Chapter 4

Polynomial and Rational Functions

4.3 Rational Functions I

1. In $R(x)=\dfrac{4x}{x-3}$, the denominator, $q(x)=x-3$, has a zero at 3. Thus, the domain of $R(x)$ is all real numbers except 3.

2. In $R(x)=\dfrac{5x^2}{3+x}$, the denominator, $q(x)=3+x$, has a zero at –3. Thus, the domain of $R(x)$ is all real numbers except –3.

3. In $H(x)=\dfrac{-4x^2}{(x-2)(x+4)}$, the denominator, $q(x)=(x-2)(x+4)$, has zeros at 2 and –4. Thus, the domain of $H(x)$ is all real numbers except 2 and –4.

4. In $G(x)=\dfrac{6}{(x+3)(4-x)}$, the denominator, $q(x)=(x+3)(4-x)$, has zeros at –3 and 4. Thus, the domain of $G(x)$ is all real numbers except –3 and 4.

5. In $F(x)=\dfrac{3x(x-1)}{2x^2-5x-3}$, the denominator, $q(x)=2x^2-5x-3=(2x+1)(x-3)$, has zeros at $\frac{-1}{2}$ and 3. Thus, the domain of $F(x)$ is all real numbers except $\frac{-1}{2}$ and 3.

6. In $Q(x)=\dfrac{-x(1-x)}{3x^2+5x-2}$, the denominator, $q(x)=3x^2+5x-2=(3x-1)(x+2)$, has zeros at $\frac{1}{3}$ and -2. Thus, the domain of $Q(x)$ is all real numbers except $\frac{1}{3}$ and -2.

7. In $R(x)=\dfrac{x}{x^3-8}$, the denominator, $q(x)=x^3-8=(x-2)(x^2+2x+4)$, has a zero at 2. (x^2+2x+4 has no real zeros.) Thus, the domain of $R(x)$ is all real numbers except 2.

8. In $R(x)=\dfrac{x}{x^4-1}$, the denominator, $q(x)=x^4-1=(x-1)(x+1)(x^2+1)$, has zeros at –1 and 1. (x^2+1 has no real zeros.) Thus, the domain of $R(x)$ is all real numbers except –1 and 1.

9. In $H(x)=\dfrac{3x^2+x}{x^2+4}$, the denominator, $q(x)=x^2+4$, has no real zeros. Thus, the domain of $H(x)$ is all real numbers.

10. In $G(x) = \dfrac{x-3}{x^4+1}$, the denominator, $q(x) = x^4 + 1$, has no real zeros. Thus, the domain of $G(x)$ is all real numbers.

11. In $R(x) = \dfrac{3(x^2-x-6)}{4(x^2-9)}$, the denominator, $q(x) = 4(x^2-9) = 4(x-3)(x+3)$, has zeros at 3 and –3. Thus, the domain of $R(x)$ is all real numbers except 3 and –3.

12. In $F(x) = \dfrac{-2(x^2-4)}{3(x^2+4x+4)}$, the denominator, $q(x) = 3(x^2+4x+4) = 3(x+2)^2$, has a zero at –2. Thus, the domain of $F(x)$ is all real numbers except –2.

13. (a) Domain: $\{x \mid x \neq 2\}$; Range: $\{y \mid y \neq 1\}$
(b) Intercept: (0, 0)
(c) Horizontal Asymptote: $y = 1$
(d) Vertical Asymptote: $x = 2$
(e) Oblique Asymptote: none

14. (a) Domain: $\{x \mid x \neq -1\}$; Range: $\{y \mid y > 0\}$
(b) Intercept: (0, 2)
(c) Horizontal Asymptote: $y = 0$
(d) Vertical Asymptote: $x = -1$
(e) Oblique Asymptote: none

15. (a) Domain: $\{x \mid x \neq 0\}$; Range: all real numbers
(b) Intercepts: (–1, 0), (1, 0)
(c) Horizontal Asymptote: none
(d) Vertical Asymptote: $x = 0$
(e) Oblique Asymptote: $y = 2x$

