Chapter 5

The Zeros of a Polynomial Function

5.3 Complex Numbers; Quadratic Equations with a Negative Discriminant

1. $(2-3i)+(6+8i)=(2+6)+(-3+8)i=8+5i$

2. $(4+5i)+(-8+2i)=(4+(-8))+(5+2)i=-4+7i$

3. $(-3+2i)-(4-4i)=(-3-4)+(2-(-4))i=-7+6i$

4. $(3-4i)-(-3-4i)=(3-(-3))+(-4-(-4))i=6+0i=6$

5. $(2-5i)-(8+6i)=(2-8)+(-5-6)i=-6-11i$

6. $(-8+4i)-(2-2i)=(-8-2)+(4-(-2))i=-10+6i$

7. $3(2-6i)=6-18i$

8. $-4(2+8i)=-8-32i$

9. $2i(2-3i)=4i-6i^2=4i-6(-1)=6+4i$

10. $3i(-3+4i)=-9i+12i^2=-9i+12(-1)=-12-9i$

11. $(3-4i)(2+i)=6+3i-8i-4i^2=6-5i-4(-1)=10-5i$

12. $(5+3i)(2-i)=10-5i+6i-3i^2=10+i-3(-1)=13+i$

13. $(-6+i)(-6-i)=36+6i-6i-i^2=36-(-1)=37$

14. $(-3+i)(3+i)=-9-3i+3i+i^2=-9+(-1)=-10$

15. $\dfrac{10}{3-4i}=\dfrac{10}{3-4i}\cdot\dfrac{3+4i}{3+4i}=\dfrac{30+40i}{9+12i-12i-16i^2}=\dfrac{30+40i}{9-16(-1)}=\dfrac{30+40i}{25}$
$=\dfrac{30}{25}+\dfrac{40}{25}i=\dfrac{6}{5}+\dfrac{8}{5}i$

16. $\frac{13}{5-12i} = \frac{13}{5-12i}\cdot\frac{5+12i}{5+12i} = \frac{65+156i}{25+60i-60i-144i^2} = \frac{65+156i}{25-144(-1)} = \frac{65+156i}{169}$
$= \frac{65}{169}+\frac{156}{169}i = \frac{5}{13}+\frac{12}{13}i$

17. $\frac{2+i}{i} = \frac{2+i}{i}\cdot\frac{-i}{-i} = \frac{-2i-i^2}{-i^2} = \frac{-2i-(-1)}{-(-1)} = \frac{1-2i}{1} = 1-2i$

18. $\frac{2-i}{-2i} = \frac{2-i}{-2i}\cdot\frac{i}{i} = \frac{2i-i^2}{-2i^2} = \frac{2i-(-1)}{-2(-1)} = \frac{1+2i}{2} = \frac{1}{2}+i$

19. $\frac{6-i}{1+i} = \frac{6-i}{1+i}\cdot\frac{1-i}{1-i} = \frac{6-6i-i+i^2}{1-i+i-i^2} = \frac{6-7i+(-1)}{1-(-1)} = \frac{5-7i}{2} = \frac{5}{2}-\frac{7}{2}i$

20. $\frac{2+3i}{1-i} = \frac{2+3i}{1-i}\cdot\frac{1+i}{1+i} = \frac{2+2i+3i+3i^2}{1+i-i-i^2} = \frac{2+5i+3(-1)}{1-(-1)} = \frac{-1+5i}{2} = -\frac{1}{2}+\frac{5}{2}i$

21. $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2 = \frac{1}{4}+2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}i\right)+\frac{3}{4}i^2 = \frac{1}{4}+\frac{\sqrt{3}}{2}i+\frac{3}{4}(-1) = \frac{-1}{2}+\frac{\sqrt{3}}{2}i$

22. $\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^2 = \frac{3}{4}-2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}i\right)+\frac{1}{4}i^2 = \frac{3}{4}-\frac{\sqrt{3}}{2}i+\frac{1}{4}(-1) = \frac{1}{2}-\frac{\sqrt{3}}{2}i$

23. $(1+i)^2 = 1+2i+i^2 = 1+2i+(-1) = 2i$

24. $(1-i)^2 = 1-2i+i^2 = 1-2i+(-1) = -2i$

25. $i^{23} = i^{22+1} = i^{22}\cdot i = \left(i^2\right)^{11}\cdot i = (-1)^{11}i = -i$

