Chapter 5

The Zeros of a Polynomial Function

5.2 The Real Zeros of a Polynomial Function

1. $f(x)=4x^3-3x^2-8x+4; \quad c=2$
   $f(2)=4(2)^3-3(2)^2-8(2)+4=32-12-16+4=8\neq 0$
   Thus, 2 is not a zero of f $x-2$ is not a factor of f.

2. $f(x)=-4x^3+5x^2+8; \quad c=-3$
   $f(-3)=-4(-3)^3+5(-3)^2+8=108+45+8=161\neq 0$
   Thus, –3 is not a zero of f $x+3$ is not a factor of f.

3. $f(x)=3x^4-6x^3-5x+10; \quad c=2$
   $f(2)=3(2)^4-6(2)^3-5(2)+10=48-48-10+10=0$
   Thus, 2 is a zero of f $x-2$ is a factor of f.

4. $f(x)=4x^4-15x^2-4; \quad c=2$
   $f(2)=4(2)^4-15(2)^2-4=64-60-4=0$
   Thus, 2 is a zero of f $x-2$ is a factor of f.
   Factoring: $f(x)=\left(4x^2+1\right)\left(x^2-4\right)=\left(4x^2+1\right)(x+2)(x-2)$

5. $f(x)=3x^6+82x^3+27; \quad c=-3$
   $f(-3)=3(-3)^6+82(-3)^3+27=2187-2214-27=0$
   Thus, –3 is a zero of f $x+3$ is a factor of f. Use synthetic division to find the factors.
   $$\begin{array}{r|rrrrrrr} -3 & 3 & 0 & 0 & 82 & 0 & 0 & 27 \\ & & -9 & 27 & -81 & -3 & 9 & -27 \\ \hline & 3 & -9 & 27 & 1 & -3 & 9 & 0 \end{array}$$
   The factored form is: $f(x)=(x+3)\left(3x^5-9x^4+27x^3+x^2-3x+9\right)$.

6. $f(x)=2x^6-18x^4+x^2-9; \quad c=-3$
   $f(-3)=2(-3)^6-18(-3)^4+(-3)^2-9=1458-1458+9-9=0$
   Thus, –3 is a zero of f $x+3$ is a factor of f.
   Factoring:
   $$f(x)=2x^4\left(x^2-9\right)+\left(x^2-9\right)=\left(x^2-9\right)\left(2x^4+1\right)=(x+3)(x-3)\left(2x^4+1\right)$$

7. $f(x) = 4x^6 - 64x^4 + x^2 - 15; \quad c = -4$

Use synthetic division to determine whether -4 is a zero.

$$\begin{array}{r|rrrrrrr} -4 & 4 & 0 & -64 & 0 & 1 & 0 & -15 \\ & & -16 & 64 & 0 & 0 & -4 & 16 \\ \hline & 4 & -16 & 0 & 0 & 1 & -4 & 1 \end{array}$$

Thus, -4 is not a zero of f $x+4$ is not a factor of f..

8. $f(x) = x^6 - 16x^4 + x^2 - 16; \quad c = -4$

$f(-4) = (-4)^6 - 16(-4)^4 + (-4)^2 - 16 = 4096 - 4096 + 16 - 16 = 0$

Thus, –4 is a zero of f $x+4$ is a factor of f.

Factoring:

$$f(x) = x^4\left(x^2 - 16\right) + \left(x^2 - 16\right) = \left(x^2 - 16\right)\left(x^4 + 1\right) = (x+4)(x-4)\left(x^4 + 1\right)$$

9. $f(x) = 2x^4 - x^3 + 2x - 1; \quad c = \frac{1}{2}$

$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^3 + 2\left(\frac{1}{2}\right) - 1 = \frac{1}{8} - \frac{1}{8} + 1 - 1 = 0$

Thus, $\frac{1}{2}$ is a zero of f $x - \frac{1}{2}$ is a factor of f.

Factoring:

$$f(x) = 2x^4 - x^3 + 2x - 1 = 2x^3\left(x - \frac{1}{2}\right) + 2\left(x - \frac{1}{2}\right) = \left(x - \frac{1}{2}\right)\left(2x^3 + 2\right)$$

$$= 2\left(x - \frac{1}{2}\right)\left(x^3 + 1\right) = 2\left(x - \frac{1}{2}\right)\left(x + 1\right)\left(x^2 - x + 1\right)$$

10. $f(x) = 3x^4 + x^3 - 3x + 1; \quad c = -\frac{1}{3}$

$f\left(-\frac{1}{3}\right) = 3\left(-\frac{1}{3}\right)^4 + \left(-\frac{1}{3}\right)^3 - 3\left(-\frac{1}{3}\right) + 1 = \frac{1}{27} - \frac{1}{27} + 1 + 1 = 2 \neq 0$

Thus, $-\frac{1}{3}$ is not a zero of f $x + \frac{1}{3}$ is not a factor of f.

11. $f(x) = -4x^7 + x^3 - x^2 + 2$

The maximum number of zeros is the degree of the polynomial which is 7.

Examining $f(x) = -4x^7 + x^3 - x^2 + 2$, there are 3 variations in sign; thus, there are 3 or 1 positive real zeros.

Examining $f(-x) = -4(-x)^7 + (-x)^3 - (-x)^2 + 2 = 4x^7 - x^3 - x^2 + 2$, there are 2 variations in sign; thus, there are 2 or 0 negative real zeros.

12. $f(x) = 5x^4 + 2x^2 - 6x - 5$

The maximum number of zeros is the degree of the polynomial which is 4.

Examining $f(x) = 5x^4 + 2x^2 - 6x - 5$, there is 1 variation in sign; thus, there is 1 positive real zero.

Examining $f(-x) = 5(-x)^4 + 2(-x)^2 - 6(-x) - 5 = 5x^4 + 2x^2 + 6x - 5$, there is 1 variation in sign; thus, there is 1 negative real zero.

13. $f(x) = 2x^6 - 3x^2 - x + 1$
The maximum number of zeros is the degree of the polynomial which is 6.
Examining $f(x) = 2x^6 - 3x^2 - x + 1$, there are 2 variations in sign; thus, there are 2 or 0 positive real zeros.
Examining $f(-x) = 2(-x)^6 - 3(-x)^2 - (-x) + 1 = 2x^6 - 3x^2 + x + 1$, there are 2 variations in sign; thus, there are 2 or 0 negative real zeros.

14. $f(x) = -3x^5 + 4x^4 + 2$
The maximum number of zeros is the degree of the polynomial which is 5.
Examining $f(x) = -3x^5 + 4x^4 + 2$, there is 1 variation in sign; thus, there is 1 positive real zeros.
Examining $f(-x) = -3(-x)^5 + 4(-x)^4 + 2 = 3x^5 + 4x^4 + 2$, there is no variation in sign; thus, there are no negative real zeros.

15. $f(x) = 3x^3 - 2x^2 + x + 2$
The maximum number of zeros is the degree of the polynomial which is 3.
Examining $f(x) = 3x^3 - 2x^2 + x + 2$, there are 2 variations in sign; thus, there are 2 or 0 positive real zeros.
Examining $f(-x) = 3(-x)^3 - 2(-x)^2 + (-x) + 2 = -3x^3 - 2x^2 - x + 2$, there is 1 variation in sign; thus, there is 1 negative real zero.

16. $f(x) = -x^3 - x^2 + x + 1$
The maximum number of zeros is the degree of the polynomial which is 3.
Examining $f(x) = -x^3 - x^2 + x + 1$, there is 1 variation in sign; thus, there is 1 positive real zero.
Examining $f(-x) = -(-x)^3 - (-x)^2 + (-x) + 1 = x^3 - x^2 - x + 1$, there are 2 variations in sign; thus, there are 2 or 0 negative real zeros.

17. $f(x) = -x^4 + x^2 - 1$
The maximum number of zeros is the degree of the polynomial which is 4.
Examining $f(x) = -x^4 + x^2 - 1$, there are 2 variations in sign; thus, there are 2 or 0 positive real zeros.
Examining $f(-x) = -(-x)^4 + (-x)^2 - 1 = -x^4 + x^2 - 1$, there are 2 variations in sign; thus, there are 2 or 0 negative real zeros.

18. $f(x) = x^4 + 5x^3 - 2$
The maximum number of zeros is the degree of the polynomial which is 4.
Examining $f(x) = x^4 + 5x^3 - 2$, there is 1 variation in sign; thus, there is 1 positive real zero.
Examining $f(-x) = (-x)^4 + 5(-x)^3 - 2 = x^4 - 5x^3 - 2$, there is 1 variation in sign; thus, there is 1 negative real zero.

19. $f(x) = x^5 + x^4 + x^2 + x + 1$
The maximum number of zeros is the degree of the polynomial which is 5.
Examining $f(x) = x^5 + x^4 + x^2 + x + 1$, there are no variations in sign; thus, there are 0 positive real zeros.
Examining $f(-x) = (-x)^5 + (-x)^4 + (-x)^2 + (-x) + 1 = -x^5 + x^4 + x^2 - x + 1$, there are 3 variations in sign; thus, there are 3 or 1 negative real zeros.

20. $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$
The maximum number of zeros is the degree of the polynomial which is 5.
Examining $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$, there are 5 variations in sign; thus, there are 5 or 3 or 1 positive real zeros.
Examining $f(-x) = (-x)^5 - (-x)^4 + (-x)^3 - (-x)^2 + (-x) - 1 = -x^5 - x^4 - x^3 - x^2 - x - 1$, there is no variation in sign; thus, there is no negative real zero.

21. $f(x) = x^6 - 1$
The maximum number of zeros is the degree of the polynomial which is 6.
Examining $f(x) = x^6 - 1$, there is 1 variation in sign; thus, there is 1 positive real zero.
Examining $f(-x) = (-x)^6 - 1 = x^6 - 1$, there is 1 variation in sign; thus, there is 1 negative real zero.

22. $f(x) = x^6 + 1$
The maximum number of zeros is the degree of the polynomial which is 6.
Examining $f(x) = x^6 + 1$, there is no variation in sign; thus, there is no positive real zero.
Examining $f(-x) = (-x)^6 + 1 = x^6 + 1$, there is no variation in sign; thus, there is no negative real zero.

