Chapter 4

Polynomial and Rational Functions

4.R Chapter Review

1. $f(x) = (x-2)^2 + 2 = x^2 - 4x + 4 + 2 = x^2 - 4x + 6$

$a = 1,\ b = -4,\ c = 6.$ Since $a = 1 > 0,$ the graph opens up.

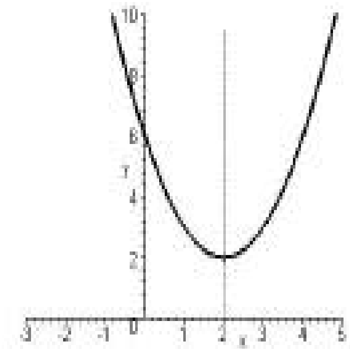
The x-coordinate of the vertex is $x = \dfrac{-b}{2a} = \dfrac{-(-4)}{2(1)} = \dfrac{4}{2} = 2.$

The y-coordinate of the vertex is $f\left(\dfrac{-b}{2a}\right) = f(2) = (2)^2 - 4(2) + 6 = 2.$

Thus, the vertex is (2, 2).
The axis of symmetry is the line $x = 2$.
The discriminant is:

$$b^2 - 4ac = (-4)^2 - 4(1)(6) = -8 < 0,$$

so the graph has no x-intercepts.
The y-intercept is $f(0) = 6$.



2. $f(x) = (x+1)^2 - 4 = x^2 + 2x + 2 - 4 = x^2 + 2x - 2$

$a = 1,\ b = 2,\ c = -2.$ Since $a = 1 > 0,$ the graph opens up.

The x-coordinate of the vertex is $x = \dfrac{-b}{2a} = \dfrac{-2}{2(1)} = -1.$

The y-coordinate of the vertex is $f\left(\dfrac{-b}{2a}\right) = f(-1) = (-1)^2 + 2(-1) - 2 = -3.$

Thus, the vertex is (- 1, - 3).
The axis of symmetry is the line $x = -1$.
The discriminant is:

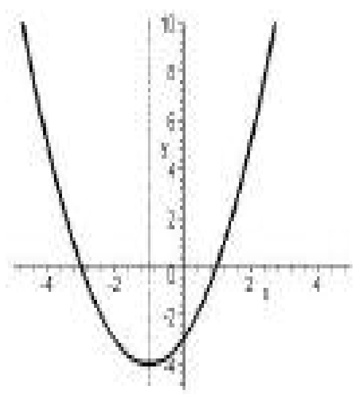
$$b^2 - 4ac = (2)^2 - 4(1)(-2) = 12 > 0,$$

so the graph has two x-intercepts.
The x-intercepts are found by solving:

$$x^2 + 2x - 2 = 0$$

$$x = \frac{-2 \pm \sqrt{12}}{2} = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$$

The x-intercepts are $-1-\sqrt{3}$ and $-1+\sqrt{3}$.
The y-intercept is $f(0) = -2$.



3. $f(x) = \frac{1}{4}x^2 - 16$

$a = \frac{1}{4}$, $b = 0$, $c = -16$. Since $a = \frac{1}{4} > 0$, the graph opens up.

The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-0}{2(\frac{1}{4})} = \frac{0}{\frac{1}{2}} = 0$.

The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f(0) = \frac{1}{4}(0)^2 - 16 = -16$.

Thus, the vertex is (0, –16).

The axis of symmetry is the line $x = 0$.

The discriminant is:

$$b^2 - 4ac = (0)^2 - 4\left(\tfrac{1}{4}\right)(-16) = 16 > 0,$$

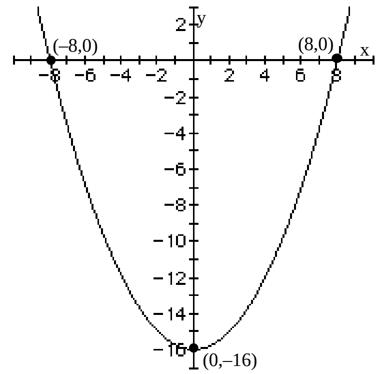
so the graph has two x-intercepts.

The x-intercepts are found by solving:

$$\frac{1}{4}x^2 - 16 = 0$$
$$x^2 - 64 = 0$$
$$x^2 = 64$$
$$x = 8 \text{ or } x = -8$$

The x-intercepts are –8 and 8.

The y-intercept is $f(0) = -16$.



4. $f(x) = -\frac{1}{2}x^2 - 2$

$a = -\frac{1}{2}$, $b = 0$, $c = -2$. Since $a = -\frac{1}{2} < 0$, the graph opens down.

The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-0}{2(-\frac{1}{2})} = \frac{0}{-1} = 0$.

The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f(0) = -\frac{1}{2}(0)^2 - 2 = -2$.

Thus, the vertex is (0, –2).

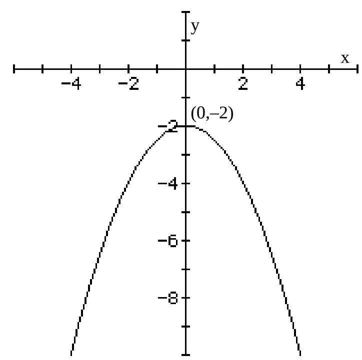
The axis of symmetry is the line $x = 0$.

The discriminant is:

$$b^2 - 4ac = (0)^2 - 4\left(-\tfrac{1}{2}\right)(-2) = -4 < 0,$$

so the graph has no x-intercepts.

The y-intercept is $f(0) = -2$.



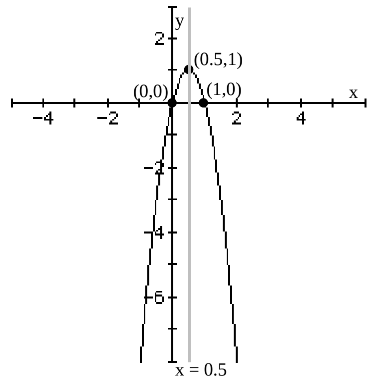
5. $f(x) = -4x^2 + 4x$

$a = -4$, $b = 4$, $c = 0$. Since $a = -4 < 0$, the graph opens down.

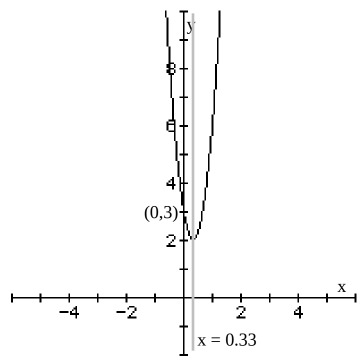
The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-4}{2(-4)} = \frac{-4}{-8} = \frac{1}{2}$.

The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f\left(\frac{1}{2}\right) = -4\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right) = -1 + 2 = 1$.

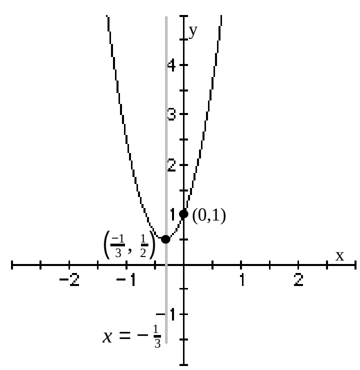
Thus, the vertex is $\left(\frac{1}{2}, 1\right)$.
The axis of symmetry is the line $x = \frac{1}{2}$.
The discriminant is:
$$b^2 - 4ac = 4^2 - 4(-4)(0) = 16 > 0,$$
so the graph has two x-intercepts.
The x-intercepts are found by solving:
$$-4x^2 + 4x = 0$$
$$-4x(x-1) = 0$$
$$x = 0 \text{ or } x = 1$$
The x-intercepts are 0 and 1.
The y-intercept is $f(0) = -4(0)^2 + 4(0) = 0$.



6. $f(x) = 9x^2 - 6x + 3$
$a = 9,\ b = -6,\ c = 3$. Since $a = 9 > 0$, the graph opens up.
The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-(-6)}{2(9)} = \frac{6}{18} = \frac{1}{3}$.
The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f\left(\frac{1}{3}\right) = 9\left(\frac{1}{3}\right)^2 - 6\left(\frac{1}{3}\right) + 3 = 1 - 2 + 3 = 2$.
Thus, the vertex is $\left(\frac{1}{3}, 2\right)$.
The axis of symmetry is the line $x = \frac{1}{3}$.
The discriminant is:
$$b^2 - 4ac = (-6)^2 - 4(9)(3) = -72 < 0,$$
so the graph has no x-intercepts.
The y-intercept is $f(0) = 9(0)^2 - 6(0) + 3 = 3$.