26. $i^{14} = \left(i^2\right)^7 = (-1)^7 = -1$

27. $i^{-15} = \frac{1}{i^{15}} = \frac{1}{i^{14+1}} = \frac{1}{i^{14}\cdot i} = \frac{1}{(i^2)^7\cdot i} = \frac{1}{(-1)^7 i} = \frac{1}{-i} = \frac{1}{-i}\cdot\frac{i}{i} = \frac{i}{-i^2} = \frac{i}{-(-1)} = i$

28. $i^{-23} = \frac{1}{i^{23}} = \frac{1}{i^{22+1}} = \frac{1}{i^{22}\cdot i} = \frac{1}{(i^2)^{11}\cdot i} = \frac{1}{(-1)^{11} i} = \frac{1}{-i} = \frac{1}{-i}\cdot\frac{i}{i} = \frac{i}{-i^2} = \frac{i}{-(-1)} = i$

29. $i^6-5 = \left(i^2\right)^3-5 = (-1)^3-5 = -1-5 = -6$

30. $4+i^3 = 4+i^2\cdot i = 4+(-1)i = 4-i$

31. $6i^3-4i^5 = i^3(6-4i^2) = i^2\cdot i(6-4(-1)) = -1\cdot i(10) = -10i$

32. $4i^3 - 2i^2 + 1 = 4i^2\, i - 2i^2 + 1 = 4(-1)i - 2(-1) + 1 = -4i + 2 + 1 = 3 - 4i$

33. $(1+i)^3 = (1+i)(1+i)(1+i) = (1+2i+i^2)(1+i) = (1+2i-1)(1+i) = 2i(1+i)$
$= 2i + 2i^2 = 2i + 2(-1) = -2 + 2i$

34. $(3i)^4 + 1 = 81i^4 + 1 = 81(1) + 1 = 82$

35. $i^7(1+i^2) = i^7(1+(-1)) = i^7(0) = 0$

36. $2i^4(1+i^2) = 2(1)(1 + (-1)) = 2(0) = 0$

37. $i^6 + i^4 + i^2 + 1 = \left(i^2\right)^3 + \left(i^2\right)^2 + i^2 + 1 = (-1)^3 + (-1)^2 + (-1) + 1 = -1 + 1 - 1 + 1 = 0$

38. $i^7 + i^5 + i^3 + i = \left(i^2\right)^3\, i + \left(i^2\right)^2\, i + i^2\, i + i = (-1)^3\, i + (-1)^2\, i + (-1)\, i + i$
$= -i + i - i + i = 0$

39. $\sqrt{-4} = 2i$ 40. $\sqrt{-9} = 3i$ 41. $\sqrt{-25} = 5i$ 42. $\sqrt{-64} = 8i$

43. $\sqrt{(3+4i)(4i-3)} = \sqrt{12i - 9 + 16i^2 - 12i} = \sqrt{-9 + 16(-1)} = \sqrt{-25} = 5i$

44. $\sqrt{(4+3i)(3i-4)} = \sqrt{12i - 16 + 9i^2 - 12i} = \sqrt{-16 + 9(-1)} = \sqrt{-25} = 5i$

45. $x^2 + 4 = 0$
$a = 1,\ b = 0,\ c = 4,\quad b^2 - 4ac = 0^2 - 4(1)(4) = -16$
$x = \dfrac{-0 \pm \sqrt{-16}}{2(1)} = \dfrac{\pm 4i}{2} = \pm 2i$ The solution set is $\{\pm 2i\}$.

46. $x^2 - 4 = 0$
$(x+2)(x-2) = 0 \to x = -2$ or $x = 2$ The solution set is $\{\pm 2\}$.

47. $x^2 - 16 = 0$
$a = 1,\ b = 0,\ c = -16,\quad b^2 - 4ac = 0^2 - 4(1)(-16) = 64$
$x = \dfrac{-0 \pm \sqrt{64}}{2(1)} = \dfrac{\pm 8}{2} = \pm 4$ The solution set is $\{\pm 4\}$.

48. $x^2 + 25 = 0$
$x^2 = -25 \to x = \pm\sqrt{-25} = \pm 5i$ The solution set is $\{\pm 5i\}$.