23. $f(x) = 3x^4 - 3x^3 + x^2 - x + 1$
p must be a factor of 1: $p = \pm 1$
q must be a factor of 3: $q = \pm 1, \pm 3$
The possible rational zeros are: $\frac{p}{q} = \pm 1, \pm \frac{1}{3}$

24. $f(x) = x^5 - x^4 + 2x^2 + 3$
p must be a factor of 3: $p = \pm 1, \pm 3$
q must be a factor of 1: $q = \pm 1$
The possible rational zeros are: $\frac{p}{q} = \pm 1, \pm 3$

25. $f(x) = x^5 - 6x^2 + 9x - 3$
p must be a factor of –3: $p = \pm 1, \pm 3$
q must be a factor of 1: $q = \pm 1$
The possible rational zeros are: $\frac{p}{q} = \pm 1, \pm 3$

26. $f(x) = 2x^5 - x^4 - x^2 + 1$
p must be a factor of 1: $p = \pm 1$
q must be a factor of 2: $q = \pm 1, \pm 2$
The possible rational zeros are: $\frac{p}{q} = \pm 1, \pm \frac{1}{2}$

27. $f(x) = -4x^3 - x^2 + x + 2$
p must be a factor of 2: $p = \pm 1, \pm 2$
q must be a factor of –4: $q = \pm 1, \pm 2, \pm 4$
The possible rational zeros are: $\frac{p}{q} = \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{4}$

28. $f(x) = 6x^4 - x^2 + 2$
p must be a factor of 2: $p = \pm 1, \pm 2$
q must be a factor of 6: $q = \pm 1, \pm 2, \pm 3, \pm 6$
The possible rational zeros are: $\frac{p}{q} = \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{1}{6}$

29. $f(x) = 6x^4 - x^2 + 9$
p must be a factor of 9: $p = \pm 1, \pm 3, \pm 9$
q must be a factor of 6: $q = \pm 1, \pm 2, \pm 3, \pm 6$
The possible rational zeros are: $\frac{p}{q} = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm 3, \pm \frac{3}{2}, \pm 9, \pm \frac{9}{2}$

30. $f(x) = -4x^3 + x^2 + x + 6$
p must be a factor of 6: $p = \pm 1, \pm 2, \pm 3, \pm 6$
q must be a factor of –4: $q = \pm 1, \pm 2, \pm 4$
The possible rational zeros are: $\frac{p}{q} = \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm 3, \pm \frac{3}{2}, \pm \frac{3}{4}, \pm 6$

31. $f(x) = 2x^5 - x^3 + 2x^2 + 12$
p must be a factor of 12: $p = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$
q must be a factor of 2: $q = \pm 1, \pm 2$
The possible rational zeros are: $\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm \frac{1}{2}, \pm 3, \pm \frac{3}{2}, \pm 6, \pm 12$

32. $f(x) = 6x^4 - x^2 + 9$
p must be a factor of 9: $p = \pm 1, \pm 3, \pm 9$
q must be a factor of 6: $q = \pm 1, \pm 2, \pm 3, \pm 6$
The possible rational zeros are: $\frac{p}{q} = \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{3}{2}, \pm 9, \pm \frac{9}{2}$

33. $f(x) = 6x^4 + 2x^3 - x^2 + 20$
p must be a factor of 20: $p = \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$
q must be a factor of 6: $q = \pm 1, \pm 2, \pm 3, \pm 6$
The possible rational zeros are:
$$\frac{p}{q} = \pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{6}, \pm 4, \pm\frac{4}{3}, \pm 5, \pm\frac{5}{2}, \pm\frac{5}{3}, \pm\frac{5}{6}, \pm 10, \pm\frac{10}{3}, \pm 20, \pm\frac{20}{3}$$

34. $f(x) = -6x^3 - x^2 + x + 10$
p must be a factor of 10: $p = \pm 1, \pm 2, \pm 5, \pm 10$
q must be a factor of –6: $q = \pm 1, \pm 2, \pm 3, \pm 6$
The possible rational zeros are:
$$\frac{p}{q} = \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm 2, \pm\frac{2}{3}, \pm 5, \pm\frac{5}{2}, \pm\frac{5}{3}, \pm\frac{5}{6}, \pm 10, \pm\frac{10}{3}$$

35. $f(x) = x^3 + 2x^2 - 5x - 6$
Step 1: $f(x)$ has at most 3 real zeros.
Step 2: By Descartes Rule of Signs, there is 1 positive real zero.
Also because $f(-x) = (-x)^3 + 2(-x)^2 - 5(-x) - 6 = -x^3 + 2x^2 + 5x - 6$, there are 2 or 0 negative real zeros.
Step 3: Possible rational zeros:
$$p = \pm 1, \pm 2, \pm 3, \pm 6; \quad q = \pm 1; \quad \frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6$$
Step 4: Using synthetic division:
$$\begin{array}{r|rrrr} -3 & 1 & 2 & -5 & -6 \\ & & -3 & 3 & 6 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$
Since the remainder is 0, $x - (-3) = x + 3$ is a factor. The other factor is the quotient: $x^2 - x - 2$.
Thus, $f(x) = (x+3)\left(x^2 - x - 2\right) = (x+3)(x+1)(x-2)$.
The zeros are –3, -1, and 2.

36. $f(x) = x^3 + 8x^2 + 11x - 20$
Step 1: $f(x)$ has at most 3 real zeros.
Step 2: By Descartes Rule of Signs, there is 1 positive real zero.
Also because $f(-x) = (-x)^3 + 8(-x)^2 + 11(-x) - 20 = -x^3 + 8x^2 - 11x - 20$, there are 2 or 0 negative real zeros.
Step 3: Possible rational zeros: $p = \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20; \quad q = \pm 1;$
$$\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$$
Step 4: Using synthetic division:
$$\begin{array}{r|rrrr} -5 & 1 & 8 & 11 & -20 \\ & & -5 & -15 & 20 \\ \hline & 1 & 3 & -4 & 0 \end{array}$$

Since the remainder is 0, $x-(-5)=x+5$ is a factor. The other factor is the quotient: x^2+3x-4.
Thus, $f(x)=(x+5)\left(x^2+3x-4\right)=(x+5)(x+4)(x-1)$.
The zeros are –5, –4, and 1.

37. $f(x)=2x^3-x^2+2x-1$; $f(-x)=2(-x)^3-(-x)^2+2(-x)-1=-2x^3-x^2-2x-1$

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: By Descartes Rule of Signs, there are 3 or 1 positive real zeros; thus, there are no negative real zeros.

Step 3: Possible rational zeros:

$$p=\pm 1 \quad q=\pm 1, \pm 2; \quad \frac{p}{q}=\pm 1, \pm\frac{1}{2}$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrr} -1 & 2 & -1 & 2 & -1 \\ & & -2 & 3 & -5 \\ \hline & 2 & -3 & 5 & \{-6\} \end{array} \rightarrow$$

$x+1$ is **not** a factor

So we try $x-1$

$$\begin{array}{r|rrrr} 1 & 2 & -1 & 2 & -1 \\ & & 2 & 1 & 3 \\ \hline & 2 & 1 & 3 & \{2\} \end{array} \rightarrow$$

$x-1$ is **not** a factor

Let's try $x-\frac{1}{2}$

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -1 & 2 & -1 \\ & & 1 & 0 & 1 \\ \hline & 2 & 0 & 2 & 0 \end{array}$$

$x-\frac{1}{2}$ **is** a factor the quotient is $2x^2+2$

Thus, $f(x)=2x^3-x^2+2x-1=\left(x-\frac{1}{2}\right)\left(2x^2+2\right)$.

Since $2x^2+2=0$ has no real solutions, $x=\frac{1}{2}$ is the only real zero.

38. $f(x)=2x^3+x^2+2x+1$

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: By Descartes Rule of Signs, there are no positive real zeros.
$f(-x)=2(-x)^3+(-x)^2+2(-x)+1=-2x^3+x^2-2x+1$; thus, there 3 or 1 negative real zeros.

Step 3: Possible rational zeros: $p=\pm 1$ $q=\pm 1, \pm 2$;

$$\frac{p}{q}=\pm 1, \pm\frac{1}{2}$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrr} -1 & 2 & 1 & 2 & 1 \\ & & -2 & 1 & -3 \\ \hline & 2 & -1 & 3 & \{-2\} \end{array} \rightarrow$$

$x+1$ is **not** a factor

$$-\frac{1}{2}\Big) \begin{array}{rrrr} 2 & 1 & 2 & 1 \\ & -1 & 0 & -1 \\ \hline 2 & 0 & 2 & 0 \rightarrow \end{array}$$ $x+\frac{1}{2}$ is a factor

The other factor is the quotient: $2x^2+2$.

Thus, $f(x)=2x^3+x^2+2x+1=\left(x+\frac{1}{2}\right)\left(x^2+2\right)$. The only real zero is $-\frac{1}{2}$.

39. $f(x)=x^4+x^2-2$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, this is one positive real zero.

$f(-x)=(-x)^4+(-x)^2-2=x^4+x^2-2$; thus, there is 1 negative real zero.

Step 3: Possible rational zeros:

$$p=\pm 1,\pm 2;\quad q=\pm 1;\quad \frac{p}{q}=\pm 1,\pm 2$$

Step 4: Using synthetic division:

$$-1\Big) \begin{array}{rrrrr} 1 & 0 & 1 & 0 & 2 \\ & -1 & 1 & -2 & 2 \\ \hline 1 & -1 & 2 & -2 & 0 \rightarrow \end{array}$$

Since the remainder is 0, $x-(-1)=x+1$ is a factor. The other factor is the quotient: x^3-x^2+2x-2.

Thus, $f(x)=(x+1)\left(x^3-x^2+2x-2\right)$. We can factor x^3-x^2+2x-2 by grouping terms $x^3-x^2+2x-2=x^2(x-1)+2(x-1)=(x-1)\left(x^2+2\right)$

Thus, $f(x)=(x+1)(x-1)\left(x^2+2\right)$. Since $x^2+2=0$ has no real solutions, we have two real zeros for *f*, namely –1 and 1.

40. $f(x)=x^4-3x^2-4$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there is 1 positive real zero.

$f(-x)=(-x)^4-3(-x)^2-4=x^4-3x^2-4$; thus, there is one negative real zero.

Step 3: Possible rational zeros: $p=\pm 1,\pm 2,\pm 4;\quad q=\pm 1$

$$\frac{p}{q}=\pm 1,\pm 2,\pm 4$$

We can factor f as follows:

$$f(x)=x^4-3x^2-4=\left(x^2-4\right)\left(x^2+1\right)=(x+2)(x-2)\left(x^2+1\right).$$

Thus, we have two real zeros, - 2 and 2.

41. $f(x)=4x^4+7x^2-2$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there is 1 positive real zero.

$f(-x)=4(-x)^4+7(-x)^2-2=4x^4+7x^2-2$; thus, there is one negative real zero.

Step 3: Possible rational zeros: $p = \pm 1, \pm 2$; $q = \pm 1, \pm 2, \pm 4$

$$\frac{p}{q} = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{4} \pm 2$$

We can factor f as follows:

$$f(x) = 4x^4 + 7x^2 - 2 = (4x^2 - 1)(x^2 + 2) = (2x + 1)(2x - 1)(x^2 + 2).$$

Thus, we have two real zeros, $-\frac{1}{2}$ and $\frac{1}{2}$.

42. $f(x) = 4x^4 + 15x^2 - 4$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there is 1 positive real zero.
$f(-x) = 4(-x)^4 + 15(-x)^2 - 4 = 4x^4 + 15x^2 - 4$; thus, there is one negative real zero.

Step 3: Possible rational zeros: $p = \pm 1, \pm 2, \pm 4$; $q = \pm 1, \pm 2, \pm 4$

$$\frac{p}{q} = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{4} \pm 2, \pm 4$$

We can factor f as follows:

$$f(x) = 4x^4 + 15x^2 - 4 = (4x^2 - 1)(x^2 + 4) = (2x + 1)(2x - 1)(x^2 + 4).$$

Thus, we have two real zeros, $-\frac{1}{2}$ and $\frac{1}{2}$.

43. $f(x) = x^4 + x^3 - 3x^2 - x + 2$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there are 2 or 0 positive real zeros.
$f(-x) = (-x)^4 + (-x)^3 - 3(-x)^2 - (-x) + 2 = x^4 - x^3 - 3x^2 + x + 2$; thus, there are 2 or 0 negative real zeros.

Step 3: Possible rational zeros:

$$p = \pm 1, \pm 2; \quad q = \pm 1; \quad \frac{p}{q} = \pm 1, \pm 2$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrr} -2 & 1 & 1 & -3 & -1 & 2 \\ & & -2 & 2 & 2 & -2 \\ \hline & 1 & -1 & -1 & 1 & 0 \end{array} \qquad \begin{array}{r|rrrr} -1 & 1 & -1 & -1 & 1 \\ & & -1 & 2 & -1 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

Since the remainder is 0, $x + 2$ and $x + 1$ are factors. The other factor is the quotient: $x^2 - 2x + 1$.