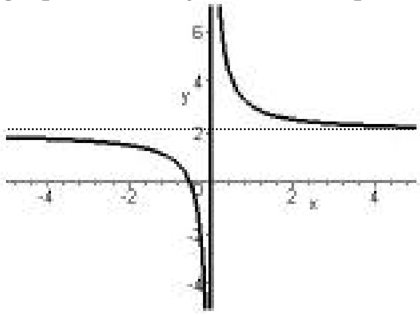
16. (a) Domain: $\{x \mid x \neq 0\}$; Range: $\{y \mid y > 2 \text{ or } y < -2\}$
(b) Intercepts: none
(c) Horizontal Asymptote: none
(d) Vertical Asymptote: $x = 0$
(e) Oblique Asymptote: $y = -x$

17. (a) Domain: $\{x \mid x \neq -2, x \neq 2\}$; Range: $\{y \mid y \leq 0 \text{ or } y > 1\}$
(b) Intercept: (0, 0)
(c) Horizontal Asymptote: $y = 1$
(d) Vertical Asymptotes: $x = -2$, $x = 2$
(e) Oblique Asymptote: none

18. (a) Domain: $\{x \mid x \neq -1, x \neq 1\}$; Range: all real numbers
(b) Intercept: (0, 0)
(c) Horizontal Asymptote: $y = 0$
(d) Vertical Asymptotes: $x = -1$, $x = 1$
(e) Oblique Asymptote: none

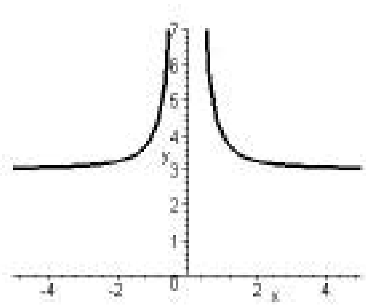
19. $F(x) = 2 + \dfrac{1}{x}$

Using the function, $y = \dfrac{1}{x}$, shift the graph vertically 2 units to up.



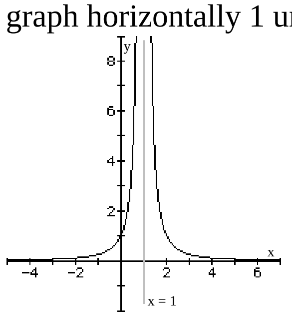
20. $R(x) = 3 + \dfrac{1}{x^2}$

Using the function $y = \dfrac{1}{x^2}$, shift the graph vertically 3 units to up.



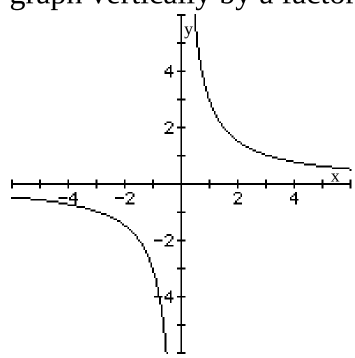
21. $R(x) = \dfrac{1}{(x-1)^2}$

Using the function, $y = \dfrac{1}{x^2}$, shift the graph horizontally 1 unit to the right.



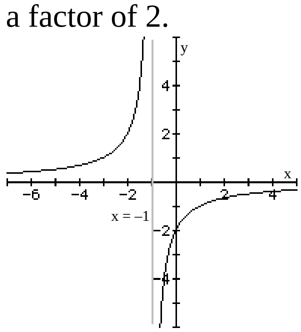
22. $R(x) = \dfrac{3}{x}$

Using the function $y = \dfrac{1}{x}$, stretch the graph vertically by a factor of 3.



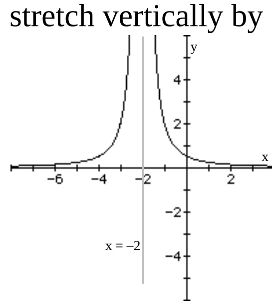
23. $H(x) = \dfrac{-2}{x+1}$

Using the function $y = \dfrac{1}{x}$, shift the graph horizontally 1 unit to the left, reflect about the x-axis, and stretch vertically by a factor of 2.



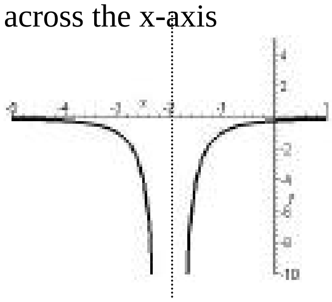
24. $G(x) = \dfrac{2}{(x+2)^2}$

Using the function $y = \dfrac{1}{x^2}$, shift the graph horizontally 2 units to the left, and stretch vertically by a factor of 2.