7. $f(x) = \frac{9}{2}x^2 + 3x + 1$
$a = \frac{9}{2},\ b = 3,\ c = 1$. Since $a = \frac{9}{2} > 0$, the graph opens up.
The x-coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-3}{2\left(\frac{9}{2}\right)} = \frac{-3}{9} = \frac{-1}{3}$.
The y-coordinate of the vertex is $f\left(\frac{-b}{2a}\right) = f\left(\frac{-1}{3}\right) = \frac{9}{2}\left(\frac{-1}{3}\right)^2 + 3\left(\frac{-1}{3}\right) + 1 = \frac{1}{2} - 1 + 1 = \frac{1}{2}$.
Thus, the vertex is $\left(\frac{-1}{3}, \frac{1}{2}\right)$.
The axis of symmetry is the line $x = \frac{-1}{3}$.
The discriminant is:
$$b^2 - 4ac = 3^2 - 4\left(\frac{9}{2}\right)(1) = 9 - 18 = -9 < 0,$$
so the graph has no x-intercepts.
The y-intercept is $f(0) = \frac{9}{2}(0)^2 + 3(0) + 1 = 1$.



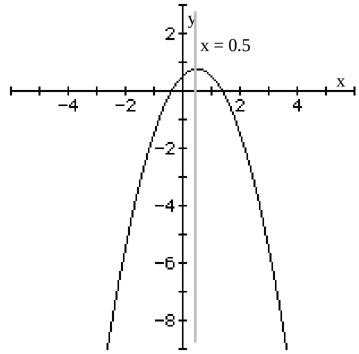
8. $f(x) = -x^2 + x + \frac{1}{2}$

$a = -1,\ b = 1,\ c = \frac{1}{2}$. Since $a = -1 < 0$, the graph opens down.

The x-coordinate of the vertex is $x = \dfrac{-b}{2a} = \dfrac{-1}{2(-1)} = \dfrac{-1}{-2} = \dfrac{1}{2}$.

The y-coordinate of the vertex is $f\left(\dfrac{-b}{2a}\right) = f\left(\dfrac{1}{2}\right) = -\left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{2}\right) + \dfrac{1}{2} = -\dfrac{1}{4} + 1 = \dfrac{3}{4}$.

Thus, the vertex is $\left(\dfrac{1}{2}, \dfrac{3}{4}\right)$.



The axis of symmetry is the line $x = \frac{1}{2}$.

The discriminant is:

$$b^2 - 4ac = 1^2 - 4(-1)\left(\tfrac{1}{2}\right) = 1 + 2 = 3 > 0,$$

so the graph has two x-intercepts.

The x-intercepts are found by solving:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{3}}{2(-1)}$$
$$= \frac{-1 \pm \sqrt{3}}{-2} = \frac{1 \pm \sqrt{3}}{2} = \frac{1 \pm 1.732}{2}$$

The x-intercepts are –0.37 and 1.37.

The y-intercept is $f(0) = -(0)^2 + (0) + \frac{1}{2} = \frac{1}{2}$.

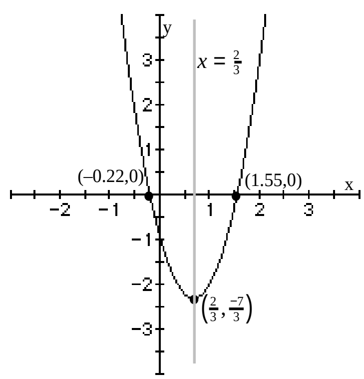
9. $f(x) = 3x^2 - 4x - 1$

$a = 3,\ b = -4,\ c = -1$. Since $a = 3 > 0$, the graph opens up.

The x-coordinate of the vertex is $x = \dfrac{-b}{2a} = \dfrac{-(-4)}{2(3)} = \dfrac{4}{6} = \dfrac{2}{3}$.

The y-coordinate of the vertex is $f\left(\dfrac{-b}{2a}\right) = f\left(\dfrac{2}{3}\right) = 3\left(\dfrac{2}{3}\right)^2 - 4\left(\dfrac{2}{3}\right) - 1 = \dfrac{4}{3} - \dfrac{8}{3} - 1 = \dfrac{-7}{3}$.

Thus, the vertex is $\left(\frac{2}{3}, \frac{-7}{3}\right)$.



The axis of symmetry is the line $x = \frac{2}{3}$.

The discriminant is:

$$b^2 - 4ac = (-4)^2 - 4(3)(-1) = 16 + 12 = 28 > 0,$$

so the graph has two x-intercepts.

The x-intercepts are found by solving:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{28}}{2(3)}$$
$$= \frac{4 \pm 2\sqrt{7}}{6} = \frac{2 \pm \sqrt{7}}{3} = \frac{-3 \pm 2.646}{3}$$

The x-intercepts are –0.22 and 1.55.

The y-intercept is $f(0) = 3(0)^2 - 4(0) - 1 = -1$.

10. $f(x) = -2x^2 - x + 4$

$a = -2,\ b = -1,\ c = 4$. Since $a = -2 < 0$, the graph opens down.

The x-coordinate of the vertex is $x = \dfrac{-b}{2a} = \dfrac{-(-1)}{2(-2)} = \dfrac{1}{-4} = -\dfrac{1}{4}$.

The y-coordinate of the vertex is

$$f\left(\frac{-b}{2a}\right) = f\left(-\frac{1}{4}\right) = -2\left(-\frac{1}{4}\right)^2 - \left(-\frac{1}{4}\right) + 4 = -\frac{1}{8} + \frac{1}{4} + 4 = \frac{33}{8}.$$

Thus, the vertex is $\left(-\frac{1}{4}, \frac{33}{8}\right)$.

The axis of symmetry is the line $x = -\frac{1}{4}$.

The discriminant is:

$$b^2 - 4ac = (-1)^2 - 4(-2)(4) = 1 + 32 = 33 > 0,$$

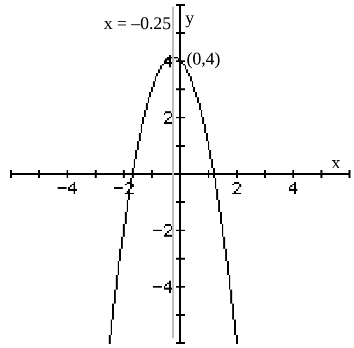
so the graph has two x-intercepts.

The x-intercepts are found by solving:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{33}}{2(-2)}$$
$$= \frac{1 \pm \sqrt{33}}{-4} = \frac{-1 \pm \sqrt{33}}{4} = \frac{-1 \pm 5.745}{4}$$

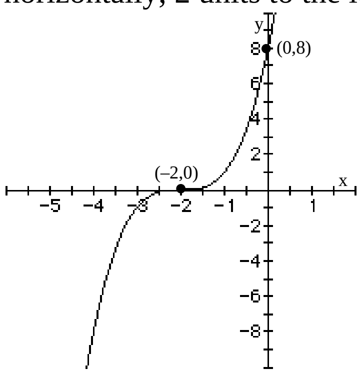
The x-intercepts are –1.69 and 1.19.

The y-intercept is $f(0) = -2(0)^2 - (0) + 4 = 4$.



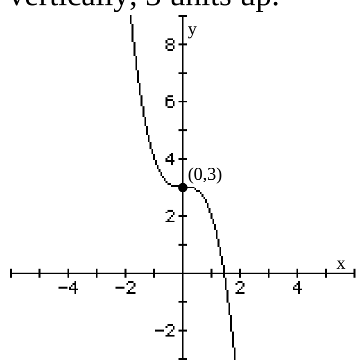
11. $f(x) = (x+2)^3$

Using the graph of $y = x^3$, shift the graph horizontally, 2 units to the left.



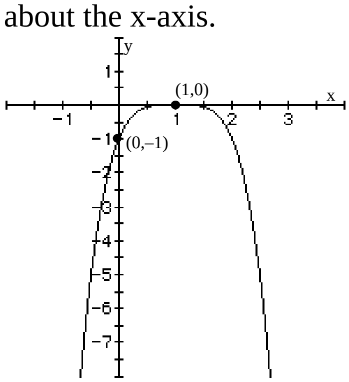
12. $f(x) = -x^3 + 3$

Using the graph of $y = x^3$, reflect the graph about the x-axis, and shift vertically, 3 units up.



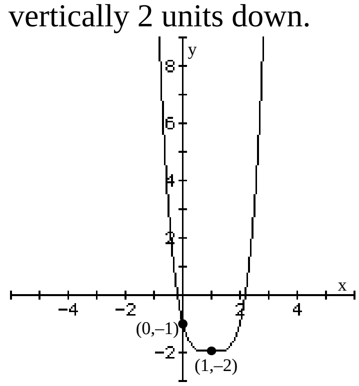
13. $f(x) = -(x-1)^4$

Using the graph of $y = x^4$, shift the graph horizontally, 1 unit right, and reflect about the x-axis.