Thus, $f(x) = (x + 2)(x + 1)(x - 1)^2$.

The zeros are –2, –1, and 1 (multiplicity 2).

44. $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there are 2 or 0 positive real zeros.
$f(-x) = (-x)^4 - (-x)^3 - 6(-x)^2 + 4(-x) + 8 = x^4 + x^3 - 6x^2 - 4x + 8$; thus, there are 2 or 0 negative real zeros.

Step 3: Possible rational zeros:

$$p = \pm 1, \pm 2, \pm 4, \pm 8; \quad q = \pm 1; \quad \frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 8$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -6 & 4 & 8 \\ & & -2 & 6 & 0 & -8 \\ \hline & 1 & -3 & 0 & 4 & 0 \end{array} \qquad \begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

Since the remainder is 0, $x+2$ and $x+1$ are factors. The other factor is the quotient: $x^2 - 4x + 4$.
Thus, $f(x) = (x+2)(x+1)(x-2)^2$.
The zeros are –2, –1, and 2 (multiplicity 2).

45. $f(x) = 4x^5 - 8x^4 - x + 2$

Step 1: $f(x)$ has at most 5 real zeros.

Step 2: By Descartes Rule of Signs, there are 2 or 0 positive real zeros.
$f(-x) = 4(-x)^5 - 8(-x)^4 - (-x) + 2 = -4x^5 - 8x^4 + x + 2$;
thus, there is 1 negative real zero.

Step 3: Possible rational zeros:

$$p = \pm 1, \pm 2; \quad q = \pm 1, \pm 2, \pm 4; \quad \frac{p}{q} = \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{4}$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrrr} 2 & 4 & -8 & 0 & 0 & -1 & 2 \\ & & 8 & 0 & 0 & 0 & -2 \\ \hline & 4 & 0 & 0 & 0 & -1 & 0 \end{array}$$

Since the remainder is 0, $x-2$ is a factor. The other factor is the quotient: $4x^4 - 1$.

Factoring,

$$f(x) = (x-2)\left(4x^4 - 1\right) = (x-2)(2x^2-1)(2x^2+1)$$
$$= (x-2)\left(\sqrt{2}x - 1\right)\left(\sqrt{2}x + 1\right)\left(2x^2+1\right)$$

The zeros are $\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$, and or –0.71, –0.71, and 2.

46. $f(x) = 4x^5 + 12x^4 - x - 3$

Step 1: $f(x)$ has at most 5 real zeros.

Step 2: By Descartes Rule of Signs, there is 1 positive real zeros.
$f(-x) = 4(-x)^5 + 12(-x)^4 - (-x) - 3 = -4x^5 + 12x^4 + x - 3$;
thus, there are 2 or 0 negative real zero.

Step 3: Possible rational zeros:

$$p = \pm 1, \pm 3; \quad q = \pm 1, \pm 2, \pm 4; \quad \frac{p}{q} = \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{4}, \pm \frac{3}{4}$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrrr} -3 & 4 & 12 & 0 & 0 & -1 & -3 \\ & & -12 & 0 & 0 & 0 & 3 \\ \hline & 4 & 0 & 0 & 0 & -1 & 0 \end{array}$$

Since the remainder is 0, $x+3$ is a factor. The other factor is the quotient: $4x^4-1$.

Factoring,

$$f(x)=(x+3)\left(4x^4-1\right)=(x+3)(2x^2-1)(2x^2+1)$$
$$=(x+3)\left(\sqrt{2}x-1\right)\left(\sqrt{2}x+1\right)\left(2x^2+1\right)$$

The zeros are $\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$, and -3 or –0.71, –0.71, and –3.

47. $x^4-x^3+2x^2-4x-8=0$

The solutions of the equation are the zeros of $f(x)=x^4-x^3+2x^2-4x-8$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there are 3 or 1 positive real zeros.
$f(-x)=(-x)^4-(-x)^3+2(-x)^2-4(-x)-8=x^4+x^3+2x^2+4x-8$;
thus, there is 1 negative real zero.

Step 3: Possible rational zeros:

$$p=\pm1, \pm2, \pm4, \pm8; \quad q=\pm1; \quad \frac{p}{q}=\pm1, \pm2, \pm4, \pm8$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 2 & -4 & -8 \\ & & -1 & 2 & -4 & 8 \\ \hline & 1 & -2 & 4 & -8 & 0 \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & -2 & 4 & -8 \\ & & 2 & 0 & 8 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

Since the remainder is 0, $x+1$ and $x-2$ are factors. The other factor is the quotient: x^2+4.

The zeros are –1 and 2. ($x^2+4=0$ has no real solutions.)

48. $2x^3+3x^2+2x+3=0$

Solve by factoring:

$$x^2(2x+3)+(2x+3)=0$$
$$(2x+3)\left(x^2+1\right)=0$$
$$x=-\frac{3}{2}$$

The zero is $-\frac{3}{2}$. ($x^2+1=0$ has no real solutions.)

49. $3x^3+4x^2-7x+2=0$

The solutions of the equation are the zeros of $f(x)=3x^3+4x^2-7x+2$.

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: By Descartes Rule of Signs, there are 2 or 0 positive real zeros.
$f(-x)=3(-x)^3+4(-x)^2-7(-x)+2=-3x^3+4x^2+7x+2$;
thus, there is 1 negative real zero.

Step 3: Possible rational zeros:

$$p=\pm1, \pm2; \quad q=\pm1, \pm3; \quad \frac{p}{q}=\pm1, \pm2, \pm\frac{1}{3}, \pm\frac{2}{3}$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrr} \frac{2}{3} & 3 & 4 & -7 & 2 \\ & & 2 & 4 & -2 \\ \hline & 3 & 6 & -3 & 0 \end{array}$$

Since the remainder is 0, $x-\frac{2}{3}$ is a factor. The other factor is the quotient: $3x^2+6x-3$.

$f(x)=\left(x-\frac{2}{3}\right)\left(3x^2+6x-3\right)=3\left(x-\frac{2}{3}\right)\left(x^2+2x-1\right)$

Using the quadratic formula to solve $x^2+2x-1=0$:

$$x=\frac{-2\pm\sqrt{4-4(1)(-1)}}{2(1)}=\frac{-2\pm\sqrt{8}}{2}=\frac{-2\pm2\sqrt{2}}{2}=-1\pm\sqrt{2}$$

The zeros are $\frac{2}{3}$, $-1+\sqrt{2}$, and $-1-\sqrt{2}$ or 0.67, 0.41, and –2.41.

50. $2x^3-3x^2-3x-5=0$

The solutions of the equation are the zeros of $f(x)=2x^3-3x^2-3x-5$.

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: By Descartes Rule of Signs, there is 1 positive real zero.

$f(-x)=2(-x)^3-3(-x)^2-3(-x)-5=-2x^3-3x^2+3x-5$;

thus, there are 2 or 0 negative real zeros.

Step 3: Possible rational zeros:

$$p=\pm1,\ \pm5;\quad q=\pm1,\ \pm2;\quad \frac{p}{q}=\pm1,\ \pm5,\ \pm\frac{1}{2},\ \pm\frac{5}{2}$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrr} \frac{5}{2} & 2 & -3 & -3 & -5 \\ & & 5 & 5 & 5 \\ \hline & 2 & 2 & 2 & 0 \end{array}$$

Since the remainder is 0, $x-\frac{5}{2}$ is a factor. The other factor is the quotient: $2x^2+2x+2$.

$f(x)=\left(x-\frac{5}{2}\right)\left(2x^2+2x+2\right)=2\left(x-\frac{5}{2}\right)\left(x^2+x+1\right)$

$x^2+x+1=0$ has no real zeros.

The zero is $\frac{5}{2}$.

51. $3x^3-x^2-15x+5=0$

Solving by factoring:

$$x^2(3x-1)-5(3x-1)=0\rightarrow(3x-1)\left(x^2-5\right)=0\rightarrow(3x-1)\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)=0$$

The solutions of the equation are $\frac{1}{3}$, $\sqrt{5}$, and $-\sqrt{5}$ or 0.33, 2.24, and –2.24.

52. $2x^3-11x^2+10x+8=0$

The solutions of the equation are the zeros of $f(x)=2x^3-11x^2+10x+8$.

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: By Descartes Rule of Signs, there are 2 or 0 positive real zeros.

$f(-x)=2(-x)^3-11(-x)^2+10(-x)+8=-2x^3-11x^2-10x+8$;

thus, there is 1 negative real zero.

Step 3: Possible rational zeros:

$$p = \pm 1, \pm 2, \pm 4, \pm 8; \quad q = \pm 1, \pm 2; \quad \frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{2}$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrr} 4 & 2 & -11 & 10 & 8 \\ & & 8 & -12 & -8 \\ \hline & 2 & -3 & -2 & 0 \end{array}$$

Since the remainder is 0, $x-4$ is a factor. The other factor is the quotient: $2x^2 - 3x - 2 = (2x+1)(x-2)$. The zeros are $-\frac{1}{2}$, 2, and 4.

53. $x^4 + 4x^3 + 2x^2 - x + 6 = 0$

The solutions of the equation are the zeros of $f(x) = x^4 + 4x^3 + 2x^2 - x + 6$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there are 2 or 0 positive real zeros.
$f(-x) = (-x)^4 + 4(-x)^3 + 2(-x)^2 - (-x) + 6 = x^4 - 4x^3 + 2x^2 + x + 6$;
thus, there are 2 or 0 negative real zeros.

Step 3: Possible rational zeros:

$$p = \pm 1, \pm 2, \pm 3, \pm 6; \quad q = \pm 1; \quad \frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrr} -3 & 1 & 4 & 2 & -1 & 6 \\ & & -3 & -3 & 3 & -6 \\ \hline & 1 & 1 & -1 & 2 & 0 \end{array} \qquad \begin{array}{r|rrrr} -2 & 1 & 1 & -1 & 2 \\ & & -2 & 2 & -2 \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

Since the remainder is 0, $x+3$ and $x+2$ are factors. The other factor is the quotient: $x^2 - x + 1$.

The zeros are –3 and –2. ($x^2 - x + 1 = 0$ has no real solutions.)

54. $x^4 - 2x^3 + 10x^2 - 18x + 9 = 0$

The solutions of the equation are the zeros of $f(x) = x^4 - 2x^3 + 10x^2 - 18x + 9$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there are 4 or 2 or 0 positive real zeros.
$f(-x) = (-x)^4 - 2(-x)^3 + 10(-x)^2 - 18(-x) + 9$
$= x^4 + 2x^3 + 10x^2 + 18x + 9$
thus, there are no negative real zeros.

Step 3: Possible rational zeros:

$$p = \pm 1, \pm 3, \pm 9; \quad q = \pm 1; \quad \frac{p}{q} = \pm 1, \pm 3, \pm 9$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrr} 1 & 1 & -2 & 10 & -18 & 9 \\ & & 1 & -1 & 9 & -9 \\ \hline & 1 & -1 & 9 & -9 & 0 \end{array} \qquad \begin{array}{r|rrrr} 1 & 1 & -1 & 9 & -9 \\ & & 1 & 0 & 9 \\ \hline & 1 & 0 & 9 & 0 \end{array}$$

Since the remainder is 0, $x-1$ and $x-1$ are factors. The other factor is the quotient: $x^2 + 9$.

The zeros are 1 (multiplicity 2). ($x^2 + 9 = 0$ has no real solutions.)

55. $x^3 - \frac{2}{3}x^2 + \frac{8}{3}x + 1 = 0$

The solutions of the equation are the zeros of $f(x) = x^3 - \frac{2}{3}x^2 + \frac{8}{3}x + 1$.

