Chapter 5

The Zeros of a Polynomial Function

5.R Chapter Review

1.
$$\begin{array}{r|rrrr} 1 & 8 & -3 & 1 & 4 \\ & & 8 & 5 & 6 \\ \hline & 8 & 5 & 6 & 10 \end{array}$$

$$8x^3 - 3x^2 + x + 4 = (x-1)(8x^2 + 5x + 6) + \frac{10}{x-1}$$

$$q(x) = 8x^2 + 5x + 6; \qquad R = \frac{10}{x-1}$$

2.
$$\begin{array}{r|rrrr} 2 & 2 & 8 & -5 & 5 \\ & & 4 & 24 & 38 \\ \hline & 2 & 12 & 19 & 43 \end{array}$$

$$2x^3 + 8x^2 - 5x + 5 = (x-2)(2x^2 + 12x + 19) + \frac{43}{x-2}$$

$$q(x) = 2x^2 + 12x + 19; \qquad R = \frac{43}{x-2}$$

3.
$$\begin{array}{r|rrrrr} -2 & 1 & -2 & 0 & 1 & -1 \\ & & -2 & 8 & -16 & 30 \\ \hline & 1 & -4 & 8 & -15 & 29 \end{array}$$

$$x^4 - 2x^3 + x - 1 = (x+2)(x^3 - 4x^2 + 8x - 15) + \frac{29}{x+2}$$

$$q(x) = x^3 - 4x^2 + 8x - 15; \qquad R = \frac{29}{x+2}$$

4.
$$\begin{array}{r|rrrrr} -1 & 1 & 0 & -1 & 3 & 0 \\ & & -1 & 1 & 0 & -3 \\ \hline & 1 & -1 & 0 & 3 & -3 \end{array}$$

$$x^4 - x^2 + 3x = (x+1)(x^3 - x^2 + 3) + \frac{-3}{x+1}$$

$$q(x) = x^3 - x^2 + 3; \qquad R = \frac{-3}{x+1}$$

5. $f(x)=12x^6-8x^4+1$ at $x=4$

$$\begin{array}{r|rrrrrrr} 4 & 12 & 0 & -8 & 0 & 0 & 0 & 1 \\ & & 48 & 192 & 736 & 2944 & 11776 & 47104 \\ \hline & 12 & 48 & 184 & 736 & 2944 & 11776 & 47105 \end{array}$$

$f(4)=47105$

6. $f(x)=-16x^3+18x^2-x+2$ at $x=-2$

$$\begin{array}{r|rrrr} -2 & -16 & 18 & -1 & 2 \\ & & 32 & -100 & 202 \\ \hline & -16 & 50 & -101 & 204 \end{array}$$

$f(-2)=204$

7. $f(x)=12x^8-x^7+8x^4-2x^3+x+3$
Examining $f(x)$, there are 4 variations in sign; thus, there are 4 or 2 or 0 positive real zeros.
Examining $f(-x)=12(-x)^8-(-x)^7+8(-x)^4-2(-x)^3+(-x)+3$
$=12x^8+x^7+8x^4+2x^3-x+3$, there are 2 variations in sign; thus, there are 2 or 0 negative real zeros.

8. $f(x)=-6x^5+x^4+5x^3+x+1$
Examining $f(x)$, there is 1 variation in sign; thus, there is 1 positive real zero.
Examining $f(-x)=-6(-x)^5+(-x)^4+5(-x)^3+(-x)+1=6x^5+x^4-5x^3-x+1$, there are 2 variations in sign; thus, there are 2 or 0 negative real zeros.

9. $f(x)=12x^8-x^7+6x^4-x^3+x-3$
p must be a factor of –3: $p=\pm1, \pm3$
q must be a factor of 12: $q=\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$
The possible rational zeros are: $\dfrac{p}{q}=\pm1, \pm3, \pm\dfrac{1}{2}, \pm\dfrac{3}{2}, \pm\dfrac{1}{3}, \pm\dfrac{1}{4}, \pm\dfrac{3}{4}, \pm\dfrac{1}{6}, \pm\dfrac{1}{12}$

10. $f(x)=-6x^5+x^4+2x^3-x+1$
p must be a factor of 1: $p=\pm1$
q must be a factor of –6: $q=\pm1, \pm2, \pm3, \pm6$
The possible rational zeros are: $\dfrac{p}{q}=\pm1, \pm\dfrac{1}{2}, \pm\dfrac{1}{3}, \pm\dfrac{1}{6}$

11. $f(x)=x^3-3x^2-6x+8$

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: By Descartes Rule of Signs, there are 2 or 0 positive real zeros. Also because $f(-x)=(-x)^3-3(-x)^2-6(-x)+8=-x^3-3x^2+6x+8$, there is 1 negative real zero.

Step 3: Possible rational zeros:

$$p=\pm1,\ \pm2,\ \pm4,\ \pm8;\quad q=\pm1;\quad \frac{p}{q}=\pm1,\ \pm2,\ \pm4,\ \pm8$$

Step 4: Using the Bounds on Zeros Theorem:

$a_2=-3,\quad a_1=-6,\quad a_0=8$

$\text{Max}\{1,\ |8|+|-6|+|-3|\}=\text{Max}\{1,\ 17\}=17$

$1+\text{Max}\{|8|,|-6|,|-3|\}=1+8=9$

The smaller of the two numbers is 9. Thus, every zero of f lies between –9 and 9.

Step 5: Using synthetic division:

$$\begin{array}{r|rrrr} -2 & 1 & -3 & -6 & 8 \\ & & -2 & 10 & -8 \\ \hline & 1 & -5 & 4 & 0 \end{array}$$

Since the remainder is 0, $x-(-2)=x+2$ is a factor. The other factor is the quotient: x^2-5x+4.