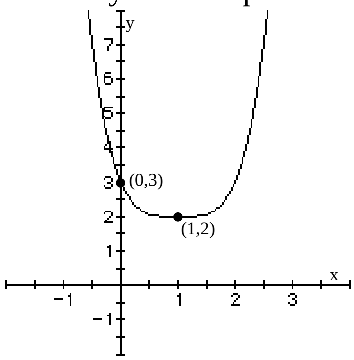


14. $f(x) = (x-1)^4 - 2$

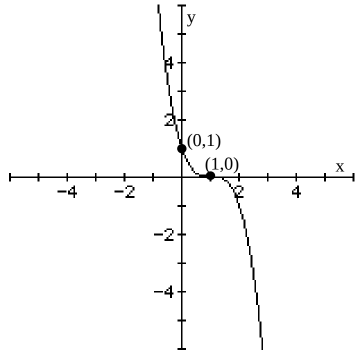
Using the graph of $y = x^4$, shift the graph horizontally, 1 unit right, and shift vertically 2 units down.



15. $f(x) = (x-1)^4 + 2$
Using the graph of $y = x^4$, shift the graph horizontally, 1 unit to the right, and shift vertically 2 units up.



16. $f(x) = (1-x)^3 = -(x-1)^3$
Using the graph of $y = x^3$, shift the graph horizontally, 1 unit to the right, and reflect on the x-axis.



17. $f(x) = 3x^2 - 6x + 4$
$a = 3$, $b = -6$, $c = 4$. Since $a = 3 > 0$, the graph opens up, so the vertex is a minimum point. The minimum occurs at $x = \frac{-b}{2a} = \frac{-(-6)}{2(3)} = \frac{6}{6} = 1$. The minimum value is
$f\left(\frac{-b}{2a}\right) = f(1) = 3(1)^2 - 6(1) + 4 = 3 - 6 + 4 = 1$.

18. $f(x) = 2x^2 + 8x + 5$
$a = 2$, $b = 8$, $c = 5$. Since $a = 2 > 0$, the graph opens up, so the vertex is a minimum point. The minimum occurs at $x = \frac{-b}{2a} = \frac{-8}{2(2)} = \frac{-8}{4} = -2$. The minimum value is
$f\left(\frac{-b}{2a}\right) = f(-2) = 2(-2)^2 + 8(-2) + 5 = 8 - 16 + 5 = -3$

19. $f(x) = -x^2 + 8x - 4$
$a = -1$, $b = 8$, $c = -4$. Since $a = -1 < 0$, the graph opens down, so the vertex is a maximum point. The maximum occurs at $x = \frac{-b}{2a} = \frac{-8}{2(-1)} = \frac{-8}{-2} = 4$. The maximum value is $f\left(\frac{-b}{2a}\right) = f(4) = -(4)^2 + 8(4) - 4 = -16 + 32 - 4 = 12$.

20. $f(x) = -x^2 - 10x - 3$
$a = -1$, $b = -10$, $c = -3$. Since $a = -1 < 0$, the graph opens down, so the vertex is a maximum point. The maximum occurs at $x = \frac{-b}{2a} = \frac{-(-10)}{2(-1)} = \frac{10}{-2} = -5$. The maximum value is $f\left(\frac{-b}{2a}\right) = f(-5) = -(-5)^2 - 10(-5) - 3 = -25 + 50 - 3 = 22$.

21. $f(x) = -3x^2 + 12x + 4$
$a = -3$, $b = 12$, $c = 4$. Since $a = -3 < 0$, the graph opens down, so the vertex is a maximum point. The maximum occurs at $x = \frac{-b}{2a} = \frac{-12}{2(-3)} = \frac{-12}{-6} = 2$. The maximum value is $f\left(\frac{-b}{2a}\right) = f(2) = -3(2)^2 + 12(2) + 4 = -12 + 24 + 4 = 16$.

22. $f(x) = -2x^2 + 4$

$a = -2, b = 0, c = 4$. Since $a = -2 < 0$, the graph opens down, so the vertex is a maximum point. The maximum occurs at $x = \frac{-b}{2a} = \frac{-0}{2(-2)} = 0$. The maximum value is

$f\left(\frac{-b}{2a}\right) = f(0) = -2(0)^2 + 4 = 4$

23. $f(x) = x(x+2)(x+4)$

(a) x-intercepts: –4, –2, 0;
y-intercept: 0

(b) crosses x axis at x = –4, –2, 0

(c) $y = x^3$ (d) 2

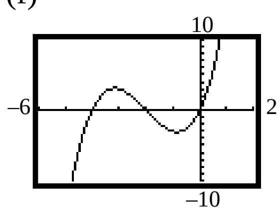
(e)

	$x < -4$	$-4 < x < -2$	$-2 < x < 0$	$x > 0$
f	-	+	-	+
Above or below x-axis	below	above	below	above

Graph of f is above the x-axis for $(-4,-2)$ $(0,\infty)$

Graph of f is below the x-axis for $(-\infty,-4)$ $(-2,0)$

(f)



24. $f(x) = x(x-2)(x-4)$

(a) x-intercepts: 4, 2, 0;
y-intercept: 0

(b) crosses x axis at x = 4, 2, 0

(c) $y = x^3$ (d) 2

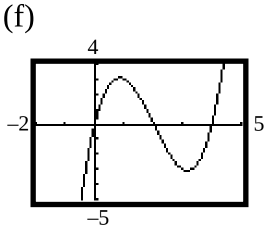
(e)

	$x < 0$	$0 < x < 2$	$2 < x < 4$	$x > 4$
f	-	+	-	+
Above or below x-axis	below	above	below	above

Graph of f is above the x-axis for $(0,2)$ $(4,\infty)$

Graph of f is below the x-axis for $(-\infty,0)$ $(2,4)$

(f)



25. $f(x) = (x-2)^2(x+4)$

(a) x-intercepts: - 4, 2;
y-intercept: 16

(b) crosses x axis at x = - 4 and touches the x axis at x = 2

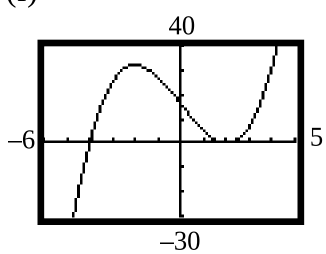
(c) $y = x^3$ (d) 2

(e)

	$x <$ -4	-4 $< x <$ 2	2 $< x$
f	-	+	+
Above or below x-axis	below	above	above

Graph of f is above the x-axis for $(-4,2)$ $(2,\infty)$
Graph of f is below the x-axis for $(-\infty,-4)$

(f)



26. $f(x) = (x-2)(x+4)^2$

(a) x-intercepts: - 4, 2;
y-intercept: - 32

(b) crosses x axis at x = 2 and touches the x axis at x = - 4

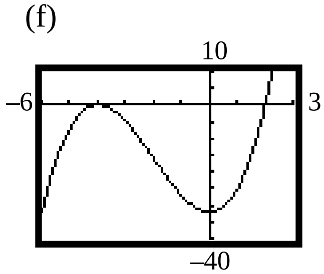
(c) $y = x^3$ (d) 2

(e)

	$x <$ - 4	-4 $< x <$ 2	2 $< x$
f	-	-	+
Above or below x-axis	below	below	above

Graph of f is above the x-axis for $(2,\infty)$
Graph of f is below the x-axis for $(-\infty,-4)$ $(-4,2)$

(f)



27. $f(x) = -2x^3 + 4x^2 = -2x^2(x-2)$

(a) x-intercepts: 0, 2;
y-intercept: 0

(b) crosses x axis at x = 2 and touches the x axis at x = 0

(c) $y = -2x^3$ (d) 2

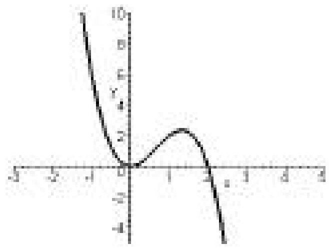
(e)

	$x < 0$	$0 < x < 2$	$2 < x$
f	+	+	-
Above or below x-axis	above	above	below

Graph of f is above the x-axis for $(-\infty,0)$ $(0,2)$

Graph of f is below the x-axis for $(2,\infty)$

(f)