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: By Descartes Rule of Signs, there are 2 or 0 positive real zeros.
$f(-x) = (-x)^3 - \frac{2}{3}(-x)^2 + \frac{8}{3}(-x) + 1 = -x^3 - \frac{2}{3}x^2 - \frac{8}{3}x + 1$;
thus, there is 1 negative real zero.

Step 3: Use the equivalent equation $3x^3 - 2x^2 + 8x + 3 = 0$ to find the possible rational zeros:

$$p = \pm 1, \pm 3; \quad q = \pm 1, \pm 3; \quad \frac{p}{q} = \pm 1, \pm 3, \pm \frac{1}{3}$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrr} \frac{-1}{3} & 1 & \frac{-2}{3} & \frac{8}{3} & 1 \\ & & \frac{-1}{3} & \frac{1}{3} & -1 \\ \hline & 1 & -1 & 3 & 0 \end{array}$$

Since the remainder is 0, $x + \frac{1}{3}$ is a factor. The other factor is the quotient: $x^2 - x + 3$.

The real zero is $\frac{-1}{3}$. ($x^2 - x + 3 = 0$ has no real solutions.)

56. $x^3 + \dfrac{3}{2}x^2 + 3x - 2 = 0$

The solutions of the equation are the zeros of $f(x) = x^3 + \dfrac{3}{2}x^2 + 3x - 2$.

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: By Descartes Rule of Signs, there is 1 positive real zero.
$f(-x) = (-x)^3 + \frac{3}{2}(-x)^2 + 3(-x) - 2 = -x^3 - \frac{3}{2}x^2 - 3x - 2$;
thus, there are 2 or no negative real zeros.

Step 3: Use the equivalent equation $2x^3 + 3x^2 + 6x - 4 = 0$ to find the possible rational zeros:

$$p = \pm 1, \pm 2, \pm 4; \quad q = \pm 1, \pm 2; \quad \frac{p}{q} = \pm 1, \pm \frac{1}{2}, \pm 2, \pm 4$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrr} 1 & 2 & 3 & 6 & -4 \\ & & 2 & 5 & 11 \\ \hline & 2 & 5 & 11 & \{7\} \end{array} \qquad \begin{array}{r|rrrr} -1 & 2 & 3 & 6 & -4 \\ & & -2 & -1 & -5 \\ \hline & 2 & 1 & 5 & \{-9\} \end{array} \qquad \begin{array}{r|rrrr} \frac{1}{2} & 2 & 3 & 6 & -4 \\ & & 1 & 2 & 4 \\ \hline & 2 & 4 & 8 & 0 \end{array}$$

Thus, $x - \frac{1}{2}$ is a factor. The other factor is the quotient: $2x^2 + 4x + 8$.

So we have $2x^3 + 3x^2 + 6x - 4 = \left(x - \frac{1}{2}\right)\left(2x^2 + 4x + 8\right) = 2\left(x - \frac{1}{2}\right)\left(x^2 + 2x + 4\right)$

Since $f(x) = x^3 + \dfrac{3}{2}x^2 + 3x - 2 = \dfrac{1}{2}\left(2x^3 + 3x^2 + 6x - 4\right)$ we conclude that

$$f(x) = x^3 + \frac{3}{2}x^2 + 3x - 2 = \frac{1}{2}\left(2x^3 + 3x^2 + 6x - 4\right) = \frac{1}{2}\left(2\left(x - \frac{1}{2}\right)\left(x^2 + 2x + 4\right)\right) = \left(x - \frac{1}{2}\right)\left(x^2 + 2x + 4\right)$$

and since $x^2 + 2x + 4 = 0$ has no real solutions, the only real zero for f is $\frac{1}{2}$.

57. $2x^4 - 19x^3 + 57x^2 - 64x + 20 = 0$

The solutions of the equation are the zeros of $f(x) = 2x^4 - 19x^3 + 57x^2 - 64x + 20$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there are 4, 2 or 0 positive real zeros.

$$f(-x) = 2(-x)^4 - 19(-x)^3 + 57(-x)^2 - 64(-x) + 20$$

$$= 2x^4 + 19x^3 + 57x^2 + 64x + 20 \qquad ;$$

thus, there are no negative real zeros.

Step 3: To find the possible rational zeros:

$$p = \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20; \quad q = \pm 1, \pm 2;$$

$$\frac{p}{q} = \pm 1, \pm \frac{1}{2}, \pm 2, \pm 4, \pm 5, \pm \frac{5}{2}, \pm 10, \pm 20$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrr} 1 & 2 & -19 & 57 & -64 & 20 \\ & & 2 & -17 & 40 & -24 \\ \hline & 2 & -17 & 40 & -24 & \{-4\} \end{array} \qquad \begin{array}{r|rrrrr} \frac{1}{2} & 2 & -19 & 57 & -64 & 20 \\ & & 1 & -9 & 24 & -20 \\ \hline & 2 & -18 & 48 & -40 & 0 \end{array}$$

Thus, $x - \frac{1}{2}$ is a factor.

So :

$$f(x) = 2x^4 - 19x^3 + 57x^2 - 64x + 20 = \left(x - \frac{1}{2}\right)\left(2x^3 - 18x^2 + 48x - 40\right)$$

$$= 2\left(x - \frac{1}{2}\right)\left(x^3 - 9x^2 + 24x - 20\right) .$$

Now try x = 2 as a solution to the equation $x^3 - 9x^2 + 24x - 20 = 0$.

$$\begin{array}{r|rrrr} 2 & 1 & -9 & 24 & -20 \\ & & 2 & -14 & 20 \\ \hline & 1 & -7 & 10 & 0 \end{array}$$

Thus, $x^3 - 9x^2 + 24x - 20 = (x-2)\left(x^2 - 7x + 10\right) = (x-2)(x-2)(x-5)$

So we have

$$f(x) = 2x^4 - 19x^3 + 57x^2 - 64x + 20 = 2\left(x - \frac{1}{2}\right)(x-2)^2(x-5) .$$

Therefore, f has real zeros $\frac{1}{2}$,2,5, and 2 is a zero of multiplicity 2.

58. $2x^4 + x^3 - 24x^2 + 20x + 16 = 0$

The solutions of the equation are the zeros of $f(x) = 2x^4 + x^3 - 24x^2 + 20x + 16$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there are 2 or 0 positive real zeros.

$f(-x) = 2(-x)^4 + (-x)^3 - 24(-x)^2 + 20(-x) + 16 = 2x^4 - x^3 - 24x^2 - 20x + 16;$

thus, there are 2 or 0 negative real zeros.

Step 3: To find the possible rational zeros:

$p = \pm 1, \pm 2, \pm 4, \pm 8, \pm 16; \quad q = \pm 1, \pm 2;$

$$\frac{p}{q} = \pm 1, \pm\frac{1}{2}, \pm 2, \pm 4, \pm 8, \pm 16$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrr} 2 & 2 & 1 & -24 & 20 & 16 \\ & & 4 & 10 & -28 & -16 \\ \hline & 2 & 5 & -14 & -8 & 0 \end{array}$$

Thus, $x - 2$ is a factor.

So :

$f(x) = 2x^4 + x^3 - 24x^2 + 20x + 16 = (x-2)(2x^3 + 5x^2 - 14x - 8)$.

Now try x = -4 as a solution to the equation $2x^3 + 5x^2 - 14x - 8 = 0$.

$$\begin{array}{r|rrrr} -4 & 2 & 5 & -14 & -8 \\ & & -8 & 12 & 8 \\ \hline & 2 & -3 & -2 & 0 \end{array}$$

Thus, $2x^3 + 5x^2 - 14x - 8 = (x+4)(2x^2 - 3x - 2) = (x+4)(2x+1)(x-2)$

So we have

$f(x) = 2x^4 + x^3 - 24x^2 + 20x + 16 = (x-2)(x+4)(2x+1)(x-2)$

$= (x-2)^2(x+4)(2x+1)$.

Therefore, f has real zeros $-\frac{1}{2}, -4, 2$, and 2 is a zero of multiplicity 2.

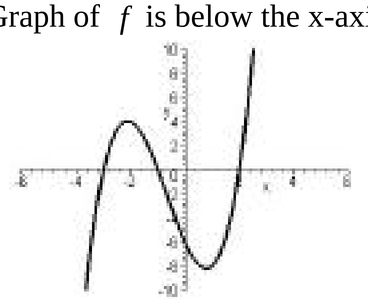
59. $f(x) = x^3 + 2x^2 + 5x - 6 = (x+3)(x+1)(x-2)$.

x-intercepts: –3, –1, 2; y-intercept: -6; crosses x axis at x = –3, –1, 2

	$x < -3$	$-3 < x < -1$	$-1 < x < 2$	$x > 2$
f	-	+	-	+
Above or below x-axis	below	above	below	above

Graph of f is above the x-axis for $(-3,-1)$ $(2,\infty)$

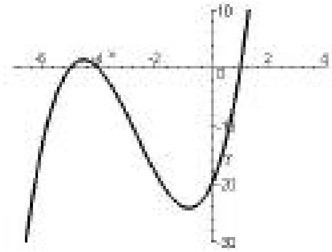
Graph of f is below the x-axis for $(-\infty,-3)$ $(-1,-2)$



60. $f(x)=x^3+8x^2+11x-20=(x+5)(x+4)(x-1)\cdot$

x-intercepts: –5, –4, 1; y-intercept: -20; crosses x axis at x =–5, –4, 1

	$x < -5$	$-5 < x < -4$	$-4 < x < 1$	$x > 1$
f	-	+	-	+
Above or below x-axis	below	above	below	above



Graph of f is above the x-axis for $(-5,-4)$ $(1,\infty)$

Graph of f is below the x-axis for $(-\infty,-5)$ $(-4,1)$

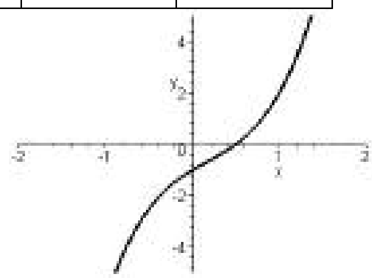
61. $f(x)=2x^3-x^2+2x-1=\left(x-\frac{1}{2}\right)(2x^2+2)$.

x-intercepts: $\frac{1}{2}$; y-intercept: -1; crosses x axis at x = $\frac{1}{2}$

	$x < \frac{1}{2}$	$x > \frac{1}{2}$
f	-	+
Above or below x-axis	below	above

Graph of f is above the x-axis for $\left(\frac{1}{2},\infty\right)$

Graph of f is below the x-axis for $\left(-\infty,\frac{1}{2}\right)$



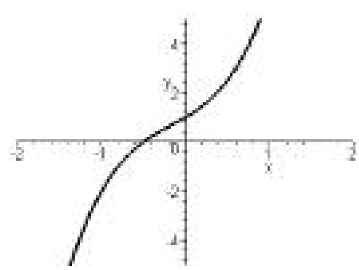
62. $f(x)=2x^3+x^2+2x+1=\left(x+\frac{1}{2}\right)(x^2+2)$.

x-intercepts: $-\frac{1}{2}$; y-intercept: 1; crosses x axis at x = $-\frac{1}{2}$

	$x < -\frac{1}{2}$	$x > -\frac{1}{2}$
f	-	+
Above or below x-axis	below	above

Graph of f is above the x-axis for $\left(-\frac{1}{2},\infty\right)$

Graph of f is below the x-axis for $\left(-\infty,-\frac{1}{2}\right)$



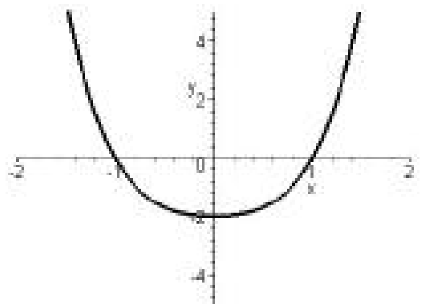
63. $f(x) = x^4 + x^2 - 2 = (x+1)(x-1)(x^2+2)$.