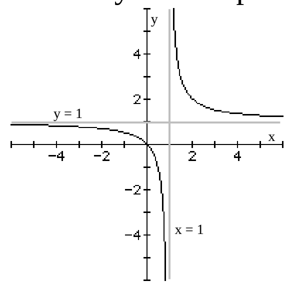
25. $R(x)=\dfrac{-1}{x^2+4x+4}=\dfrac{-1}{(x+2)^2}$

Using the function $y=\dfrac{1}{x^2}$, shift the graph horizontally 2 units to the left, then reflect across the x-axis



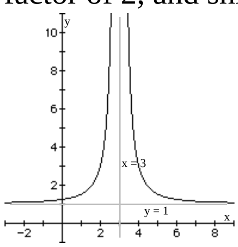
26. $R(x)=\dfrac{1}{x-1}+1$

Using the function $y=\dfrac{1}{x}$, shift the graph horizontally 1 unit to the right, and shift vertically 1 unit up.



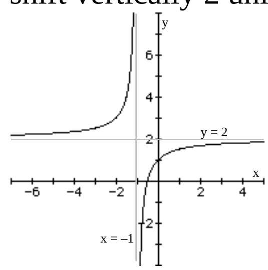
27. $G(x)=1+\dfrac{2}{(x-3)^2}=\dfrac{2}{(x-3)^2}+1$

Using the function $y=\dfrac{1}{x^2}$, shift the graph 3 units right, stretch vertically by a factor of 2, and shift vertically 1 unit up.



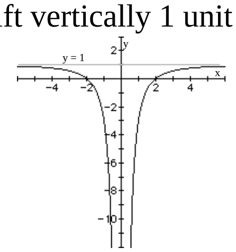
28. $F(x)=2-\dfrac{1}{x+1}=\dfrac{-1}{x+1}+2$

Using the function $y=\dfrac{1}{x}$, shift the graph 1 unit left, reflect about the x-axis, and shift vertically 2 units up.



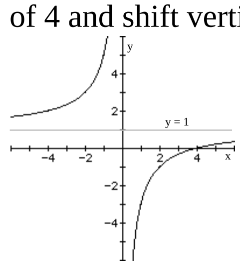
29. $R(x)=\dfrac{x^2-4}{x^2}=1-\dfrac{4}{x^2}$

Using the function $y=\dfrac{1}{x^2}$, reflect about the x-axis, stretch vertically by a factor of 4 and shift vertically 1 unit up.



30. $R(x)=\dfrac{x-4}{x}=1-\dfrac{4}{x}$

Using the function $y=\dfrac{1}{x}$, reflect about the x-axis, stretch vertically by a factor of 4 and shift vertically 1 unit up.



31. $R(x)=\dfrac{3x}{x+4}$

The degree of the numerator, $p(x)=3x$, is $n=1$. The degree of the denominator, $q(x)=x+4$, is $m=1$. Since $n=m$, the line $y=\dfrac{3}{1}=3$ is a horizontal asymptote. The denominator is zero at $x=-4$, so $x=-4$ is a vertical asymptote.

32. $R(x)=\dfrac{3x+5}{x-6}$

The degree of the numerator, $p(x)=3x+5$, is $n=1$. The degree of the denominator, $q(x)=x-6$, is $m=1$. Since $n=m$, the line $y=\dfrac{3}{1}=3$ is a horizontal asymptote. The denominator is zero at $x=6$, so $x=6$ is a vertical asymptote.

33. $H(x)=\dfrac{x^4+2x^2+1}{x^2-x+1}$

The degree of the numerator, $p(x)=x^4+2x^2+1$, is $n=4$. The degree of the denominator, $q(x)=x^2-x+1$, is $m=2$. Since $n>m+1$, there is no horizontal asymptote or oblique asymptote. The denominator has no real zeros, so there is no vertical asymptote.

34. $G(x)=\dfrac{-x^2+1}{x+5}$

The degree of the numerator, $p(x)=-x^2+1$, is $n=2$. The degree of the denominator, $q(x)=x+5$, is $m=1$. Since $n=m+1$, there is an oblique asymptote.