28. $f(x) = -4x^3 + 4x = -4x(x^2-1) = -4x(x+1)(x-1)$

(a) x-intercepts: 0, - 1, 1;
y-intercept: 0

(b) crosses x axis at x = 0, - 1, 1

(c) $y = -4x^3$ (d) 2

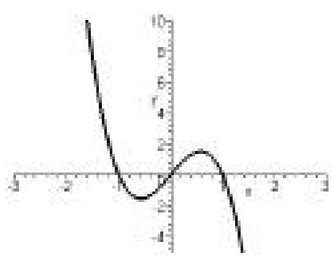
(e)

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
f	+	-	+	-
Above or below x-axis	above	below	above	below

Graph of f is above the x-axis for $(-\infty,-1)$ $(0,1)$

Graph of f is below the x-axis for $(-1,0)$ $(1,\infty)$

(f)



29. $f(x) = (x-1)^2(x+3)(x+1)$

(a) x-intercepts: - 3, - 1, 1;
y-intercept: 3

(b) crosses x axis at x = - 3, - 1 and touches x-axis at x = 1

(c) $y = x^4$(d) 3

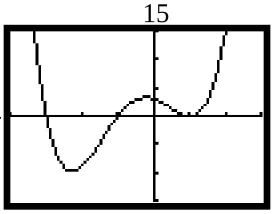
(e)

	$x < -3$	$-3 < x < -1$	$-1 < x < 1$	$1 < x$
f	+	-	+	+
Above or below x-axis	above	below	above	above

Graph of f is above the x-axis for $(-\infty,-3)$ $(-1,1)$ $(1,\infty)$

Graph of f is below the x-axis for $(-3,-1)$

(f)



30. $f(x) = (x-4)(x+2)^2(x-2)$

(a) x-intercepts: - 2, 2, 4;
y-intercept: 32

(b) crosses x axis at x = 2, 4 and touches x-axis at x = -2

(c) $y = x^4$ (d) 3

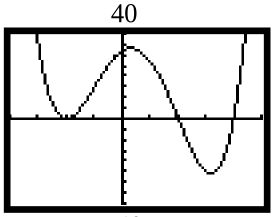
(e)

	$x < -2$	$-2 < x < 2$	$2 < x < 4$	$4 < x$
f	+	+	-	+
Above or below x-axis	above	above	below	above

Graph of f is above the x-axis for $(-\infty,-2)$ $(-2,2)$ $(4,\infty)$

Graph of f is below the x-axis for $(2,4)$

(f)



31. $R(x) = \dfrac{2x-6}{x}$ $p(x) = 2x-6$; $q(x) = x$; $n = 1$; $m = 1$

Step 1: Domain: $\{x \mid x \neq 0\}$

Step 2: (a) The x-intercept is the zero of $p(x)$: 3

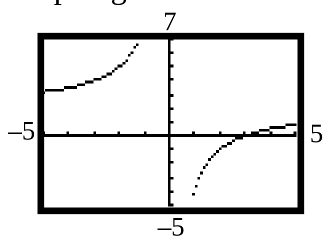
(b) There is no y-intercept because 0 is not in the domain.

Step 3: $R(-x) = \dfrac{2(-x)-6}{-x} = \dfrac{-2x-6}{-x} = \dfrac{2x+6}{x}$; this is neither $R(x)$ nor $-R(x)$, so there is no symmetry.

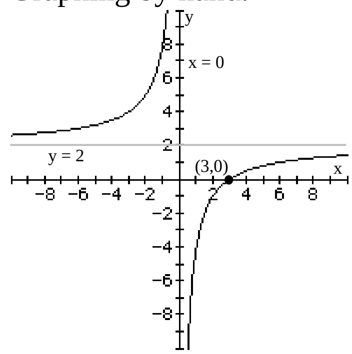
Step 4: The vertical asymptote is the zero of $q(x)$: $x = 0$

Step 5: Since $n = m$, the line $y = 2$ is the horizontal asymptote.
$R(x)$ does not intersect $y = 2$.

Step 6: Graphing:



Step 7: Graphing by hand:



32. $R(x) = \dfrac{4-x}{x}$ $p(x) = 4 - x;\ q(x) = x;\ n = 1;\ m = 1$

Step 1: Domain: $\{x \mid x \neq 0\}$

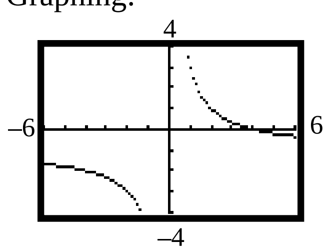
Step 2: (a) The x-intercept is the zero of $p(x)$: 4
(b) There is no y-intercept because 0 is not in the domain.

Step 3: $R(-x) = \dfrac{4-(-x)}{-x} = \dfrac{4+x}{-x}$; this is neither $R(x)$ nor $-R(x)$, so there is no symmetry.

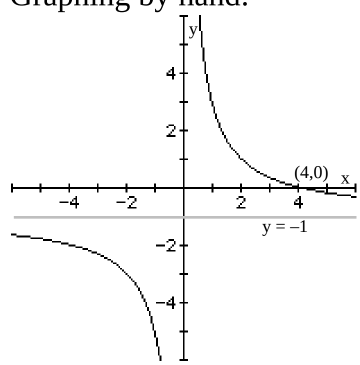
Step 4: The vertical asymptote is the zero of $q(x)$: $x = 0$

Step 5: Since $n = m$, the line $y = -1$ is the horizontal asymptote.
$R(x)$ does not intersect $y = -1$.

Step 6: Graphing:



Step 7: Graphing by hand:



33. $H(x) = \dfrac{x+2}{x(x-2)}$ $p(x) = x + 2;\ q(x) = x(x-2) = x^2 - 2x;\ n = 1;\ m = 2$

Step 1: Domain: $\{x \mid x \neq 0,\ x \neq 2\}$

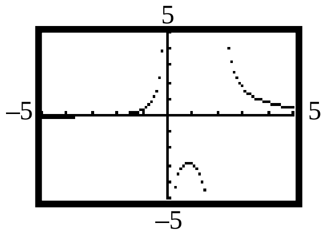
Step 2: (a) The x-intercept is the zero of $p(x)$: –2
(b) There is no y-intercept because 0 is not in the domain.

Step 3: $H(-x) = \dfrac{-x+2}{-x(-x-2)} = \dfrac{-x+2}{x^2+2x}$; this is neither $H(x)$ nor $-H(x)$, so there is no symmetry.

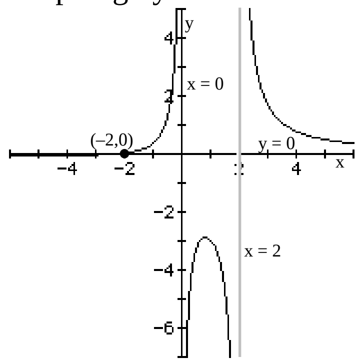
Step 4: The vertical asymptotes are the zeros of $q(x)$: $x = 0$ and $x = 2$

Step 5: Since $n < m$, the line $y = 0$ is the horizontal asymptote.
$R(x)$ intersects $y = 0$ at (–2, 0).

Step 6: Graphing:



Step 7: Graphing by hand:



34. $H(x) = \dfrac{x}{x^2 - 1} = \dfrac{x}{(x-1)(x+1)}$ $p(x) = x$; $q(x) = x^2 - 1$; $n = 1$; $m = 2$

Step 1: Domain: $\{x \mid x \neq -1, x \neq 1\}$

Step 2: (a) The x-intercept is the zero of $p(x)$: 0

(b) The y-intercept is 0.

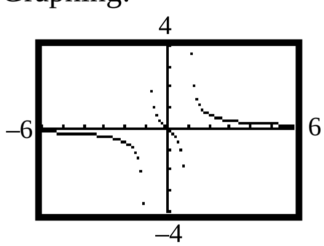
Step 3: $H(-x) = \dfrac{-x}{(-x)^2 - 1} = \dfrac{-x}{x^2 - 1} = -H(x)$; the graph is symmetric to the origin.

Step 4: The vertical asymptotes are the zeros of $q(x)$: $x = -1$ and $x = 1$

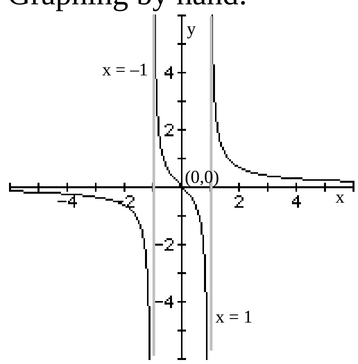
Step 5: Since $n < m$, the line $y = 0$ is the horizontal asymptote.

$R(x)$ intersects $y = 0$ at (0, 0).

