pm \sqrt{1^2 - 4\left(\frac{-32}{2500}\right)(200)}}{2\left(\frac{-32}{2500}\right)} = \frac{-1 \pm \sqrt{11.24}}{-0.0256}$$

$x = -91.90$ or $x = 170.02$

Since the distance cannot be negative, the projectile strikes the water 170.02 feet from the base of the cliff.

(d) Graphing:



(e) Solving when $h(x) = 100$:

$\frac{-32}{2500}x^2 + x + 200 = 100$

$\frac{-32}{2500}x^2 + x + 100 = 0$

$$x = \frac{-1 \pm \sqrt{1^2 - 4\left(\frac{-32}{2500}\right)(100)}}{2\left(\frac{-32}{2500}\right)} = \frac{-1 \pm \sqrt{6.12}}{-0.0256}; \quad x = -57.57 \text{ or } x = 135.70$$

Since the distance cannot be negative, the projectile is 100 feet above the water 135.70 feet from the base of the cliff.

62. (a) $a = \frac{-32}{10000}$, $b = 1$, $c = 0$. The maximum height occurs when

$$x = \frac{-b}{2a} = \frac{-1}{2\left(\frac{-32}{10000}\right)} = \frac{10000}{64} = 156.25 \text{ feet.}$$

(b) The maximum height is

$$h(156.25) = \frac{-32(156.25)^2}{10000} + 156.25 = 78.125 \text{ feet.}$$

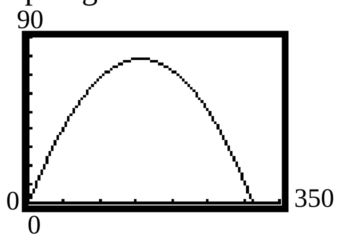
(c) Solving when $h(x) = 0$:

$$\frac{-32}{2500}x^2 + x = 0$$

$$x\left(\frac{-32}{10000}x + 1\right) = 0 \rightarrow x = 0 \text{ or } x = 312.5$$

Since the distance cannot be zero, the projectile lands 312.5 feet from where it was fired.

(d) Graphing:



(e) Solving when $h(x) = 50$:

$$\frac{-32}{10000}x^2 + x = 50$$

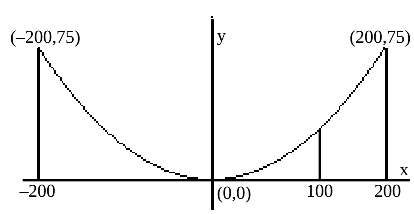
$$\frac{-32}{10000}x^2 + x - 50 = 0$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4\left(\frac{-32}{10000}\right)(-50)}}{2\left(\frac{-32}{10000}\right)} = \frac{-1 \pm \sqrt{0.36}}{-0.0064} = \frac{-1 \pm 0.6}{-0.0064}$$

$$x = 62.5 \text{ or } x = 250$$

The projectile is 50 feet above the ground 62.5 feet and 250 feet from where it was fired.

63. Locate the origin at the point where the cable touches the road. Then the equation of the parabola is of the form: $y = ax^2$, where $a > 0$. Since the point (200, 75) is on the parabola, we can find the constant a:

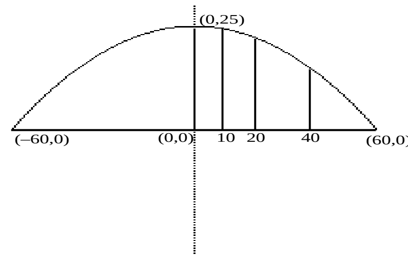


$$75 = a(200)^2 \quad \rightarrow \quad a = \frac{75}{200^2} = 0.001875$$

When $x = 100$, we have:

$$y = 0.001875(100)^2 = 18.75 \text{ meters}.$$

64. Locate the origin at the point directly under the highest point of the arch. Then the equation of the parabola is of the form: $y = -ax^2 + k$, where $a > 0$. Since the maximum height is 25 feet, when $x = 0$, $y = k = 25$. Since the point (60, 0) is on the parabola, we can find the constant a:



$$0 = -a(60)^2 + 25 \quad \to \quad a = \frac{25}{60^2} = 0.006944$$

The equation of the parabola is:

$$y = -\frac{25}{60^2}x^2 + 25.$$

At $x = 10$: $y = -\frac{25}{60^2}(10)^2 + 25 = -\frac{25}{36} + 25 = 24.31$ feet

At $x = 20$: $y = -\frac{25}{60^2}(20)^2 + 25 = -\frac{25}{9} + 25 = 22.22$ feet

At $x = 40$: $y = -\frac{25}{60^2}(40)^2 + 25 = -\frac{100}{9} + 25 = 13.89$ feet