x-intercepts: –1, 1; y-intercept: -2; crosses x axis at x = –1, 1

	$x < -1$	$-1 < x < 1$	$x > 1$
f	+	-	+
Above or below x-axis	above	below	above

Graph of f is above the x-axis for $(-\infty,-1)$ $(1,\infty)$

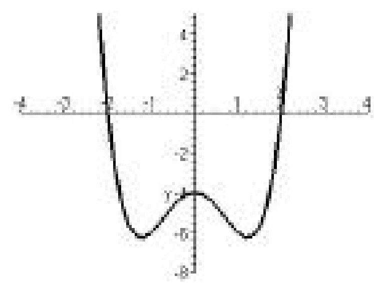
Graph of f is below the x-axis for $(-1,-1)$



64. $f(x) = x^4 - 3x^2 - 4 = (x+2)(x-2)(x^2+1)$.

x-intercepts: –2, 2; y-intercept: -4; crosses x axis at x = –2, 2

	$x < -2$	$-2 < x < -2$	$x > 2$
f	+	-	+
Above or below x-axis	above	below	above



Graph of f is above the x-axis for $(-\infty,-2)$ $(2,\infty)$

Graph of f is below the x-axis for $(-2,2)$

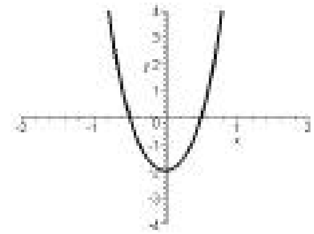
65. $f(x) = 4x^4 + 7x^2 - 2 = (2x+1)(2x-1)(x^2+2)$.

x-intercepts: $-\frac{1}{2},\frac{1}{2}$; y-intercept: -2; crosses x axis at x = $-\frac{1}{2},\frac{1}{2}$

	$x < -\frac{1}{2}$	$-\frac{1}{2} < x < \frac{1}{2}$	$x > \frac{1}{2}$
f	+	-	+
Above or below x-axis	above	below	above

Graph of f is above the x-axis for $\left(-\infty,-\frac{1}{2}\right)$ $\left(\frac{1}{2},\infty\right)$

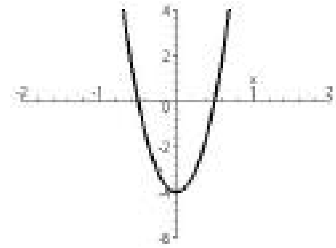
Graph of f is below the x-axis for $\left(-\frac{1}{2},\frac{1}{2}\right)$



66. $f(x) = 4x^4 + 15x^2 - 4 = (2x+1)(2x-1)(x^2+4)$.

x-intercepts: $-\frac{1}{2}, \frac{1}{2}$; y-intercept: -4; crosses x axis at x = $-\frac{1}{2}, \frac{1}{2}$

	$x < -\frac{1}{2}$	$-\frac{1}{2} < x < \frac{1}{2}$	$x > \frac{1}{2}$
f	+	-	+
Above or below x-axis	above	below	above



Graph of f is above the x-axis for $\left(-\infty, -\frac{1}{2}\right)$ $\left(\frac{1}{2}, \infty\right)$

Graph of f is below the x-axis for $\left(-\frac{1}{2}, \frac{1}{2}\right)$

67. $f(x) = x^4 + x^3 - 3x^2 - x + 2 = (x+2)(x+1)(x-1)^2$.

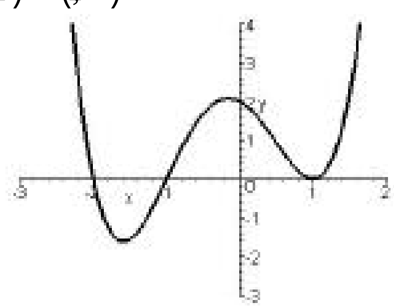
x-intercepts: –2, –1, 1; y-intercept: 2

crosses x axis at x = –2, –1; touches x axis at x = 1

	$x < -2$	$-2 < x < -1$	$-1 < x < 1$	$x > 1$
f	+	-	+	+
Above or below x-axis	above	below	above	above

Graph of f is above the x-axis for $(-\infty, -2)$ $(-1, 1)$ $(1, \infty)$

Graph of f is below the x-axis for $(-2, -1)$



68. $f(x) = x^4 - 3x^3 - 6x^2 + 4x + 8 = (x+2)(x+1)(x-2)^2$.

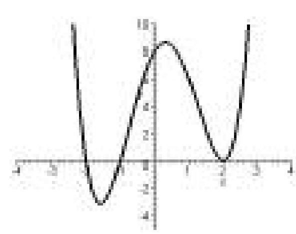
x-intercepts: –2, –1, 2; y-intercept: 8

crosses x axis at x = –2, –1; touches x axis at x = 2

	$x < -2$	$-2 < x < -1$	$-1 < x < 2$	$x > 2$
f	+	-	+	+
Above or below x-axis	above	below	above	above

Graph of f is above the x-axis for $(-\infty, -2)$ $(-1, 2)$ $(2, \infty)$

Graph of f is below the x-axis for $(-2, -1)$



69. $f(x) = 4x^5 - 8x^4 - x + 2 = (x-2)\left(\sqrt{2}x - 1\right)\left(\sqrt{2}x + 1\right)\left(2x^2 + 1\right)$

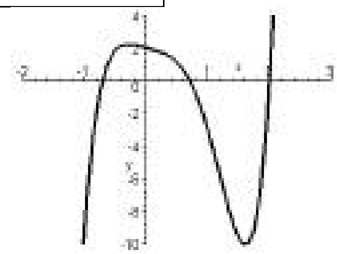
x-intercepts: $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 2$; y-intercept: 2

crosses x axis at x = $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 2$

	$x < -\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}} < x < 2$	$x > 2$
f	-	+	-	+
Above or below x-axis	below	above	below	above

Graph of f is above the x-axis for $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $(2, \infty)$

Graph of f is below the x-axis for $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ $\left(\frac{1}{\sqrt{2}}, 2\right)$



70. $f(x) = 4x^5 + 12x^4 - x - 3 = (x+3)\left(\sqrt{2}x - 1\right)\left(\sqrt{2}x + 1\right)\left(2x^2 + 1\right)$

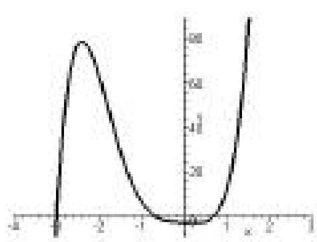
x-intercepts: $-3, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$; y-intercept: -3

crosses x axis at x = $-3, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

	$x < -3$	$-3 < x < -\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$	$x > \frac{1}{\sqrt{2}}$
f	-	+	-	+
Above or below x-axis	below	above	below	above

Graph of f is above the x-axis for $\left(-3, -\frac{1}{\sqrt{2}}\right)$ $\left(\frac{1}{\sqrt{2}}, \infty\right)$

Graph of f is below the x-axis for $(-\infty, -3)$ $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$



71. $f(x)=x^4-3x^2-4$
$a_3=0, a_2=-3, a_1=0, a_0=-4$

$Max\{1, |-4|+|0|+|-3|+|0|\}=Max\{1, 4+0+3+0\}=Max\{1, 7\}=7$

$1+Max\{|-4|, |0|, |-3|, |0|\}=1+Max\{4, 0, 3, 0\}=1+4=5$
The smaller of the two numbers is 5. Thus, every zero of f lies between – 5 and 5.

72. $f(x)=x^4-5x^2-36$
$a_3=0, a_2=-5, a_1=0, a_0=-36$

$Max\{1, |-36|+|0|+|-5|+|0|\}=Max\{1, 36+0+5+0\}=Max\{1, 41\}=41$

$1+Max\{|-36|, |0|, |-5|, |0|\}=1+Max\{36, 0, 5, 0\}=1+36=37$
The smaller of the two numbers is 37. Thus, every zero of f lies between – 37 and 37.

73. $f(x)=x^4+x^3-x-1$
$a_3=1, a_2=0, a_1=-1, a_0=-1$

$Max\{1, |-1|+|-1|+|0|+|1|\}=Max\{1, 1+1+0+1\}=Max\{1, 3\}=3$

$1+Max\{|-1|, |-1|, |0|, |1|\}=1+Max\{1, 1, 0, 1\}=1+1=2$
The smaller of the two numbers is 2. Thus, every zero of f lies between – 2 and 2.

74. $f(x)=x^4+x^3+x-1$
$a_3=1, a_2=0, a_1=1, a_0=-1$

$Max\{1, |-1|+|1|+|0|+|1|\}=Max\{1, 1+1+0+1\}=Max\{1, 3\}=3$

$1+Max\{|-1|, |1|, |0|, |1|\}=1+Max\{1, 1, 0, 1\}=1+1=2$
The smaller of the two numbers is 2. Thus, every zero of f lies between – 2 and 2.

75. $f(x)=3x^4+3x^3-x^2-12x-12=3\left(x^4+x^3-\frac{1}{3}x^2-4x-4\right)$

Note: The leading coefficient must be 1. $a_3=1, a_2=-\frac{1}{3}, a_1=-4, a_0=-4$

$Max\left\{1, |-4|+|-4|+\left|-\frac{1}{3}\right|+|1|\right\}=Max\left\{1, 4+4+\frac{1}{3}+1\right\}=Max\left\{1, \frac{28}{3}\right\}=\frac{28}{3}$

$1+Max\left\{|-4|, |-4|, \left|-\frac{1}{3}\right|, |1|\right\}=1+Max\left\{4, 4, \frac{1}{3}, 1\right\}=1+4=5$
The smaller of the two numbers is 5. Thus, every zero of f lies between – 5 and 5.

76. $f(x) = 3x^4 - 3x^3 - 5x^2 + 27x - 36 = 3\left(x^4 + x^3 - \frac{5}{3}x^2 + 9x - 12\right)$

Note: The leading coefficient must be 1.

$a_3 = 1, a_2 = -\frac{5}{3}, a_1 = 9, a_0 = -12$

$Max\left\{1, |-12| + |9| + \left|-\frac{5}{3}\right| + |1|\right\} = Max\left\{1, 12 + 9 + \frac{5}{3} + 1\right\} = Max\left\{1, \frac{71}{3}\right\} = \frac{71}{3}$

$1 + Max\left\{|-12|, |9|, \left|-\frac{5}{3}\right|, |1|\right\} = 1 + Max\left\{12, 9, \frac{5}{3}, 1\right\} = 1 + 12 = 13$

The smaller of the two numbers is 13. Thus, every zero of f lies between – 13 and 13.