49. $x^2 - 6x + 13 = 0$
$a = 1,\ b = -6,\ c = 13,\quad b^2 - 4ac = (-6)^2 - 4(1)(13) = 36 - 52 = -16$
$x = \dfrac{-(-6) \pm \sqrt{-16}}{2(1)} = \dfrac{6 \pm 4i}{2} = 3 \pm 2i$ The solution set is $\{3 - 2i, 3 + 2i\}$.

50. $x^2 + 4x + 8 = 0$
$a = 1,\ b = 4,\ c = 8,\quad b^2 - 4ac = 4^2 - 4(1)(8) = 16 - 32 = -16$
$x = \dfrac{-4 \pm \sqrt{-16}}{2(1)} = \dfrac{-4 \pm 4i}{2} = -2 \pm 2i$
The solution set is $\{-2 - 2i, -2 + 2i\}$.

51. $x^2 - 6x + 10 = 0$
$a = 1,\ b = -6,\ c = 10,\quad b^2 - 4ac = (-6)^2 - 4(1)(10) = 36 - 40 = -4$
$x = \dfrac{-(-6) \pm \sqrt{-4}}{2(1)} = \dfrac{6 \pm 2i}{2} = 3 \pm i$
The solution set is $\{3 - i, 3 + i\}$.

52. $x^2 - 2x + 5 = 0$
$a = 1,\ b = -2,\ c = 5,\quad b^2 - 4ac = (-2)^2 - 4(1)(5) = 4 - 20 = -16$
$x = \dfrac{-(-2) \pm \sqrt{-16}}{2(1)} = \dfrac{2 \pm 4i}{2} = 1 \pm 2i$
The solution set is $\{1 - 2i, 1 + 2i\}$.

53. $8x^2 - 4x + 1 = 0$
$a = 8,\ b = -4,\ c = 1,\quad b^2 - 4ac = (-4)^2 - 4(8)(1) = 16 - 32 = -16$
$x = \dfrac{-(-4) \pm \sqrt{-16}}{2(8)} = \dfrac{4 \pm 4i}{16} = \dfrac{1}{4} \pm \dfrac{1}{4}i$
The solution set is $\left\{\dfrac{1}{4} - \dfrac{1}{4}i,\ \dfrac{1}{4} + \dfrac{1}{4}i\right\}$.

54. $10x^2 + 6x + 1 = 0$
$a = 10,\ b = 6,\ c = 1,\quad b^2 - 4ac = 6^2 - 4(10)(1) = 36 - 40 = -4$
$x = \dfrac{-6 \pm \sqrt{-4}}{2(10)} = \dfrac{-6 \pm 2i}{20} = -\dfrac{3}{10} \pm \dfrac{1}{10}i$
The solution set is $\left\{-\dfrac{3}{10} - \dfrac{1}{10}i,\ -\dfrac{3}{10} + \dfrac{1}{10}i\right\}$.

55. $5x^2 + 1 = 2x \rightarrow 5x^2 - 2x + 1 = 0$
$a = 5, b = -2, c = 1,\quad b^2 - 4ac = (-2)^2 - 4(5)(1) = 4 - 20 = -16$
$x = \dfrac{-(-2) \pm \sqrt{-16}}{2(5)} = \dfrac{2 \pm 4i}{10} = \dfrac{1}{5} \pm \dfrac{2}{5}i$
The solution set is $\left\{\dfrac{1}{5} - \dfrac{2}{5}i,\ \dfrac{1}{5} + \dfrac{2}{5}i\right\}$.

56. $13x^2 + 1 = 6x \rightarrow 13x^2 - 6x + 1 = 0$

$a = 13,\ b = -6,\ c = 1,\quad b^2 - 4ac = (-6)^2 - 4(13)(1) = 36 - 52 = -16$

$$x = \frac{-(-6) \pm \sqrt{-16}}{2(13)} = \frac{6 \pm 4i}{26} = \frac{3}{13} \pm \frac{2}{13} i$$

The solution set is $\left\{ \frac{3}{13} - \frac{2}{13} i,\ \frac{3}{13} + \frac{2}{13} i \right\}$.