65. Let x = the depth of the gutter and y = the width of the gutter.
Then $A = xy$ is the cross-sectional area of the gutter.
Since the aluminum sheets for the gutter are 12 inches wide, we have

$$2x + y = 12 \text{ or } y = 12 - 2x.$$

The area is to be maximized, so: $A = xy = x(12 - 2x) = -2x^2 + 12x$.
This equation is a parabola opening down; thus, it has a maximum when

$$x = \frac{-b}{2a} = \frac{-12}{2(-2)} = \frac{-12}{-4} = 3.$$

Thus, a depth of 3 inches produces a maximum cross-sectional area.

66. Let x = width of the window and y = height of the rectangular part of the window.
The perimeter of the window is: $x + 2y + \frac{\pi x}{2} = 20 \quad \to \quad y = \frac{40 - 2x - \pi x}{4}$
The area of the window is:

$$A(x) = x\left(\frac{40 - 2x - \pi x}{4}\right) + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = 10x - \frac{x^2}{2} - \frac{\pi x^2}{4} + \frac{\pi x^2}{8} = \left(-\frac{1}{2} - \frac{\pi}{8}\right)x^2 + 10x$$

This equation is a parabola opening down; thus, it has a maximum when

$$x = \frac{-b}{2a} = \frac{-10}{2\left(-\frac{1}{2} - \frac{\pi}{8}\right)} = \frac{10}{1 + \frac{\pi}{4}} \approx 5.60 \text{ feet}$$

$$y = \frac{40 - 2(5.60) - \pi(5.60)}{4} \approx 2.80 \text{ feet}$$

The width of the window is about 5.60 feet and the height of the rectangular part is about 2.80 feet.

67. Let x = the width of the rectangle or the diameter of the semicircle.
Let y = the length of the rectangle.
The perimeter of each semicircle is $\frac{\pi x}{2}$.
The perimeter of the track is given by: $\frac{\pi x}{2}+\frac{\pi x}{2}+y+y=1500$.
Solving for x:

$$\begin{aligned}\frac{\pi x}{2}+\frac{\pi x}{2}+y+y&=1500\\ \pi x+2y&=1500\\ \pi x&=1500-2y\\ x&=\frac{1500-2y}{\pi}\end{aligned}$$

The area of the rectangle is: $A=xy=\left(\frac{1500-2y}{\pi}\right)y=\frac{-2}{\pi}y^2+\frac{1500}{\pi}y$

This equation is a parabola opening down; thus, it has a maximum when

$$y=\frac{-b}{2a}=\frac{\frac{-1500}{\pi}}{2\left(\frac{-2}{\pi}\right)}=\frac{-1500}{-4}=375.\quad \text{Thus, } x=\frac{1500-2(375)}{\pi}=\frac{750}{\pi}\approx 238.73.$$

The dimensions for the rectangle with maximum area are $\frac{750}{\pi}\approx 238.73$ meters by 375 meters.

68. Let x = width of the window and y = height of the rectangular part of the window.
The perimeter of the window is: $3x+2y=16 \quad\rightarrow\quad y=\frac{16-3x}{2}$.
The area of the window is:

$$A(x)=x\left(\frac{16-3x}{2}\right)+\frac{\sqrt{3}}{4}x^2=8x-\frac{3}{2}x^2+\frac{\sqrt{3}}{4}x^2=\left(-\frac{3}{2}+\frac{\sqrt{3}}{4}\right)x^2+8x.$$

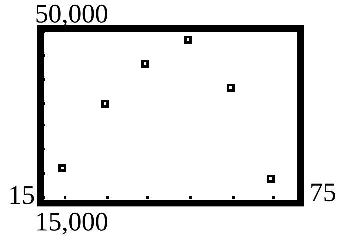
This equation is a parabola opening down; thus, it has a maximum when

$$x=\frac{-b}{2a}=\frac{-8}{2\left(-\frac{3}{2}+\frac{\sqrt{3}}{4}\right)}=\frac{-8}{-3+\frac{\sqrt{3}}{2}}\approx 3.75\text{ feet.}\qquad y=\frac{16-3(3.75)}{2}\approx 2.38.$$

The window is about 3.75 feet wide and 2.38 feet high (rectangular part).

Problems 69 – 73. The equations for the curves that are graphed on the screens use many more decimal places in order to get the desired accuracy.

69. (a) Graphing: The data appear to be quadratic with $a < 0$.