Dividing:

$$\begin{array}{r|l} & -x+5 \\ \hline x+5 & -x^2+0x+1 \\ & \underline{-x^2-5x} \\ & \qquad 5x+1 \\ & \qquad \underline{5x+25} \\ & \qquad\ -24 \end{array} \qquad G(x)=-x+5+\frac{-24}{x+5}$$

Thus, the oblique asymptote is $y=-x+5$.

The denominator is zero at $x=-5$, so $x=-5$ is a vertical asymptote.

35. $T(x)=\dfrac{x^3}{x^4-1}$

The degree of the numerator, $p(x)=x^3$, is $n=3$. The degree of the denominator, $q(x)=x^4-1$ is $m=4$. Since $n<m$, the line $y=0$ is a horizontal asymptote. The denominator is zero at $x=-1$ and $x=1$, so $x=-1$ and $x=1$ are vertical asymptotes.

36. $P(x)=\dfrac{4x^5}{x^3-1}$

The degree of the numerator, $p(x)=4x^5$, is $n=5$. The degree of the denominator, $q(x)=x^3-1$ is $m=3$. Since $n>m+1$, there is no horizontal asymptote and there is no oblique asymptote. The denominator is zero at $x=1$, so $x=1$ is a vertical asymptote.

37. $Q(x) = \dfrac{5 - x^2}{3x^4}$

The degree of the numerator, $p(x) = 5 - x^2$, is $n = 2$. The degree of the denominator, $q(x) = 3x^4$ is $m = 4$. Since $n < m$, the line $y = 0$ is a horizontal asymptote. The denominator is zero at $x = 0$, so $x = 0$ is a vertical asymptote.

38. $F(x) = \dfrac{-2x^2 + 1}{2x^3 + 4x^2} = \dfrac{-2x^2 + 1}{2x^2(x+2)}$

The degree of the numerator, $p(x) = -2x^2 + 1$, is $n = 2$. The degree of the denominator, $q(x) = 2x^3 + 4x^2$ is $m = 3$. Since $n < m$, the line $y = 0$ is a horizontal asymptote. The denominator is zero at $x = 0$ and $x = -2$, so $x = 0$ and $x = -2$ are vertical asymptotes.

39. $R(x) = \dfrac{3x^4 + 4}{x^3 + 3x}$

The degree of the numerator, $p(x) = 3x^4 + 4$, is $n = 4$. The degree of the denominator, $q(x) = x^3 + 3x$ is $m = 3$. Since $n = m + 1$, there is an oblique asymptote.

Dividing:

$$\begin{array}{r|l} & 3x \\ \hline x^3 + 3x & 3x^4 + 0x^3 + 0x^2 + 0x + 4 \\ & \underline{3x^4 \qquad\quad + 9x^2} \\ & \qquad -9x^2 + 0x + 4 \end{array} \qquad R(x) = 3x + \frac{-9x^2 + 4}{x^3 + 3x}$$

Thus, the oblique asymptote is $y = 3x$.

The denominator is zero at $x = 0$, so $x = 0$ is a vertical asymptote.

40. $R(x) = \dfrac{6x^2 + x + 12}{3x^2 - 5x - 2} = \dfrac{6x^2 + x + 12}{(3x+1)(x-2)}$

The degree of the numerator, $p(x) = 6x^2 + x + 12$, is $n = 2$. The degree of the denominator, $q(x) = 3x^2 - 5x - 2$ is $m = 2$. Since $n = m$, the line $y = \frac{6}{3} = 2$ is a horizontal asymptote.

The denominator is zero at $x = -\frac{1}{3}$ and $x = 2$, so $x = -\frac{1}{3}$ and $x = 2$ are vertical asymptotes.

41. $G(x) = \dfrac{x^3 - 1}{x - x^2}$, $x \neq 1$

The degree of the numerator, $p(x) = x^3 - 1$ is $n = 3$. The degree of the denominator, $q(x) = x - x^2$ is $m = 2$. Since $n = m + 1$, there is an oblique asymptote.