Step 6: Graphing:



Step 7: Graphing by hand:



35. $R(x) = \dfrac{x^2 + x - 6}{x^2 - x - 6} = \dfrac{(x+3)(x-2)}{(x-3)(x+2)}$ $p(x) = x^2 + x - 6$; $q(x) = x^2 - x - 6$;

$n = 2$; $m = 2$

Step 1: Domain: $\{x \mid x \neq -2, x \neq 3\}$

Step 2: (a) The x-intercepts are the zeros of $p(x)$: –3 and 2

(b) The y-intercept is $R(0) = \dfrac{0^2 + 0 - 6}{0^2 - 0 - 6} = \dfrac{-6}{-6} = 1$.

Step 3: $R(-x) = \dfrac{(-x)^2 + (-x) - 6}{(-x)^2 - (-x) - 6} = \dfrac{x^2 - x - 6}{x^2 + x - 6}$; this is neither $R(x)$ nor $-R(x)$, so there is no symmetry.

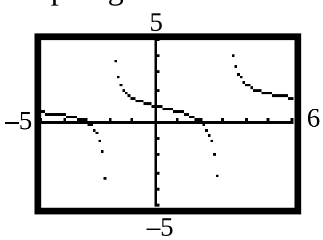
Step 4: The vertical asymptotes are the zeros of $q(x)$: $x = -2$ and $x = 3$

Step 5: Since $n = m$, the line $y = 1$ is the horizontal asymptote.

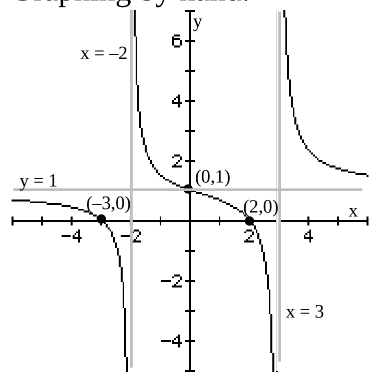
$R(x)$ intersects $y = 1$ at (0, 1), since:

$$\frac{x^2+x-6}{x^2-x-6}=1$$
$$x^2+x-6=x^2-x-6$$
$$2x=0$$
$$x=0$$

Step 6: Graphing:



Step 7: Graphing by hand:



36. $R(x)=\dfrac{x^2-6x+9}{x^2}=\dfrac{(x-3)^2}{x^2}$ $p(x)=x^2-6x+9;\ q(x)=x^2;\ \ n=2;\ m=2$

Step 1: Domain: $\{x \mid x \neq 0\}$

Step 2: (a) The x-intercept is the zero of $p(x)$: 3

(b) There is no y-intercept because 0 is not in the domain.

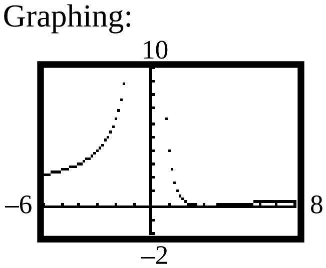
Step 3: $R(-x)=\dfrac{(-x)^2-6(-x)+9}{(-x)^2}=\dfrac{x^2+6x+9}{x^2}$; this is neither $R(x)$ nor $-R(x)$, so there is no symmetry.

Step 4: The vertical asymptote is the zero of $q(x)$: $x=0$

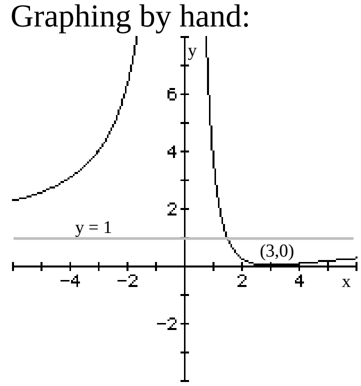
Step 5: Since $n=m$, the line $y=1$ is the horizontal asymptote.
$R(x)$ intersects $y=1$ at $\left(\frac{3}{2},1\right)$, since:

$$\frac{x^2-6x+9}{x^2}=1 \rightarrow x^2-6x+9=x^2 \rightarrow -6x=-9 \rightarrow x=\frac{3}{2}$$

Step 6: Graphing:



Step 7: Graphing by hand:



37. $F(x)=\dfrac{x^3}{x^2-4}$ $p(x)=x^3;\ q(x)=x^2-4;\ n=3;\ m=2$

Step 1: Domain: $\{x \mid x \neq -2,\ x \neq 2\}$

Step 2: (a) The x-intercept is the zero of $p(x)$: 0.

(b) The y-intercept is $F(0) = \dfrac{0^3}{0^2-4} = \dfrac{0}{-4} = 0$.

Step 3: $F(-x) = \dfrac{(-x)^3}{(-x)^2-4} = \dfrac{-x^3}{x^2-4} = -F(x)$; $F(x)$ is symmetric to the origin.

Step 4: The vertical asymptotes are the zeros of $q(x)$: $x=-2$ and $x=2$

Step 5: Since $n = m+1$, there is an oblique asymptote. Dividing:

$$\begin{array}{r|l} & x \\ \hline x^2-4 & x^3+0x^2+0x+0 \\ & \underline{x^3 \qquad\;\; -4x} \\ & \qquad\qquad 4x \end{array} \qquad F(x) = x + \frac{4x}{x^2-4}$$

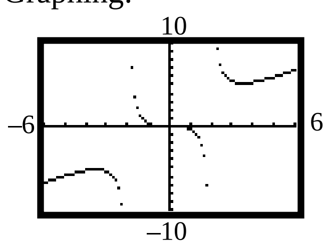
The oblique asymptote is $y = x$.

Solve to find intersection points:

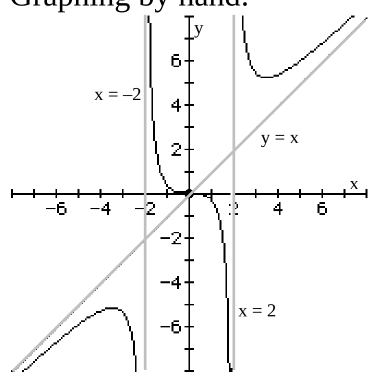
$$\frac{x^3}{x^2-4} = x \to x^3 = x^3 - 4x \to 0 = -4x \to x = 0$$

The oblique asymptote intersects $F(x)$ at (0, 0).

Step 6: Graphing:



Step 7: Graphing by hand:



38. $F(x) = \dfrac{3x^3}{(x-1)^2}$ $\quad p(x) = 3x^3$; $q(x) = (x-1)^2$; $n=3$; $m=2$

Step 1: Domain: $\{x \mid x \neq 1\}$

Step 2: (a) The x-intercept is the zero of $p(x)$: 0.

(b) The y-intercept is $F(0) = \dfrac{3\ 0^3}{(0-1)^2} = \dfrac{0}{1} = 0$.

Step 3: $F(-x) = \dfrac{3(-x)^3}{(-x-1)^2} = \dfrac{-3x^3}{(x+1)^2}$; this is neither $F(x)$ nor $-F(x)$, so there is no symmetry.

Step 4: The vertical asymptote is the zero of $q(x)$: $x = 1$

Step 5: Since $n = m+1$, there is an oblique asymptote. Dividing:

$$\begin{array}{r} 3x+6 \\ x^2-2x+1\overline{)\,3x^3+0x^2+0x+0} \\ \underline{3x^3-6x^2+3x} \\ 6x^2-3x \\ \underline{6x^2-12x+6} \\ 9x-6 \end{array}$$

$$F(x)=3x+6+\frac{9x-6}{x^2-2x+1}$$

The oblique asymptote is $y=3x+6$.

Solve to find intersection points:

$$\frac{3x^3}{x^2-2x+1}=3x+6$$

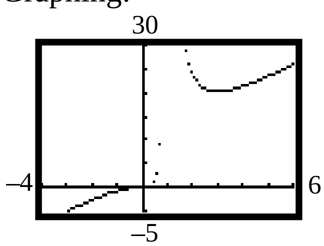
$$3x^3=3x^3-9x+6$$

$$9x=6$$

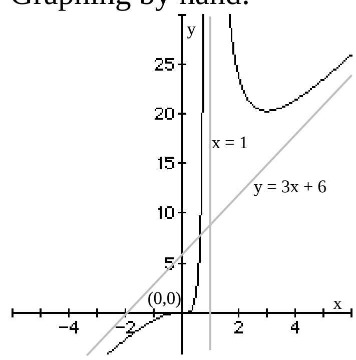
$$x=\frac{2}{3}$$

The oblique asymptote intersects $F(x)$ at $\left(\frac{2}{3},8\right)$.