(b) $I(x) = -42.6x^2 + 3806x - 38526$

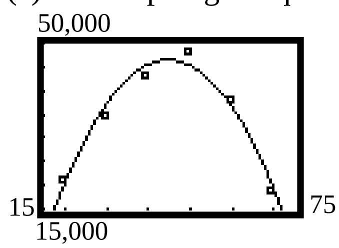
$$x = \frac{-b}{2a} = \frac{-3806}{2(-42.6)} = 44.695$$

An individual will earn the most income at an age of 44.7 years.

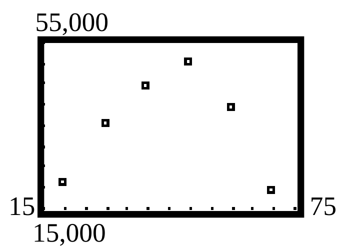
(c) The maximum income will be:

$$I(44.7) = -42.6(44.7)^2 + 3805(44.7) - 38526 = \$46{,}438.87$$

(d) and (e) Graphing the quadratic function of best fit:



70. (a) Graphing: The data appear to be quadratic with $a < 0$.



(b) $I(x) = -44.8x^2 + 4009x - 41392$

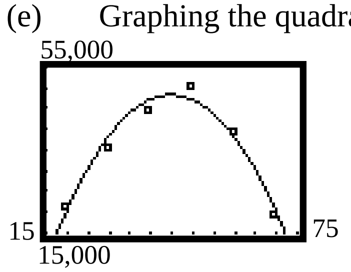
$$x = \frac{-b}{2a} = \frac{-4009}{2(-44.8)} \approx 44.74$$

An individual will earn the most income at an age of 44.8 years.

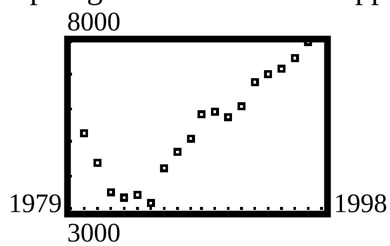
(c) The maximum income will be:

$$I(44.7) = -44.8(44.7)^2 + 4009(44.7) - 41392 = \$48{,}295.87$$

(d) and (e) Graphing the quadratic function of best fit:



71. (a) Graphing: The data appears to be quadratic with $a > 0$.



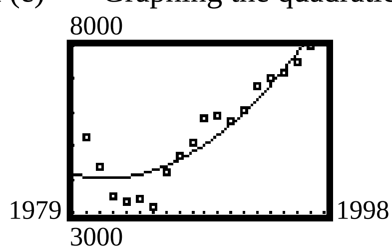
(b) Note: The coefficients given in the text are incorrect. The correct function is:

$$I(x) = 17.199x^2 - 68147.78x + 67507955$$

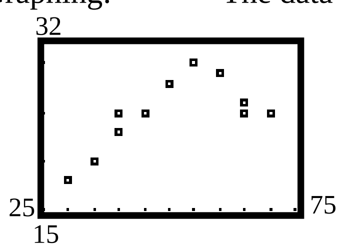
$$x = \frac{-b}{2a} = \frac{-(-68147.78)}{2(17.199)} \approx 1981.16$$ Imports were lowest in 1981.

(c) The predicted number of barrels imported in 1998 (using the equation to many more decimal places) is: $I(1998) \approx 3165$ thousand barrels

(d) and (e) Graphing the quadratic function of best fit:



72. (a) Graphing: The data appears to be quadratic with $a < 0$.

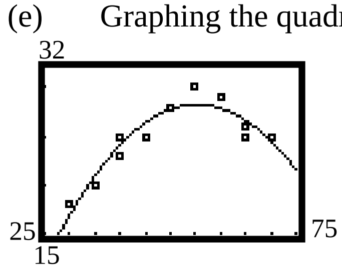


(b) $M(s) = -0.018s^2 + 1.93s - 25.34$

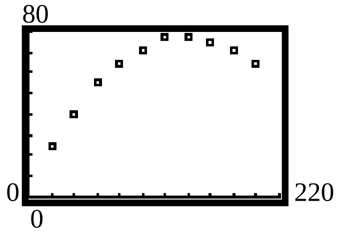
$$s = \frac{-b}{2a} = \frac{-1.93}{2(-0.018)} = 53.61$$ The speed that maximizes miles per gallon is about 54 miles per hour.

(c) The predicted miles per gallon when the speed is 63 miles per hour is: $M(63) = 24.81$ miles per gallon.

(d) and (e) Graphing the quadratic function of best fit:



73. (a) Graphing: The data appears to be quadratic with $a < 0$.