77. $f(x) = 4x^5 - x^4 + 2x^3 - 2x^2 + x - 1 = 4\left(x^5 - \frac{1}{4}x^4 + \frac{1}{2}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x - \frac{1}{4}\right)$

Note: The leading coefficient must be 1. $a_4 = -\frac{1}{4}, a_3 = \frac{1}{2}, a_2 = -\frac{1}{2}, a_1 = \frac{1}{4}, a_0 = -\frac{1}{4}$

$Max\left\{1, \left|-\frac{1}{4}\right| + \left|\frac{1}{4}\right| + \left|-\frac{1}{2}\right| + \left|\frac{1}{2}\right| + \left|-\frac{1}{4}\right|\right\} = Max\left\{1, \frac{1}{4} + \frac{1}{4} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4}\right\} = Max\left\{1, \frac{7}{4}\right\} = \frac{7}{4}$

$1 + Max\left\{\left|-\frac{1}{4}\right|, \left|\frac{1}{4}\right|, \left|-\frac{1}{2}\right|, \left|\frac{1}{2}\right|, \left|-\frac{1}{4}\right|\right\} = 1 + Max\left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right\} = 1 + \frac{1}{2} = \frac{3}{2}$

The smaller of the two numbers is $\frac{3}{2}$. Thus, every zero of f lies between $-\frac{3}{2}$ and $\frac{3}{2}$

78. $f(x) = 4x^5 + x^4 + x^3 + x^2 - 2x - 2 = 4\left(x^5 + \frac{1}{4}x^4 + \frac{1}{4}x^3 + \frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{2}\right)$

Note: The leading coefficient must be 1. $a_4 = \frac{1}{4}, a_3 = \frac{1}{4}, a_2 = \frac{1}{4}, a_1 = -\frac{1}{2}, a_0 = -\frac{1}{2}$

$Max\left\{1, \left|-\frac{1}{2}\right| + \left|-\frac{1}{2}\right| + \left|\frac{1}{4}\right| + \left|\frac{1}{4}\right| + \left|\frac{1}{4}\right|\right\} = Max\left\{1, \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right\} = Max\left\{1, \frac{7}{4}\right\} = \frac{7}{4}$

$1 + Max\left\{\left|-\frac{1}{2}\right|, \left|-\frac{1}{2}\right|, \left|\frac{1}{4}\right|, \left|\frac{1}{4}\right|, \left|\frac{1}{4}\right|\right\} = 1 + Max\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\} = 1 + \frac{1}{2} = \frac{3}{2}$

The smaller of the two numbers is $\frac{3}{2}$. Thus, every zero of f lies between $-\frac{3}{2}$ and $\frac{3}{2}$.

79. $f(x) = 8x^4 - 2x^2 + 5x - 1;\quad [0, 1]$
$f(0) = -1 < 0$ and $f(1) = 10 > 0$
Since one is positive and one is negative, there is a zero in the interval.

80. $f(x) = x^4 + 8x^3 - x^2 + 2;\quad [-1, 0]$
$f(-1) = -6 < 0$ and $f(0) = 2 > 0$
Since one is positive and one is negative, there is a zero in the interval.

81. $f(x) = 2x^3 + 6x^2 - 8x + 2; \quad [-5, -4]$
$f(-5) = -58 < 0$ and $f(-4) = 2 > 0$
Since one is positive and one is negative, there is a zero in the interval.

82. $f(x) = 3x^3 - 10x + 9; \quad [-3, -2]$
$f(-3) = -42 < 0$ and $f(-2) = 5 > 0$
Since one is positive and one is negative, there is a zero in the interval.

83. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 - 18x + 18; \quad [1.4, 1.5]$
$f(1.4) = -0.1754 < 0$ and $f(1.5) = 1.4063 > 0$
Since one is positive and one is negative , there is a zero in the interval .

84. $f(x) = x^5 - 3x^4 - 2x^3 + 6x^2 + x + 2; \quad [1.7, 1.8]$
$f(1.7) = 0.35627 > 0$ and $f(1.8) = -1.021 < 0$
Since one is positive and one is negative , there is a zero in the interval .

85. $8x^4 - 2x^2 + 5x - 1 = 0; \quad 0 \le r \le 1$

Consider the function $f(x) = 8x^4 - 2x^2 + 5x - 1$
Subdivide the interval [0,1] into 10 equal subintervals:

[0,0.1]; [0.1,0.2]; [0.2,0.3]; [0.3,0.4]; [0.4,0.5]; [0.5,0.6]; [0.6,0.7]; [0.7,0.8]; [0.8,0.9]; [0.9,1]

$f(0) = -1; f(0.1) = -0.5192$
$f(0.1) = -0.5192; f(0.2) = -0.0672$
$f(0.2) = -0.0672; f(0.3) = 0.3848$ so f has a real zero on the interval [0.2,0.3].
Subdivide the interval [0.2,0.3] into 10 equal subintervals:

[0.2,0.21]; [0.21,0.22]; [0.22,0.23]; [0.23,0.24]; [0.24,0.25]; [0.25,0.26];[0.26,0.27]; [0.27,0.28]; [0.28,0.29]; [0.29,0.3]

$f(0.2) = -0.0672; f(0.21) = -0.02264$
$f(0.21) = -0.02264; f(0.22) = 0.0219$ so f has a real zero on the interval [0.21,0.22], therefore $r = 0.21$, correct to 2 decimal places.

86. $x^4 + 8x^3 - x^2 + 2 = 0; \quad -1 \le r \le 0$

Consider the function $f(x) = x^4 + 8x^3 - x^2 + 2$
Subdivide the interval [- 1, 0] into 10 equal subintervals:

[-1,-0.9]; [-0.9,-0.8]; [-0.8,-0.7]; [-0.7,-0.6]; [-0.6,-0.5]; [-0.5,-0.4]; [-0.4,-0.3]; [-0.3,-0.2]; [-0.2,-0.1]; [-0.1,0]

$f(-1) = -6; f(-0.9) = -3.9859$
$f(-0.9) = -3.9859; f(-0.8) = -2.3264$
$f(-0.8) = -2.3264; f(-0.7) = -0.9939$
$f(-0.7) = -0.9939;\ f(-0.6) = 0.0416$ so f has a real zero on the interval [-0.7,-0.6].

Subdivide the interval [-0.7,-0.6] into 10 equal subintervals:

[-0.7,-0.69]; [-0.69,-0.68]; [-0.68,-0.67]; [-0.67,-0.66]; [-0.66,-0.65]; [-0.65,-0.64]; [-0.64,-0.63]; [-0.63,-0.62]; [-0.62,-0.61]; [-0.61,-0.6]

$f(-0.7) = -0.9939; f(-0.69) = -0.8775$
$f(-0.69) = -0.8775; f(-0.68) = -0.7640$
$f(-0.68) = -0.7640; f(-0.67) = -0.6535$
$f(-0.67) = -0.6535; f(-0.66) = -0.5458$
$f(-0.66) = -0.5458; f(-0.65) = -0.4410$
$f(-0.65) = -0.4410; f(-0.64) = -0.3390$
$f(-0.64) = -0.3390; f(-0.63) = -0.2397$
$f(-0.63) = -0.2397; f(-0.62) = -0.1433$
$f(-0.62) = -0.1433;\ f(-0.61) = -0.0495$
$f(-0.61) = -0.0495; f(-0.60) = 0.0416$ so f has a real zero on the interval [-0.61,-0.6], therefore $r = -0.61$, correct to 2 decimal places.

87. $2x^3 + 6x^2 - 8x + 2 = 0; \quad -5 \le r \le -4$

Consider the function $f(x) = 2x^3 + 6x^2 - 8x + 2$
Subdivide the interval [-5,-4] into 10 equal subintervals:

[-5,-4.9]; [-4.9,-4.8]; [-4.8,-4.7]; [-4.7,-4.6]; [-4.6,-4.5]; [-4.5,-4.4]; [-4.4,-4.3]; [-4.3,-4.2]; [-4.2,-4.1]; [-4.1,-4]

$f(-5) = -58;\ f(-4.9) = -50.038$
$f(-4.9) = -50.038; f(-4.8) = -42.544$
$f(-4.8) = -42.544; f(-4.7) = -35.506$

$f(-4.7) = -35.506; f(-4.6) = -28.912$
$f(-4.6) = -28.912;\ f(-4.5) = -22.75$
$f(-4.5) = -22.75;\ f(-4.4) = -17$
$f(-4.4) = -17; f(-4.3) = -11.674$
$f(-4.3) = -11.674; f(-4.2) = -6.736$
$f(-4.2) = -6.736; f(-4.1) = -2.182$
$f(-4.1) = -2.182; f(-4) = 2$ so f has a real zero on the interval [-4.1,-4].

Subdivide the interval [-4.1,-4] into 10 equal subintervals:

[-4.1,-4.09]; [-4.09,-4.08]; [-4.08,-4.07]; [-4.07,-4.06]; [-4.06,-4.05]; [-4.05,-4.04];
[-4.04,-4.03]; [-4.03,-4.02]; [-4.02,-4.01]; [-4.01,-4]

$f(-4.1) = -2.182;\ f(-4.09) = -1.7473$
$f(-4.09) = -1.7473; f(-4.08) = -1.3162$
$f(-4.08) = -1.3162; f(-4.07) = -0.8889$
$f(-4.07) = -0.8889; f(-4.06) = -0.4652$
$f(-4.06) = -0.4652; f(-4.05) = -0.0452$
$f(-4.05) = -0.4652; f(-4.04) = 0.3711$ so f has a real zero on the interval [-4.05,-4.04], therefore $r = -4.05$, correct to 2 decimal places.

88. $3x^3 - 10x + 9 = 0; \quad -3 \le r \le -2$

Consider the function $f(x) = 3x^3 - 10x + 9$
Subdivide the interval [-3,-2] into 10 equal subintervals:

[-3,-2.9]; [-2.9,-2.8]; [-2.8,-2.7]; [-2.7,-2.6]; [-2.6,-2.5]; [-2.5,-2.4]; [-2.4,-2.3];
[-2.3,-2.2]; [-2.2,-2.1]; [-2.1,-2]

$f(-3) = -42; f(-2.9) = -35.167$
$f(-2.9) = -35.167; f(-2.8) = -28.856$
$f(-2.8) = -28.856; f(-2.7) = -23.049$
$f(-2.7) = -23.049; f(-2.6) = -17.728$
$f(-2.6) = -17.728; f(-2.5) = -12.875$
$f(-2.5) = -12.875; f(-2.4) = -8.472$
$f(-2.4) = -8.472; f(-2.3) = -4.501$
$f(-2.3) = -4.501; f(-2.2) = -0.944$
$f(-2.2) = -0.944; f(-2.1) = 2.217$ so f has a real zero on the interval [-2.2,-2.1].

Subdivide the interval [-2.2,-2.1] into 10 equal subintervals:

[-2.2,-2.19]; [-2.19,-2.18]; [-2.18,-2.17]; [-2.17,-2.16]; [-2.16,-2.15]; [-2.15,-2.14];
[-2.14,-2.13]; [-2.13,-2.12]; [-2.12,-2.11]; [-2.11,-2.1]

$f(-2.2) = -0.944; f(-2.19) = -0.6104$
$f(-2.19) = -0.6104; f(-2.18) = -0.2807$
$f(-2.18) = -0.2807; f(-2.17) = 0.0451$ so f has a real zero on the interval [-2.18,-2.17], therefore $r = -2.18$, correct to 2 decimal places.

89. $f(x) = x^3 + x^2 + x - 4$

$f(1) = -1; f(2) = 10$ so f has a real zero on the interval [1,2],

Subdivide the interval [1,2] into 10 equal subintervals:

[1,1.1]; [1.1,1.2]; [1.2,1.3]; [1.3,1.4]; [1.4,1.5]; [1.5,1.6]; [1.6,1.7]; [1.7,1.8]; [1.8,1.9]; [1.9,2]

$f(1) = -1; f(1.1) = -0.359$
$f(1.1) = -0.359; f(1.2) = 0.368$ so f has a real zero on the interval [1.1,1.2].
Subdivide the interval [1,1.2] into 10 equal subintervals:

[1,1.11]; [1.11,1.12]; [1.12,1.13]; [1.13,1.14]; [1.14,1.15]; [1.15,1.16];[1.16,1.17]; [1.17,1.18]; [1.18,1.19]; [1.19,1.2]

$f(1) = -1; f(1.11) = -0.2903$
$f(1.11) = -0.2903; f(1.12) = -0.2207$
$f(1.12) = -0.2207; f(1.13) = -0.1502$
$f(1.13) = -0.1502; f(1.14) = -0.0789$
$f(1.14) = -0.0789; f(1.15) = -0.0066$
$f(1.15) = -0.0066; f(1.16) = 0.0665$

so f has a real zero on the interval [1.15,1.16], therefore $r = 1.15$, correct to 2 decimal places.