57. $x^2 + x + 1 = 0$

$a = 1,\ b = 1,\ c = 1,\quad b^2 - 4ac = 1^2 - 4(1)(1) = 1 - 4 = -3$

$$x = \frac{-1 \pm \sqrt{-3}}{2(1)} = \frac{-1 \pm \sqrt{3}\, i}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$$

The solution set is $\left\{ \frac{-1}{2} - \frac{\sqrt{3}}{2} i,\ \frac{-1}{2} + \frac{\sqrt{3}}{2} i \right\}$.

58. $x^2 - x + 1 = 0$

$a = 1,\ b = -1,\ c = 1,\quad b^2 - 4ac = (-1)^2 - 4(1)(1) = 1 - 4 = -3$

$$x = \frac{-(-1) \pm \sqrt{-3}}{2(1)} = \frac{1 \pm \sqrt{3}\, i}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

The solution set is $\left\{ \frac{1}{2} - \frac{\sqrt{3}}{2} i,\ \frac{1}{2} + \frac{\sqrt{3}}{2} i \right\}$.

59. $x^3 - 8 = 0$

$(x - 2)\left(x^2 + 2x + 4\right) = 0$

$x - 2 = 0$ or $x^2 + 2x + 4 = 0$

$x = 2$

$a = 1,\ b = 2,\ c = 4,\quad b^2 - 4ac = 2^2 - 4(1)(4) = 4 - 16 = -12$

$$x = \frac{-2 \pm \sqrt{-12}}{2(1)} = \frac{-2 \pm 2\sqrt{3}\, i}{2} = -1 \pm \sqrt{3} i$$

The solution set is $\left\{ 2,\ -1 - \sqrt{3} i,\ -1 + \sqrt{3} i \right\}$.

60. $x^3 + 27 = 0$

$(x + 3)\left(x^2 - 3x + 9\right) = 0$

$x + 3 = 0$ or $x^2 - 3x + 9 = 0$

$x = -3$

$a = 1,\ b = -3,\ c = 9,\quad b^2 - 4ac = (-3)^2 - 4(1)(9) = 9 - 36 = -27$

$$x = \frac{-(-3) \pm \sqrt{-27}}{2(1)} = \frac{3 \pm 3\sqrt{3}\, i}{2} = \frac{3}{2} \pm \frac{3\sqrt{3}}{2} i$$

The solution set is $\left\{ -3,\ \frac{3}{2} - \frac{3\sqrt{3}}{2} i,\ \frac{3}{2} + \frac{3\sqrt{3}}{2} i \right\}$.

61. $x^4 = 16 \to x^4 - 16 = 0$

$$(x^2 - 4)(x^2 + 4) = 0$$
$$(x-2)(x+2)(x^2 + 4) = 0$$
$$x - 2 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x^2 + 4 = 0$$
$$x = 2 \quad \text{or} \quad x = -2$$
$$a = 1,\ b = 0,\ c = 4, \quad b^2 - 4ac = 0^2 - 4(1)(4) = 0 - 16 = -16$$
$$x = \frac{-0 \pm \sqrt{-16}}{2(1)} = \frac{\pm 4i}{2} = \pm 2i$$

The solution set is $\{-2,\ 2,\ -2i,\ 2i\}$.

62. $x^4 = 1 \to x^4 - 1 = 0$

$$(x^2 - 1)(x^2 + 1) = 0$$
$$(x-1)(x+1)(x^2 + 1) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{or} \quad x^2 + 1 = 0$$
$$x = 1 \quad \text{or} \quad x = -1 \quad \text{or} \quad x^2 = -1$$
$$x = 1 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = \pm i$$

The solution set is $\{-1,\ 1,\ -i,\ i\}$.

63. $x^4 + 13x^2 + 36 = 0$

$$(x^2 + 9)(x^2 + 4) = 0$$
$$x^2 + 9 = 0 \quad \text{or} \quad x^2 + 4 = 0$$
$$a = 1,\ b = 0,\ c = 9, \quad b^2 - 4ac = 0^2 - 4(1)(9) = 0 - 36 = -36$$
$$x = \frac{-0 \pm \sqrt{-36}}{2(1)} = \frac{\pm 6i}{2} = \pm 3i$$
$$a = 1,\ b = 0,\ c = 4, \quad b^2 - 4ac = 0^2 - 4(1)(4) = 0 - 16 = -16$$
$$x = \frac{-0 \pm \sqrt{-16}}{2(1)} = \frac{\pm 4i}{2} = \pm 2i$$

The solution set is $\{-3i,\ 3i,\ -2i,\ 2i\}$.