Step 6: Graphing:



Step 7: Graphing by hand:



39. $R(x)=\frac{2x^4}{(x-1)^2}$ $p(x)=2x^4$; $q(x)=(x-1)^2$; $n=4$; $m=2$

Step 1: Domain: $\{x \mid x \neq 1\}$

Step 2: (a) The x-intercept is the zero of $p(x)$: 0

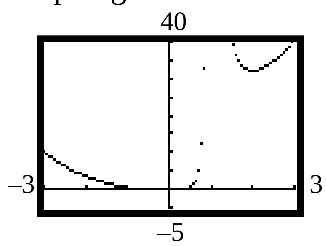
(b) The y-intercept is $R(0)=\frac{2(0)^4}{(0-1)^2}=\frac{0}{1}=0$.

Step 3: $R(-x)=\frac{2(-x)^4}{(-x-1)^2}=\frac{2x^4}{(x+1)^2}$; this is neither $R(x)$ nor $-R(x)$, so there is no symmetry.

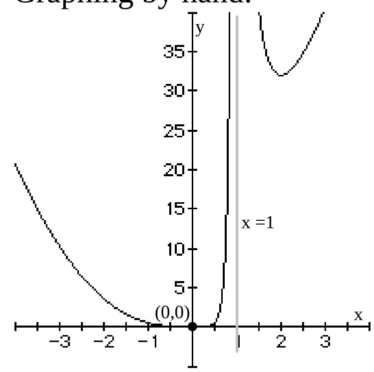
Step 4: The vertical asymptote is the zero of $q(x)$: $x=1$

Step 5: Since $n>m+1$, there is no horizontal asymptote and no oblique asymptote.

Step 6: Graphing:



Step 7: Graphing by hand:



40. $R(x)=\dfrac{x^4}{x^2-9}=\dfrac{x^4}{(x+3)(x-3)}$ $\quad p(x)=x^4;\ q(x)=x^2-9;\ n=4;\ m=2$

Step 1: Domain: $\{x \mid x \neq -3,\ x \neq 3\}$

Step 2: (a) The x-intercept is the zero of $p(x)$: 0

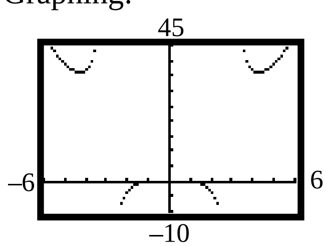
(b) The y-intercept is $R(0)=\dfrac{(0)^4}{0^2-9}=\dfrac{0}{-9}=0$.

Step 3: $R(-x)=\dfrac{(-x)^4}{(-x)^2-9}=\dfrac{x^4}{x^2-9}=R(x)$; $R(x)$ is symmetric to the y-axis.

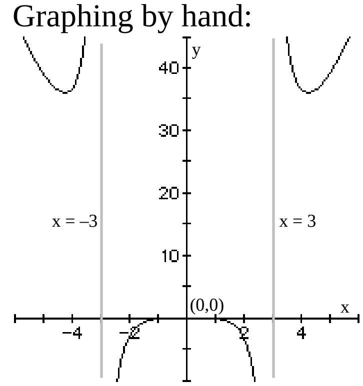
Step 4: The vertical asymptotes are the zeros of $q(x)$: $x=-3$ and $x=3$

Step 5: Since $n>m+1$, there is no horizontal asymptote and no oblique asymptote.

Step 6: Graphing:



Step 7: Graphing by hand:



41. $G(x)=\dfrac{x^2-4}{x^2-x-2}=\dfrac{(x+2)(x-2)}{(x-2)(x+1)}=\dfrac{x+2}{x+1}$ $\quad p(x)=x^2-4;\ q(x)=x^2-x-2;$

$n=2;\ m=2$

Step 1: Domain: $\{x \mid x \neq -1,\ x \neq 2\}$

Step 2: (a) The x-intercept is the zero of $p(x)$: –2 (2 is not a zero because reduced form must be used to find the zeros.)

(b) The y-intercept is $G(0)=\dfrac{0^2-4}{0^2-0-2}=\dfrac{-4}{-2}=2$.

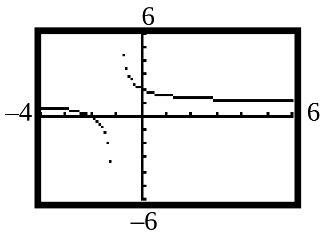
Step 3: $G(-x)=\dfrac{(-x)^2-4}{(-x)^2-(-x)-2}=\dfrac{x^2-4}{x^2+x-2}$; this is neither $G(x)$ nor $-G(x)$, so there is no symmetry.

Step 4: The vertical asymptote is the zero of $q(x)$: $x = -1$ ($x = 2$ is not a vertical asymptote because reduced form must be used to find the them.)

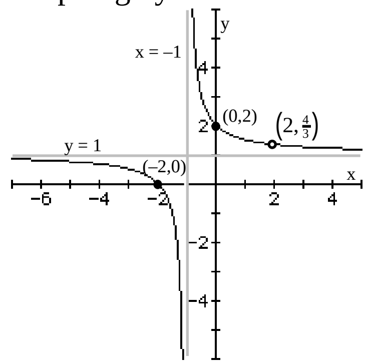
Step 5: Since $n = m$, the line $y = 1$ is the horizontal asymptote.
$G(x)$ does not intersect $y = 1$ because $G(x)$ is not defined at $x = 2$.

$$\frac{x^2-4}{x^2-x-2} = 1 \rightarrow x^2 - 4 = x^2 - x - 2 \rightarrow -2 = -x \rightarrow x = 2$$

Step 6: Graphing:



Step 7: Graphing by hand:



42. $F(x) = \dfrac{(x-1)^2}{x^2-1} = \dfrac{(x-1)^2}{(x-1)(x+1)} = \dfrac{x-1}{x+1}$ $\quad p(x) = (x-1)^2$; $q(x) = x^2 - 1$;
$n = 2$; $m = 2$

Step 1: Domain: $\{x \mid x \neq -1, \ x \neq 1\}$

Step 2: (a) There is no x-intercept since 1 is not in the domain.

(b) The y-intercept is $F(0) = \dfrac{(0-1)^2}{0^2-1} = \dfrac{1}{-1} = -1$.

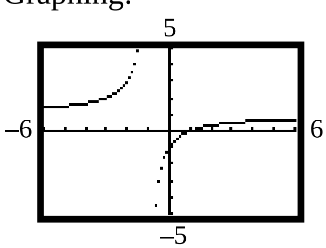
Step 3: $F(-x) = \dfrac{(-x-1)^2}{(-x)^2-1} = \dfrac{(x+1)^2}{x^2-1}$; this is neither $F(x)$ nor $-F(x)$, so there is no symmetry.

Step 4: The vertical asymptote is the zero of $q(x)$: $x = -1$ ($x = 1$ is not a vertical asymptote because reduced form must be used to find them.)

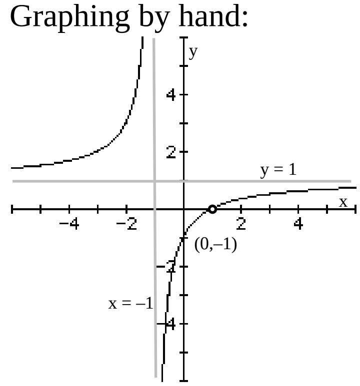
Step 5: Since $n = m$, the line $y = 1$ is the horizontal asymptote.
$F(x)$ does not intersect $y = 1$ because $F(x)$ is not defined at $x = 1$.

$$\frac{x^2-2x+1}{x^2-1} = 1 \rightarrow x^2 - 2x + 1 = x^2 - 1 \rightarrow -2x = -2 \rightarrow x = 1$$

Step 6: Graphing:



Step 7: Graphing by hand:



43. $2x^2+5x-12<0$ $f(x)=2x^2+5x-12$

$(x+4)(2x-3)<0$

$x=-4,\ x=\frac{3}{2}$ are the zeros.

Interval	Test Number	$f(x)$	Positive/Negative
$-\infty<x<-4$	–5	13	Positive
$-4<x<3/2$	0	–12	Negative
$3/2<x<\infty$	2	6	Positive

The solution set is $\left\{x \middle| -4<x<\frac{3}{2}\right\}$.

44. $3x^2-2x-1\ge 0$ $f(x)=3x^2-2x-1$

$(3x+1)(x-1)\ge 0$

$x=-\frac{1}{3},\ x=1$ are the zeros.

Interval	Test Number	$f(x)$	Positive/Negative
$-\infty<x<-1/3$	–1	4	Positive
$-1/3<x<1$	0	–1	Negative
$1<x<\infty$	2	7	Positive

The solution set is $\left\{x \middle| x\le -\frac{1}{3} \text{ or } x\ge 1\right\}$.