90. $f(x) = 2x^4 + x^2 - 1$

$f(0) = -1; f(1) = 2$ so f has a real zero on the interval [0,1],
Subdivide the interval [0,1] into 10 equal subintervals:

[0,0.1]; [0.1,0.2]; [0.2,0.3]; [0.3,0.4]; [0.4,0.5]; [0.5,0.6]; [0.6,0.7]; [0.7,0.8]; [0.8,0.9]; [0.9,1]

$f(0) = -1; f(0.1) = -0.9898$
$f(0.1) = -0.9898; f(0.2) = -0.9568$
$f(0.2) = -0.9568; f(0.3) = -0.8938$
$f(0.3) = -0.8938; f(0.4) = -0.7888$
$f(0.4) = -0.7888; f(0.5) = -0.625$
$f(0.5) = -0.625; f(0.6) = -0.3808$
$f(0.6) = -0.3808; f(0.7) = -0.0298$
$f(0.7) = -0.0298; f(0.8) = 0.4592$ so f has a real zero on the interval [0.7,0.8].
Subdivide the interval [0.7,0.8] into 10 equal subintervals:

[0.7,0.71]; [0.71,0.72]; [0.72,0.73]; [0.73,0.74]; [0.74,0.75]; [0.75,0.76];[0.76,0.77]; [0.77,0.78]; [0.78,0.79]; [0.79,0.8]

$f(0.7) = -0.298; f(0.71) = 0.123$ so f has a real zero on the interval [0.7,0.71], therefore $r = 0.70$, correct to 2 decimal places.

91. $f(x) = 2x^4 - 3x^3 - 4x^2 - 8$
$f(2) = -16; f(3) = 37$ so f has a real zero on the interval [2,3],
Subdivide the interval [2,3] into 10 equal subintervals:

[2,2.1]; [2.1,2.2]; [2.2,2.3]; [2.3,2.4]; [2.4,2.5]; [2.5,2.6]; [2.6,2.7]; [2.7,2.8]; [2.8,2.9]; [2.9,3]

$f(2) = -16; f(2.1) = -14.5268$
$f(2.1) = -14.5268; f(2.2) = -12.4528$
$f(2.2) = -12.4528; f(2.3) = -9.6928$
$f(2.3) = -9.6928; f(2.4) = -6.1568$
$f(2.4) = -6.1568; f(2.5) = -1.75$
$f(2.5) = -1.75; f(2.6) = 3.6272$ so f has a real zero on the interval [2.5,2.6].
Subdivide the interval [2.5,2.6] into 10 equal subintervals:

[2.5,2.51]; [2.51,2.52]; [2.52,2.53]; [2.53,2.54]; [2.54,2.55]; [2.55,2.56];[2.56,2.57]; [2.57,2.58]; [2.58,2.59]; [2.59,2.6]

$f(2.5) = -1.75; f(2.51) = -1.2576$
$f(2.51) = -1.2576; f(2.52) = -0.7555$
$f(2.52) = -0.7555; f(2.53) = -0.2434$
$f(2.53) = -0.2434; f(2.54) = 0.2787$ so f has a real zero on the interval [2.53,2.54], therefore $r = 2.53$, correct to 2 decimal places.

92. $f(x) = 3x^3 - 2x^2 - 20$

$f(2) = -4; f(3) = 43$ so f has a real zero on the interval [2,3],
Subdivide the interval [2,3] into 10 equal subintervals:

[2,2.1]; [2.1,2.2]; [2.2,2.3]; [2.3,2.4]; [2.4,2.5]; [2.5,2.6]; [2.6,2.7]; [2.7,2.8]; [2.8,2.9]; [2.9,3]

$f(2) = -4; f(2.1) = -1.037$
$f(2.1) = -1.037; f(2.2) = 2.264$ so f has a real zero on the interval [2.1,2.1].
Subdivide the interval [2.1,2.2] into 10 equal subintervals:

[2.1,2.11]; [2.11,2.12]; [2.12,2.13]; [2.13,2.14]; [2.14,2.15]; [2.15,2.16];[2.16,2.17]; [2.17,2.18]; [2.18,2.19]; [2.19,2.2]

$f(2.1) = -1.037; f(2.11) = -0.7224$
$f(2.11) = -0.7224; f(2.12) = -0.4044$

$f(2.12) = -0.4044; f(2.13) = -0.0830$
$f(2.13) = -0.0830; f(2.14) = 0.2418$ so f has a real zero on the interval [2.13,2.14], therefore $r = 2.13$, correct to 2 decimal places.

93. $x-2$ is a factor of $f(x) = x^3 - kx^2 + kx + 2$ only if the remainder that results when $f(x)$ is divided by $x-2$ is 0. Dividing, we have:

$$\begin{array}{r|rrrr} 2 & 1 & -k & k & 2 \\ & & 2 & -2k+4 & -2k+8 \\ \hline & 1 & -k+2 & -k+4 & -2k+10 \end{array}$$

Since we want the remainder to equal 0, set the remainder equal to zero and solve:

$-2k + 10 = 0 \rightarrow -2k = -10 \rightarrow k = 5$

94. $x+2$ is a factor of $f(x) = x^4 - kx^3 + kx^2 + 1$ only if the remainder that results when $f(x)$ is divided by $x+2$ is 0. Dividing, we have:

$$\begin{array}{r|rrrrr} -2 & 1 & -k & k & 0 & 1 \\ & & -2 & 2k+4 & -6k-8 & 12k+16 \\ \hline & 1 & -k-2 & 3k+4 & -6k-8 & 12k+17 \end{array}$$

Since we want the remainder to equal 0, set the remainder equal to zero and solve:

$12k + 17 = 0 \rightarrow 12k = -17 \rightarrow k = -\frac{17}{12}$

95. By the Remainder Theorem we know that the remainder from synthetic division by c is equal to $f(c)$. Thus the easiest way to find the remainder is to evaluate:

$f(1) = 2(1)^{20} - 8(1)^{10} + 1 - 2 = 2 - 8 + 1 - 2 = -7$

The remainder is –7.

96. By the Remainder Theorem we know that the remainder from synthetic division by c is equal to $f(c)$. Thus the easiest way to find the remainder is to evaluate:

$f(-1) = -3(-1)^{17} + (-1)^9 - (-1)^5 + 2(-1) = 3 - 1 + 1 - 2 = 1$

The remainder is 1.

97. We want to prove that $x-c$ is a factor of $x^n - c^n$, for any positive integer *n*. By the Factor Theorem, $x-c$ will be a factor of $f(x)$ provided $f(c) = 0$. Here, $f(x) = x^n - c^n$, so that $f(c) = c^n - c^n = 0$. Therefore, $x-c$ is a factor of $x^n - c^n$.

98. We want to prove that $x+c$ is a factor of $x^n + c^n$, if $n \geq 1$ is an odd integer. By the Factor Theorem, $x+c$ will be a factor of $f(x)$ provided $f(-c) = 0$. Here, $f(x) = x^n + c^n$, so that $f(-c) = (-c)^n + c^n = -c^n + c^n = 0$ if $n \geq 1$ is an odd integer. Therefore, $x+c$ is a factor of $x^n + c^n$ if $n \geq 1$ is an odd integer.

99. $x^3 - 8x^2 + 16x - 3 = 0$ has solution $x = 3$, so $x - 3$ is a factor of $f(x) = x^3 - 8x^2 + 16x - 3$.

Using synthetic division

$$\begin{array}{r|rrrr} 3 & 1 & -8 & 16 & -3 \\ & & 3 & -15 & 3 \\ \hline & 1 & -5 & 1 & 0 \end{array}$$

$f(x) = x^3 - 8x^2 + 16x - 3 = (x - 3)(x^2 - 5x + 1)$.

Solving $x^2 - 5x + 1 = 0$

$$x = \frac{5 \pm \sqrt{25 - 4}}{2} = \frac{5 \pm \sqrt{21}}{2}$$

The sum of these two roots is $\frac{5 + \sqrt{21}}{2} + \frac{5 - \sqrt{21}}{2} = \frac{10}{2} = 5$.

100. $x^3 + 5x^2 + 5x - 2 = 0$ has solution $x = -2$, so $x + 2$ is a factor of $f(x) = x^3 + 5x^2 + 5x - 2$.

Using synthetic division

$$\begin{array}{r|rrrr} -2 & 1 & 5 & 5 & -2 \\ & & -2 & -6 & 2 \\ \hline & 1 & 3 & -1 & 0 \end{array}$$

$f(x) = x^3 + 5x^2 + 5x - 2 = (x + 2)(x^2 + 3x - 1)$.

Solving $x^2 + 3x - 1 = 0$

$$x = \frac{-3 \pm \sqrt{9 + 4}}{2} = \frac{-3 \pm \sqrt{13}}{2}$$

The sum of these two roots is $\frac{-3 + \sqrt{13}}{2} + \frac{-3 - \sqrt{13}}{2} = \frac{-6}{2} = -3$.

101. $f(x) = 2x^3 + 3x^2 - 6x + 7$

By the Rational Zero Theorem, the only possible rational zeros are:

$$\frac{p}{q} = \pm 1, \pm 7, \pm \frac{1}{2}, \pm \frac{7}{2}$$

Since $\frac{1}{3}$ is not in the list of possible rational zeros, it is not a zero of $f(x)$.

102. $f(x) = 4x^3 - 5x^2 - 3x + 1$

By the Rational Zero Theorem, the only possible rational zeros are:

$$\frac{p}{q} = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}$$

Since $\frac{1}{3}$ is not in the list of possible rational zeros, it is not a zero of $f(x)$.

103. $f(x) = 2x^6 - 5x^4 + x^3 - x + 1$
By the Rational Zero Theorem, the only possible rational zeros are:
$$\frac{p}{q} = \pm 1, \pm \frac{1}{2}$$
Since $\frac{3}{5}$ is not in the list of possible rational zeros, it is not a zero of $f(x)$.

104. $f(x) = x^7 + 6x^5 - x^4 + x + 2$
By the Rational Zero Theorem, the only possible rational zeros are:
$$\frac{p}{q} = \pm 1, \pm 2$$
Since $\frac{2}{3}$ is not in the list of possible rational zeros, it is not a zero of $f(x)$.

105. Let x be the length of a side of the original cube.
After removing the 1 inch slice, one dimension will be $x - 1$.
The volume of the new solid will be:
$$(x-1) \quad x \quad x = 294 \rightarrow x^3 - x^2 = 294 \rightarrow x^3 - x^2 - 294 = 0$$
By Descartes Rule of Signs, we know that there is one positive real solution.
The possible rational zeros are:
$$p = \pm 1, \pm 2, \pm 3, \pm 6, \pm 7, \pm 14, \pm 21, \pm 42, \pm 49, \pm 98, \pm 147, \pm 294; \quad q = \pm 1$$
The rational zeros are the same as the values for p.
Using synthetic division:
$$\begin{array}{r|rrrr} 7 & 1 & -1 & 0 & -294 \\ & & 7 & 42 & 294 \\ \hline & 1 & 6 & 42 & 0 \end{array}$$
7 is a zero, so the length of the original edge of the cube was 7 inches.