Dividing:

$$\begin{array}{r|l} & -x - 1 \\ \hline -x^2 + x & x^3 + 0x^2 + 0x - 1 \\ & \underline{x^3 - x^2} \\ & \qquad x^2 + 0x \\ & \qquad \underline{x^2 - x} \\ & \qquad\qquad x - 1 \end{array}$$

$$G(x) = -x - 1 + \frac{x - 1}{x - x^2} = -x - 1 - \frac{1}{x}, \; x \neq 1$$

Thus, the oblique asymptote is $y = -x - 1$.

$G(x)$ must be in lowest terms to find the vertical asymptote:

$$G(x) = \frac{x^3 - 1}{x - x^2} = \frac{(x-1)(x^2 + x + 1)}{-x(x-1)} = \frac{x^2 + x + 1}{-x}$$

The denominator is zero at $x = 0$, so $x = 0$ is a vertical asymptotes.

42. $F(x)=\dfrac{x-1}{x-x^3}=\dfrac{x-1}{-x(x^2-1)}=\dfrac{x-1}{-x(x-1)(x+1)}=\dfrac{1}{-x(x+1)}$

The degree of the numerator, $p(x)=x-1$, is $n=1$. The degree of the denominator, $q(x)=x-x^3$ is $m=3$. Since $n<m$, the line $y=0$ is a horizontal asymptote. The denominator is zero at $x=0$, and $x=-1$, so $x=0$, and $x=-1$ are vertical asymptotes.

43. $g(h)=\dfrac{3.99\times 10^{14}}{\left(6.374\times 10^{6}+h\right)^2}$

(a) $g(0)=\dfrac{3.99\times 10^{14}}{\left(6.374\times 10^{6}+0\right)^2}\approx 9.821\ m/s^2$

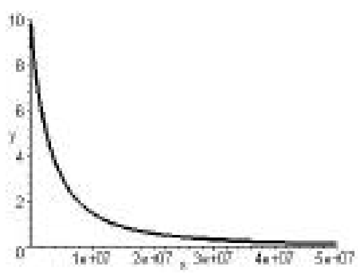
(b) $g(443)=\dfrac{3.99\times 10^{14}}{\left(6.374\times 10^{6}+443\right)^2}\approx 9.8195\ m/s^2$

(c) $g(8448)=\dfrac{3.99\times 10^{14}}{\left(6.374\times 10^{6}+8448\right)^2}\approx 9.795\ m/s^2$

(d) $g(h)=\dfrac{3.99\times 10^{14}}{\left(6.374\times 10^{6}+h\right)^2}\approx\dfrac{3.99\times 10^{14}}{h^2}\to 0$ as $h\to\infty$

$y=0$ is the horizontal asymptote.

(e)



(f) $g(h)=\dfrac{3.99\times 10^{14}}{\left(6.374\times 10^{6}+h\right)^2}=0$, to solve this equation would require that

$3.99\times 10^{14}=0$, which is impossible. Therefore, there is no height above sea level at which $g=0$.

44. $P(t)=\dfrac{50(1+0.5t)}{(2+0.01t)}$

(a) $P(0)=\dfrac{50(1+0)}{(2+0)}=\dfrac{50}{2}=25$ insects

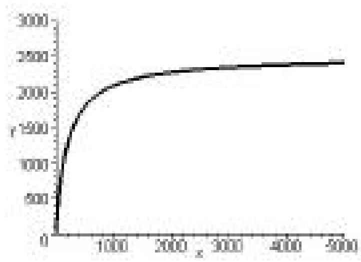
(b) $P(5) = \dfrac{50(1+0.5(5))}{(2+0.01(5))} = \dfrac{175}{2.05} \approx 85$ insects

(c) $P(t) = \dfrac{50(1+0.5t)}{(2+0.01t)} \approx \dfrac{50(0.5t)}{0.01t} = 2500 \;\; as \;\; t \to \infty$

$y = 2500$ is the horizontal asymptote.

The area can sustain a maximum population of 2500 insects.

(d)



(e) $P(10000000000) = \dfrac{50(1+0.5(10000000000))}{(2+0.01(10000000000))} \approx 2499.9999505$

45. A rational function $R(x)$ has a vertical asymptote at x = c whenever $R(c) = \dfrac{nonzero}{zero}$.
That is, whenever x = c yields a zero in the denominator of the function formula. And the denominator will equal zero only if it contains the factor $(x-c)^n$, for some $n > 0$.