45. $\frac{6}{x+3}\ge 1$ $f(x)=\frac{6}{x+3}-1$

$$\frac{6}{x+3}-1\ge 0 \rightarrow \frac{6-1(x+3)}{x+3}\ge 0 \rightarrow \frac{-x+3}{x+3}\ge 0$$

The zeros and values where the expression is undefined are $x=-3$, and $x=3$.

Interval	Test Number	$f(x)$	Positive/Negative
$-\infty<x<-3$	–4	–7	Negative
$-3<x<3$	0	1	Positive
$3<x<\infty$	4	–1/7	Negative

The solution set is $\left\{x \middle| -3<x\le 3\right\}$.

46. $\frac{-2}{1-3x}<1$ $f(x)=\frac{-2}{1-3x}-1$

$$\frac{-2}{1-3x}-1<0$$

$$\frac{-2-1+3x}{1-3x}<0 \rightarrow \frac{3x-3}{1-3x}<0$$

The zeros and values where the expression is undefined are $x=\frac{1}{3}$, and $x=1$.

Interval	Test Number	$f(x)$	Positive/Negative
$-\infty<x<1/3$	0	–3	Negative
$1/3<x<1$	0.5	3	Positive
$1<x<\infty$	2	–3/5	Negative

The solution set is $\left\{x \middle| x<\frac{1}{3} \text{ or } x>1\right\}$.

47. $$\frac{2x-6}{1-x}<2 \qquad f(x)=\frac{2x-6}{1-x}-2$$

$$\frac{2x-6}{1-x}-2<0$$

$$\frac{2x-6-2(1-x)}{1-x}<0$$

$$\frac{4x-8}{1-x}<0$$

The zeros and values where the expression is undefined are $x=1$, and $x=2$.

Interval	Test Number	$f(x)$	Positive/Negative
$-\infty<x<1$	0	–8	Negative
$1<x<2$	1.5	4	Positive
$2<x<\infty$	3	–2	Negative

The solution set is $\{x \mid x<1 \text{ or } x>2\}$.

48. $$\frac{3-2x}{2x+5}\geq 2 \qquad f(x)=\frac{3-2x}{2x+5}-2$$

$$\frac{3-2x}{2x+5}-2\geq 0 \rightarrow \frac{3-2x-4x-10}{2x+5}\geq 0 \rightarrow \frac{-6x-7}{2x+5}\geq 0$$

The zeros and values where the expression is undefined are $x=-\frac{7}{6}$, and $x=-\frac{5}{2}$.

Interval	Test Number	$f(x)$	Positive/Negative
$-\infty<x<-5/2$	–3	–11	Negative
$-5/2<x<-7/6$	–2	5	Positive
$-7/6<x<\infty$	0	–7/5	Negative

The solution set is $\{x \mid -\frac{5}{2}<x\leq-\frac{7}{6}\}$.

49. $$\frac{(x-2)(x-1)}{x-3}>0 \qquad f(x)=\frac{(x-2)(x-1)}{x-3}$$

The zeros and values where the expression is undefined are $x=1$, $x=2$, and $x=3$.

Interval	Test Number	$f(x)$	Positive/Negative
$-\infty<x<1$	0	–2/3	Negative
$1<x<2$	1.5	1/6	Positive
$2<x<3$	2.5	–3/2	Negative
$3<x<\infty$	4	6	Positive

The solution set is $\{x \mid 1<x<2 \text{ or } x>3\}$.

50. $$\frac{x+1}{x(x-5)}\leq 0 \qquad f(x)=\frac{x+1}{x(x-5)}$$

The zeros and values where the expression is undefined are $x=-1$, $x=0$, and $x=5$.

Interval	Test Number	$f(x)$	Positive/Negative
$-\infty<x<-1$	–2	–1/14	Negative
$-1<x<0$	–0.5	2/11	Positive
$0<x<5$	1	–1/2	Negative
$5<x<\infty$	6	7/6	Positive

The solution set is $\{x \mid x\leq-1 \text{ or } 0<x<5\}$.

51. $\dfrac{x^2-8x+12}{x^2-16}>0$ $\qquad f(x)=\dfrac{x^2-8x+12}{x^2-16}$

$\dfrac{(x-2)(x-6)}{(x+4)(x-4)}>0$

The zeros and values where the expression is undefined are $x=-4, x=2, x=4, x=6$.

Interval	Test Number	$f(x)$	Positive/Negative
$-\infty<x<-4$	–5	77/9	Positive
$-4<x<2$	0	–3/4	Negative
$2<x<4$	3	3/7	Positive
$4<x<6$	5	–1/3	Negative
$6<x<\infty$	7	5/33	Positive

The solution set is $\{x \mid x<-4,\ 2<x<4,\ x>6\}$.

52 $\dfrac{x\left(x^2+x-2\right)}{x^2+9x+20}\le 0$ $\qquad f(x)=\dfrac{x\left(x^2+x-2\right)}{x^2+9x+20}$

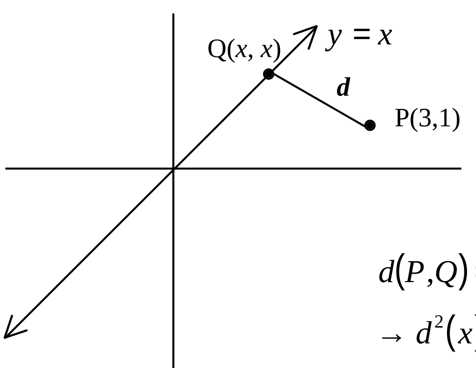
$\dfrac{x(x+2)(x-1)}{(x+5)(x+4)}\le 0$

The zeros and values where the expression is undefined are $x=-5, x=-4, x=-2, x=0, x=1$.

Interval	Test Number	$f(x)$	Positive/Negative
$-\infty<x<-5$	–6	–84	Negative
$-5<x<-4$	–4.5	247.5	Positive
$-4<x<-2$	–3	–6	Negative
$-2<x<0$	–1	1/6	Positive
$0<x<1$	0.5	–0.025	Negative
$1<x<\infty$	2	0.19	Positive

The solution set is $\{x \mid x<-5,\ -4<x\le -2, \text{ or } 0\le x\le 1\}$

53.



$$d(P,Q)=\sqrt{(x-3)^2+(x-1)^2}$$

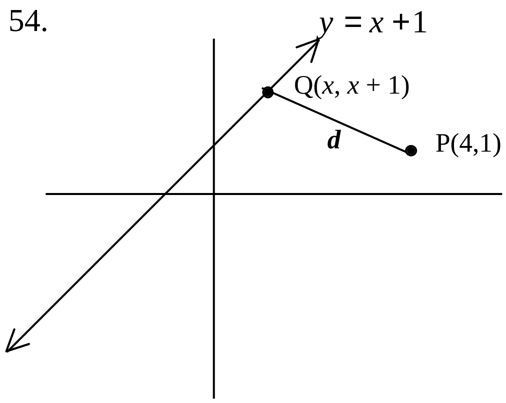
$$\to d^2(x)=(x-3)^2+(x-1)^2=x^2-6x+9+x^2-2x+1$$

$$d^2(x)=2x^2-8x+10$$

Since $d^2(x)=2x^2-8x+10$ is a quadratic function with $a=2>0$, the vertex corresponds to the minimum value for the function.

The vertex occurs at $x=-\dfrac{b}{2a}=-\dfrac{-8}{2(2)}=4$. Therefore the point Q on the line $y=x$ will be closest to the point $P=(3,1)$ when $Q=(4,4)$.

54.



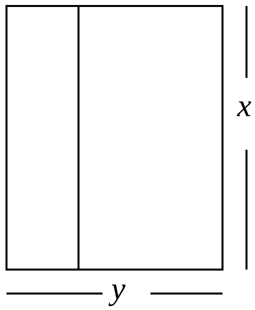
$$d(P,Q) = \sqrt{(x-4)^2 + (x+1-1)^2}$$
$$\to d^2(x) = (x-4)^2 + x^2$$
$$= x^2 - 8x + 16 + x^2$$
$$d^2(x) = 2x^2 - 8x + 16$$

Since $d^2(x) = 2x^2 - 8x + 16$ is a quadratic function with $a = 2 > 0$, the vertex corresponds to the minimum value for the function.
The vertex occurs at $x = -\frac{b}{2a} = -\frac{-8}{2(2)} = 4$. Therefore the point Q on the line $y = x + 1$ will be closest to the point $P = (4,1)$ when $Q = (4,5)$.

55. Since there are 200 feet of border, we know that $2x + 2y = 200$.
The area is to be maximized, so $A = x \quad y$.
Solving the perimeter formula for y:
$$2x + 2y = 200$$
$$2y = 200 - 2x$$
$$y = 100 - x$$
The area function is: $A(x) = x(100 - x) = -x^2 + 100x$
The maximum value occurs at the vertex:
$$x = \frac{-b}{2a} = \frac{-100}{2(-1)} = \frac{-100}{-2} = 50$$
The pond should be 50 feet by 50 feet for maximum area.