106. Let x be the length of a side of the original cube.
The volume is x^3.
The dimensions are changed to $x + 6$, $x + 12$, and $x - 4$.
The volume of the new solid will be $(x+6)(x+12)(x-4)$
Solve the volume equation:
$$(x+6)(x+12)(x-4) = 2x^3$$
$$\left(x^2 + 18x + 72\right)(x-4) = 2x^3 \rightarrow x^3 + 14x^2 - 288 = 2x^3 \rightarrow x^3 - 14x^2 + 288 = 0$$
By Descartes Rule of Signs, we know that there are 2 or 0 positive real solutions.
The possible rational zeros are:
$$p = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 9, \pm 12, \pm 16, \pm 18, \pm 24, \pm 32, \pm 36, \pm 48,$$
$$\pm 72, \pm 96, \pm 144, \pm 288; \quad q = \pm 1$$
The rational zeros are the same as the values for p.
Using synthetic division:
$$\begin{array}{r|rrrr} 6 & 1 & -14 & 0 & 288 \\ & & 6 & -48 & -288 \\ \hline & 1 & -8 & -48 & 0 \end{array}$$
6 is a zero; the other factor is $x^2 - 8x - 48 = (x-12)(x+4)$. The other zeros are 12 and –4. The length of the original edge of the cube was 6 inches or 12 inches.

107. $f(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1x + a_0$; where $a_{n-1}, a_{n-2}, ... a_1, a_0$ are integers

If r is a real zero of f, then r is either rational or irrational. We know that the rational roots of f must be of the form $\frac{p}{q}$ where p is a divisor of a_0 and q is a divisor of 1. This means that $q = \pm 1$. So if r is rational, then $r = \frac{p}{q} = \pm p$. Therefore, r is an integer or r is irrational.

108. Let $\frac{p}{q}$ be a root for the polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1x + a_0$ where $a_n, a_{n-1}, a_{n-2}, ... a_1, a_0$ are integers. Suppose also that p and q have no common factors other than 1 and – 1. Then

$$f\left(\frac{p}{q}\right) = a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + a_{n-2}\left(\frac{p}{q}\right)^{n-2} + ... + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

$$\rightarrow \frac{1}{q^n}\left(a_np^n + a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 + ... + a_1pq^{n-1} + a_0q^n\right) = 0$$

Because p is a factor of the first n terms of this equation, p must also be a factor of a_0q^n. Since p is not a factor of q, p must be a factor of a_0. Similarly, q must be a factor of a_n

109. (a) $f(x) = 8x^4 - 2x^2 + 5x - 1 \quad 0 \le r \le 1$

At Step 0 we have the interval [0,1].

$f(0) = -1; \quad f(1) = 10$

Let m = the midpoint of the interval being considered.

So $m_0 = 0.5$

n	m_{n-1}	$f(m_{n-1})$	New interval
1	0.5	$f(0.5) = 1.5 > 0$	[0,0.5]
2	0.25	$f(0.25) = 0.15625 > 0$	[0,0.25]
3	0.125	$f(0.125) = -0.4043 < 0$	[0.125,0.25]
4	0.1875	$f(0.1875) = -0.1229 < 0$	[0.1875,0.25]
5	0.21875	$f(0.21875) = 0.0164 > 0$	[0.1875,0.21875]
6	0.203125	$f(0.203125) = -0.0533 < 0$	[0.203125,0.21875]
7	0.2109375	$f(0.2109375) = -0.0185 < 0$	[0.2109375,0.21875]
8	0.21484375		

Since the midpoint value at Step 8 agrees with the midpoint value at Step 7 to two decimal places, r = 0.21, correct to 2 decimal places.

(b) $f(x) = x^4 + 8x^3 - x^2 + 2; \quad -1 \le r \le 0$

At Step 0 we have the interval [-1,0].

$f(-1) = -6; \quad f(0) = 2$

Let m = the midpoint of the interval being considered.

So $m_0 = -0.5$

n	m_{n-1}	$f(m_{n-1})$	New interval
1	- 0.5	$f(-0.5) = 0.8125 > 0$	[-1,- 0.5]
2	- 0.75	$f(-0.75) = -1.6211 < 0$	[-0.75,- 0.5]
3	-0.625	$f(-0.625) = -0.1912 < 0$	[-0.625,-0.5]
4	-0.5625	$f(-0.5625) = 0.3599 > 0$	[-0.625,-0.5625]
5	-0.59375	$f(-0.59375) = 0.0972 > 0$	[-0.625,-0.59375]
6	-0.609375	$f(-0.609375) = -0.04372 < 0$	[-0.609375,-0.59375]
7	-0.6015625		

Since the midpoint value at Step 7 agrees with the midpoint value at Step 6 to two decimal places, r = - 0.60, correct to 2 decimal places.

(c) $f(x) = 2x^3 + 6x^2 - 8x + 2; \quad -5 \le r \le -4$

At Step 0 we have the interval [-5,-4].

$f(-5) = -58; \quad f(-4) = 2$

Let m = the midpoint of the interval being considered.

So $m_0 = -4.5$

n	m_{n-1}	$f(m_{n-1})$	New interval
1	- 4.5	$f(-4.5) = -22.75 < 0$	[-4.5,- 4]
2	- 4.25	$f(-4.25) = -9.1562 < 0$	[-4.25,- 4]
3	-4.125	$f(-4.125) = -3.2852 < 0$	[-4.125,-4]
4	-4.0625	$f(-4.0625) = -0.5708 < 0$	[-4.0625,-4]
5	-4.03125	$f(-4.03125) = 0.7324 > 0$	[-4.0625, -4.03125]
6	-4.046875	$f(-4.046875) = 0.0852 > 0$	[-4.0625, -4.046875]
7	-4.0546875	$f(-4.0546875) = -0.2417 < 0$	[-4.0546875, -4.046875]
8	-4.05078125		

Since the midpoint value at Step 8 agrees with the midpoint value at Step 7 to two decimal places, r = - 4.05, correct to 2 decimal places.

(d) $f(x) = 3x^3 - 10x + 9; \quad -3 \le r \le -2$

At Step 0 we have the interval [-3,-2].

$f(-3) = -42; \quad f(-2) = 5$

Let m = the midpoint of the interval being considered.

So $m_0 = -2.5$

n	m_{n-1}	$f(m_{n-1})$	New interval
1	- 2.5	$f(-2.5)=-12.875<0$	[-2.5,- 2]
2	- 2.25	$f(-2.25)=-2.6719<0$	[-2.25,- 2]
3	-2.125	$f(-2.125)=1.4629>0$	[-2.25,- 2.125]
4	-2.1875	$f(-2.1875)=-0.5276<0$	[-2.1875,- 2.125]
5	-2.15625	$f(-2.15625)=0.4866>0$	[-2.1875,-2.15625]
6	-2.171875	$f(-2.171875)=-0.0157<0$	[-2.171875,-2.15625]
7	-2.1640625	$f(-2.1640625)=0.2366>0$	[-2.171875,- 2.1640625]
8	-2.16796875		

Since the midpoint value at Step 8 agrees with the midpoint value at Step 7 to two decimal places, r = - 2.16, correct to 2 decimal places.

(e) $f(x)=x^3+x^2+x-4;\quad 1\le r\le 2$

At Step 0 we have the interval [1,2].

$f(1)=-1;\quad f(2)=10$

Let m = the midpoint of the interval being considered.
So $m_0=1.5$

n	m_{n-1}	$f(m_{n-1})$	New interval
1	1.5	$f(1.5)=3.125>0$	[1,1.5]
2	1.25	$f(1.25)=0.7656>0$	[1,1.25]
3	1.125	$f(1.125)=-0.1855<0$	[1.125,1.25]
4	1.1875	$f(1.1875)=0.2722>0$	[1.125,1.1875]
5	1.15625	$f(1.15625)=0.0390>0$	[1.125,1.15625]
6	1.140625	$f(1.140625)=-0.0744<0$	[1.140625,1.15625]
7	1.1484375		

Since the midpoint value at Step 7 agrees with the midpoint value at Step 6 to two decimal places, r = 1.14, correct to 2 decimal places.

(f) $f(x)=2x^4+x^2-1;\quad 0\le r\le 1$

At Step 0 we have the interval [0,1].

$f(0)=-1;\quad f(1)=2$

Let m = the midpoint of the interval being considered.
So $m_0=0.5$

n	m_{n-1}	$f(m_{n-1})$	New interval
1	0.5	$f(0.5) = -0.625 < 0$	[0.5,1]
2	0.75	$f(0.75) = 0.1593 > 0$	[0.5,0.75]
3	0.625	$f(0.625) = -0.3042 < 0$	[0.625,0.75]
4	0.6875	$f(0.6875) = -0.0805 < 0$	[0.6875,0.75]
5	0.71875	$f(0.71875) = 0.0504 > 0$	[0.6875,0.71875]
6	0.703125	$f(0.703125) = -0.0168 < 0$	[0.703125,0.71875]
7	0.7109375	$f(0.7109375) = 0.0164 > 0$	[0.703125, 0.7109375]
8	0.70703125	$f(0.70703125) = -0.0032 < 0$	[0.70703125, 0.7109375]
9	0.708984375		

Since the midpoint value at Step 9 agrees with the midpoint value at Step 8 to two decimal places, r = 0.70, correct to 2 decimal places.

(g) $f(x) = 2x^4 - 3x^3 - 4x^2 - 8; \qquad 2 \le r \le 3$

At Step 0 we have the interval [2,3].

$f(2) = -16; \quad f(3) = 37$

Let m = the midpoint of the interval being considered.
So $m_0 = 2.5$

n	m_{n-1}	$f(m_{n-1})$	New interval
1	2.5	$f(2.5) = -1.75 < 0$	[2.5,3]
2	2.75	$f(2.75) = 13.7422 > 0$	[2.5,2.75]
3	2.625	$f(2.625) = 5.1352 > 0$	[2.5,2.625]
4	2.5625	$f(2.5625) = 1.4905 > 0$	[2.5,2.5625]
5	2.53125	$f(2.53125) = -0.1787 < 0$	[2.53125,2.5625]
6	2.546875	$f(2.546875) = 0.6435 > 0$	[2.53125, 2.546875]
7	2.5390625	$f(2.5390625) = 0.2293 > 0$	[2.53125, 2.5390625]
8	2.53515625		

Since the midpoint value at Step 8 agrees with the midpoint value at Step 7 to two decimal places, r = 2.53, correct to 2 decimal places.

(h) $f(x) = 3x^3 - 2x^2 - 20; \qquad 2 \le r \le 3$

At Step 0 we have the interval [2,3].

$f(2) = -4; \quad f(3) = 43$

Let m = the midpoint of the interval being considered.
So $m_0 = 2.5$

n	m_{n-1}	$f(m_{n-1})$	New interval
1	2.5	$f(2.5) = 14.375 > 0$	[2,2.5]
2	2.25	$f(2.25) = 4.0469 > 0$	[2,2.25]
3	2.125	$f(2.125) = -0.2441 < 0$	[2.125,2.25]
4	2.1875	$f(2.1875) = 1.8323 > 0$	[2.125,2.1875]
5	2.15625	$f(2.15625) = 0.7771 > 0$	[2.125,2.15625]
6	2.140625	$f(2.140625) = 0.2622 > 0$	[2.125, 2.140625]
7	2.1328125	$f(2.1328125) = 0.0080 > 0$	[2.125, 2.1328125]
8	2.1315625		

Since the midpoint value at Step 8 agrees with the midpoint value at Step 7 to two decimal places, $r = 2.13$, correct to 2 decimal places.