(b) $h(x) = -0.0037x^2 + 1.03x + 5.7$

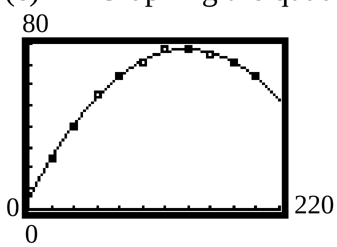
$$x = \frac{-b}{2a} = \frac{-1.03}{2(-0.0037)} \approx 139.19 \text{ feet}$$

The ball travels about 139 feet before reaching its maximum height.

(c) The maximum height will be: (using the equation to many more decimal places)

$$h(139) = -0.0037(139)^2 + 1.03(139) + 5.7 \approx 77.38 \text{ feet}$$

(d) and (e) Graphing the quadratic function of best fit:



74. We are given: $V(x) = kx(a - x) = -kx^2 + akx$

The reaction rate is a maximum when: $x = \dfrac{-b}{2a} = \dfrac{-ak}{2(-k)} = \dfrac{ak}{2k} = \dfrac{a}{2}$

75. We have:

$$a(-h)^2 + b(-h) + c = ah^2 - bh + c = y_0$$
$$a(0)^2 + b(0) + c = c = y_1$$
$$a(h)^2 + b(h) + c = ah^2 + bh + c = y_2$$

Equating the two equations for the area, we have:

$$y_0 + 4y_1 + y_2 = ah^2 - bh + c + 4c + ah^2 + bh + c = 2ah^2 + 6c$$

Therefore, Area $= \dfrac{h}{3}\left(2ah^2 + 6c\right) = \dfrac{h}{3}\left(y_0 + 4y_1 + y_2\right)$.

76. $f(x) = -5x^2 + 8 \quad h = 1$:

$$\text{Area} = \frac{h}{3}\left(2ah^2 + 6c\right) = \frac{1}{3}\left(2(-5)(1)^2 + 6(8)\right) = \frac{1}{3}(-10 + 48) = \frac{38}{3} \approx 12.67$$

77. $f(x) = 2x^2 + 8, \quad h = 2$

$$\text{Area} = \frac{2}{3}\left(2(2)(2)^2 + 6(8)\right) = \frac{2}{3}(16 + 48) = \frac{2}{3}(64) = \frac{128}{3}$$

78. $f(x) = x^2 + 3x + 5, \quad h = 4$

$$\text{Area} = \frac{h}{3}\left(2ah^2 + 6c\right) = \frac{4}{3}\left(2(1)(4)^2 + 6(5)\right) = \frac{4}{3}(32 + 30) = \frac{248}{3} \approx 82.67$$

79. $f(x) = -x^2 + x + 4, \quad h = 1$

$$\text{Area} = \frac{1}{3}\left(2(-1)(1)^2 + 6(4)\right) = \frac{1}{3}(-2+24) = \frac{1}{3}(22) = \frac{22}{3}$$

80. $A(x) = x(10-x) = -x^2 + 10x$

The area is a maximum when: $x = \frac{-b}{2a} = \frac{-10}{2(-1)} = \frac{10}{2} = 5$

$$A(5) = -(5)^2 + 10(5) = -25 + 50 = 25$$

The largest area that can be enclosed is 25 square units.

81. If x is even, then ax^2 and bx are even. When two even numbers are added to an odd number the result is odd. Thus, $f(x)$ is odd.

If x is odd, then ax^2 and bx are odd. The sum of three odd numbers results in an odd number. Thus, $f(x)$ is odd.

82. $f(x) = -(x+2)^2 = -x^2 - 4x - 4$ opens down and has only one x-intercept (- 2, 0).
In general, a quadratic function that has only one x-intercept must have the form $f(x) = ax^2 + bx + c$ such that $b^2 - 4ac = 0$, since discriminant = 0 means the associated quadratic equation has exactly one real solution.
Also, we must have $a < 0$ in order for the parabola to open downwards.

83. $f(x) = x^2 + 2x - 3;$ $\quad f(x) = x^2 + 2x + 1;$ $\quad f(x) = x^2 + 2x$

each member of this family will be a parabola with the following characteristics:

- opens upwards since a > 0
- vertex occurs at $x = \frac{-b}{2a} = \frac{-2}{2(1)} = -1$
- there is at least one x-intercept since $b^2 - 4ac \ge 0$

84. $f(x) = x^2 - 4x + 1;$ $\quad f(x) = x^2 + 1;$ $\quad f(x) = x^2 + 4x + 1$

each member of this family will be a parabola with the following characteristics:

- opens upwards since a > 0
- y-intercept occurs at (0, 1)