56. Let x represent the length and y represent the width of the rectangle.
$2x + 2y = 20 \quad \to \quad y = 10 - x$.
$x \quad y = 16 \quad \to \quad x(10 - x) = 16$.
Solving the area equation:
$$10x - x^2 = 16$$
$$x^2 - 10x + 16 = 0$$
$$(x - 8)(x - 2) = 0$$
$$x = 8 \text{ or } x = 2$$
The length and width of the rectangle are 8 feet by 2 feet.

57. Consider the diagram



Total amount of fence = $3x+2y=10000$

$$y=\frac{10000-3x}{2}=5000-\frac{3}{2}x$$

Total enclosed area = $(x)(y)=(x)\left(5000-\frac{3}{2}x\right)$

$$A(x)=5000x-\frac{3}{2}x^2=-\frac{3}{2}x^2+5000x$$ is a quadratic function with $a=-\frac{3}{2}<0$.

So the vertex corresponds to the maximum value for this function.
The vertex occurs when

$x=-\frac{b}{2a}=-\frac{5000}{2\left(-\frac{3}{2}\right)}=\frac{5000}{3}$ → the maximum area is $A\left(\frac{5000}{3}\right)=-\frac{3}{2}\left(\frac{5000}{3}\right)^2+5000\left(\frac{5000}{3}\right)$

$$=-\frac{3}{2}\left(\frac{25000000}{9}\right)+\frac{25000000}{3}=-\frac{12500000}{3}+\frac{25000000}{3}$$

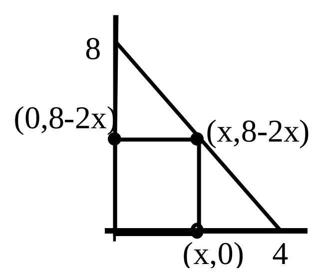
$$=\frac{12500000}{3}\approx 4166666.67 \text{ square meters}$$

58. The area function is: $A(x)=x(8-2x)=-2x^2+8x$
The maximum value occurs at the vertex:

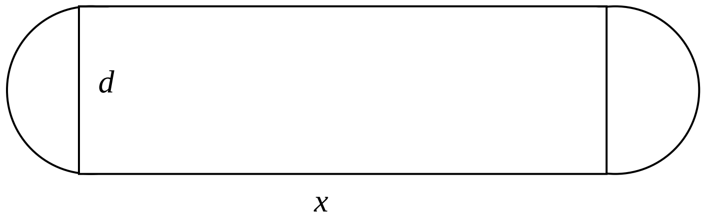
$$x=\frac{-b}{2a}=\frac{-8}{2(-2)}=\frac{-8}{-4}=2$$

The maximum area is:

$$A(2)=-2(2)^2+8(2)=-8+16=8 \text{ square units.}$$



59. Consider the diagram



$d =$ diameter of the semicircles $=$ width of the rectangle
$x =$ length of the rectangle
outside dimension length $= 2x + 2$(circumference of a semicircle)

$= 2x +$ circumference of a circle

$= 2x + \pi d = 100$

$\rightarrow x = \frac{100 - \pi d}{2} = 50 - \frac{1}{2}\pi d$

Total enclosed area = (area of the rectangle) + 2(area of a semicircle)

= (area of the rectangle) + area of a circle

$$= (x)(d) + \pi r^2 = (x)(d) + \pi\left(\frac{d}{2}\right)^2 = \left(50 - \frac{1}{2}\pi d\right)(d) + \pi\left(\frac{d}{2}\right)^2$$

$$= 50d - \frac{1}{2}\pi d^2 + \frac{1}{4}\pi d^2 = 50d - \frac{1}{4}\pi d^2 = -\frac{1}{4}\pi d^2 + 50d$$

$A(d) = -\frac{1}{4}\pi d^2 + 50d$ is a quadratic function with $a = -\frac{1}{4}\pi < 0$. Therefore the vertex corresponds to the maximum value for the function.

The vertex occurs when $x = -\frac{b}{2a} = -\frac{50}{2\left(-\frac{1}{4}\pi\right)} = \frac{100}{\pi}$

$\rightarrow$ the maximum area is $A\left(\frac{100}{\pi}\right) = -\frac{1}{4}\pi\left(\frac{100}{\pi}\right)^2 + 50\left(\frac{100}{\pi}\right) \approx 795.78$ square meters

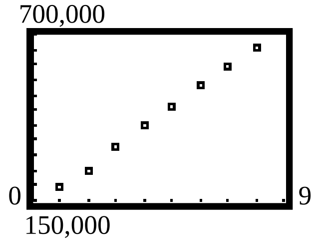
60. Locate the origin at the point directly under the highest point of the arch. Then the equation is in the form: $y = -ax^2 + k$, where $a > 0$. Since the maximum height is 10 feet, when $x = 0$, $y = k = 10$. Since the point (10, 0) is on the parabola, we can find the constant :

$$0 = -a(10)^2 + 10 \quad \rightarrow \quad a = \frac{10}{10^2} = \frac{1}{10} = 0.10$$

The equation of the parabola is:

$$y = -\frac{1}{10}x^2 + 10 \qquad \text{At } x = 8: \quad y = -\frac{1}{10}(8)^2 + 10 = -6.4 + 10 = 3.6 \text{ feet}$$

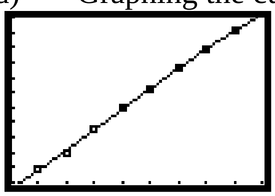
61. (a) Graphing:



$A(t) = -212t^3 + 2429t^2 + 59569t + 130003$

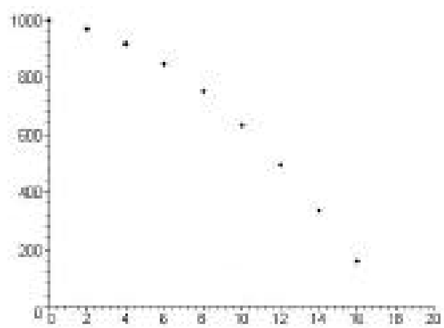
(b) $A(11) = -212(11)^3 + 2429(11)^2 + 59569(11) + 130003 = 796999$ cases

(c) and (d) Graphing the cubic function of best fit:



(e) answers will vary

62. (a) Graphing:



(b) $s(t) = -2.7t^2 - 10t + 1000$

$s(0) = -2.7(0)^2 - 10(0) + 1000 = 1000\ ft/\text{sec}$

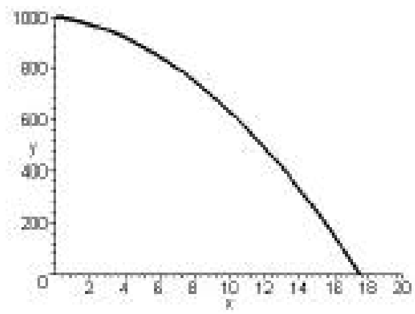
(c) $s(t) = -2.7t^2 - 10t + 1000 = 0$

$$t = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(-2.7)(1000)}}{2(-2.7)} = \frac{10 \pm \sqrt{100 + 10800}}{-5.4} = \frac{10 \pm \sqrt{10900}}{-5.4}$$

≈ 17.48 seconds or -21.19 seconds

We discard the negative time value, so the ball hits the ground after approximately 17.48 seconds.

(d) and (e) Graphing the quadratic function of best fit:



(f) $-\frac{1}{2}g = -2.7 \rightarrow g = 5.4\ ft/s^2$

63. Answers will vary, one example is $p(x) = -5(x^2+1)(x+1)^3\left(x-\frac{3}{5}\right)$

64. Answers will vary.

65. (a) The degree is even.
(b) The leading coefficient is positive.
(c) The function is even - it is symmetric to the y-axis.
(d) x^2 is a factor because the curve touches the x-axis at the origin.
(e) The minimum degree is 8.
(f) Answers will vary, 5 possibilities are:

$$p_1(x) = x^2(x+3)(x+2)(x+1)(x-1)(x-2)(x-3)$$

$$p_2(x) = x^4(x+3)(x+2)(x+1)(x-1)(x-2)(x-3)$$

$$p_3(x) = x^2(x+3)^3(x+2)(x+1)(x-1)(x-2)(x-3)$$

$$p_4(x) = 6x^2(x+3)(x+2)(x+1)(x-1)(x-2)(x-3)$$

$$p_5(x) = x^2(x+3)(x+2)^3(x+1)(x-1)(x-2)(x-3)$$