64. $x^4 + 3x^2 - 4 = 0$

$$(x^2 - 1)(x^2 + 4) = 0$$
$$(x-1)(x+1)(x^2 + 4) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{or} \quad x^2 + 4 = 0$$
$$x = 1 \quad \text{or} \quad x = -1 \quad \text{or} \quad x^2 = -4$$
$$x = 1 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = \pm 2i$$

The solution set is $\{-1,\ 1,\ -2i,\ 2i\}$.

65. $3x^2 - 3x + 4 = 0$
$a = 3,\ b = -3,\ c = 4,\quad b^2 - 4ac = (-3)^2 - 4(3)(4) = 9 - 48 = -39$
The equation has two complex conjugate solutions.

66. $2x^2 - 4x + 1 = 0$
$a = 2,\ b = -4,\ c = 1,\quad b^2 - 4ac = (-4)^2 - 4(2)(1) = 16 - 8 = 8$
The equation has two unequal real number solutions.

67. $2x^2 + 3x - 4 = 0$
$a = 2,\ b = 3,\ c = -4,\quad b^2 - 4ac = 3^2 - 4(2)(-4) = 9 + 32 = 41$
The equation has two unequal real solutions.

68. $x^2 + 6 = 2x \rightarrow x^2 - 2x + 6 = 0$
$a = 1,\ b = -2, c = 6,\quad b^2 - 4ac = (-2)^2 - 4(1)(6) = 4 - 24 = -20$
The equation has two complex conjugate solutions.

69. $9x^2 - 12x + 4 = 0$
$a = 9,\ b = -12,\ c = 4,\quad b^2 - 4ac = (-12)^2 - 4(9)(4) = 144 - 144 = 0$
The equation has a repeated real solution.

70. $4x^2 + 12x + 9 = 0$
$a = 4,\ b = 12,\ c = 9,\quad b^2 - 4ac = 12^2 - 4(4)(9) = 144 - 144 = 0$
The equation has a repeated real solution.

71. The other solution is the conjugate of $2 + 3i$, or $2 - 3i$.

72. The other solution is the conjugate of $4 - i$, or $4 + i$.

73. $z + \bar{z} = 3 - 4i + \overline{3 - 4i} = 3 - 4i + 3 + 4i = 6$

74. $w - \bar{w} = 8 + 3i - \left(\overline{8 + 3i}\right) = 8 + 3i - (8 - 3i) = 8 + 3i - 8 + 3i = 0 + 6i = 6i$

75. $z\,\bar{z} = (3 - 4i)(\overline{3 - 4i}) = (3 - 4i)(3 + 4i) = 9 + 12i - 12i - 16i^2 = 9 - 16(-1) = 25$

76. $\overline{z - w} = \overline{3 - 4i - (8 + 3i)} = \overline{3 - 4i - 8 - 3i} = \overline{-5 - 7i} = -5 + 7i$

77. $z + \bar{z} = a + bi + \overline{a + bi} = a + bi + a - bi = 2a$
$z - \bar{z} = a + bi - (\overline{a + bi}) = a + bi - (a - bi) = a + bi - a + bi = 2bi$

78. $\bar{\bar{z}} = \overline{\overline{a + bi}} = \overline{a - bi} = a + bi = z$

79. $\overline{z + w} = \overline{(a + bi) + (c + di)} = \overline{(a + c) + (b + d)i} = (a + c) - (b + d)i$
$= (a - bi) + (c - di) = \overline{a + bi} + \overline{c + di} = \bar{z} + \bar{w}$

80. $\overline{z\,w} = \overline{(a+bi)\,(c+di)} = \overline{ac + adi + bci + bdi^2} = \overline{(ac-bd)+(ad+bc)i}$
$= (ac-bd)-(ad+bc)i$
$\overline{z}\;\overline{w} = \overline{a+bi}\;\overline{c+di} = (a-bi)(c-di) = ac - adi - bci + bdi^2$
$= (ac-bd)-(ad+bc)i$

81. Answers will vary.

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