Thus, $f(x)=(x+2)(x^2-5x+4)=(x+2)(x-1)(x-4)$.

The zeros are –2, 1, and 4.

12. $f(x)=x^3-x^2-10x-8$

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: By Descartes Rule of Signs, there is 1 positive real zero. Also because $f(-x)=(-x)^3-(-x)^2-10(-x)-8=-x^3-x^2+10x-8$, there are 2 or 0 negative real zeros.

Step 3: Possible rational zeros:

$$p=\pm1,\ \pm2,\ \pm4,\ \pm8;\quad q=\pm1;\quad \frac{p}{q}=\pm1,\ \pm2,\ \pm4,\ \pm8$$

Step 4: Using the Bounds on Zeros Theorem:

$a_2=-1,\quad a_1=-10,\quad a_0=-8$

$\text{Max}\{1,\ |-8|+|-10|+|-1|\}=\text{Max}\{1,\ 19\}=19$

$1+\text{Max}\{|-8|,|-10|,|-1|\}=1+10=11$

The smaller of the two numbers is 11. Thus, every zero of f lies between –11 and 11.

Step 5: Using synthetic division:

$$\begin{array}{r|rrrr} -2 & 1 & -1 & -10 & -8 \\ & & -2 & 6 & 8 \\ \hline & 1 & -3 & -4 & 0 \end{array}$$

Since the remainder is 0, $x-(-2)=x+2$ is a factor. The other factor is the quotient: x^2-3x-4.

Thus, $f(x)=(x+2)(x^2-3x-4)=(x+2)(x+1)(x-4)$.

The zeros are –2, –1, and 4.

13. $f(x) = 4x^3 + 4x^2 - 7x + 2$

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: By Descartes Rule of Signs, there are 2 or 0 positive real zeros.
$f(-x) = 4(-x)^3 + 4(-x)^2 - 7(-x) + 2 = -4x^3 + 4x^2 + 7x + 2$; thus, there is 1 negative real zero.

Step 3: Possible rational zeros:

$$p = \pm 1, \pm 2; \quad q = \pm 1, \pm 2, \pm 4; \quad \frac{p}{q} = \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{4}$$

Step 4: Using the Bounds on Zeros Theorem:

$$f(x) = 4\left(x^3 + x^2 - \tfrac{7}{4}x + \tfrac{1}{2}\right)$$

$$a_2 = 1, \quad a_1 = \tfrac{-7}{4}, \quad a_0 = \tfrac{1}{2}$$

$$\text{Max}\left\{1, \left|\tfrac{1}{2}\right| + \left|\tfrac{-7}{4}\right| + |1|\right\} = \text{Max}\left\{1, \tfrac{13}{4}\right\} = 3.25$$

$$1 + \text{Max}\left\{\left|\tfrac{1}{2}\right|, \left|\tfrac{-7}{4}\right|, |1|\right\} = 1 + \tfrac{7}{4} = \tfrac{11}{4} = 2.75$$

The smaller of the two numbers is 2.75. Thus, every zero of f lies between –2.75 and 2.75.

Step 5: Using synthetic division:

$$\begin{array}{r|rrrr} -2 & 4 & 4 & -7 & 2 \\ & & -8 & 8 & -2 \\ \hline & 4 & -4 & 1 & 0 \end{array}$$

Since the remainder is 0, $x - (-2) = x + 2$ is a factor. The other factor is the quotient: $4x^2 - 4x + 1$.

Thus, $f(x) = (x+2)\left(4x^2 - 4x + 1\right) = (x+2)(2x-1)(2x-1)$.

The zeros are –2 and $\frac{1}{2}$ (multiplicity 2).

14. $f(x) = 4x^3 - 4x^2 - 7x - 2$

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: By Descartes Rule of Signs, there is 1 positive real zero.
$f(-x) = 4(-x)^3 - 4(-x)^2 - 7(-x) - 2 = -4x^3 - 4x^2 + 7x - 2$; thus, there are 2 or 0 negative real zeros.

Step 3: Possible rational zeros:

$$p = \pm 1, \pm 2; \quad q = \pm 1, \pm 2, \pm 4; \quad \frac{p}{q} = \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{4}$$

Step 4: Using the Bounds on Zeros Theorem:

$$f(x) = 4\left(x^3 - x^2 - \tfrac{7}{4}x - \tfrac{1}{2}\right)$$

$$a_2 = -1, \quad a_1 = \tfrac{-7}{4}, \quad a_0 = -\tfrac{1}{2}$$

$$\text{Max}\left\{1, \left|-\tfrac{1}{2}\right| + \left|-\tfrac{7}{4}\right| + |-1|\right\} = \text{Max}\left\{1, \tfrac{13}{4}\right\} = 3.25$$

$$1 + \text{Max}\left\{\left|-\tfrac{1}{2}\right|, \left|\tfrac{-7}{4}\right|, |-1|\right\} = 1 + \tfrac{7}{4} = \tfrac{11}{4} = 2.75$$

The smaller of the two numbers is 2.75. Thus, every zero of f lies between –2.75 and 2.75.

Step 5: Using synthetic division:

$$\begin{array}{r|rrrr} 2 & 4 & -4 & -7 & -2 \\ & & 8 & 8 & 2 \\ \hline & 4 & 4 & 1 & 0 \end{array}$$

Since the remainder is 0, $x-2$ is a factor. The other factor is the quotient: $4x^2+4x+1$.

Thus, $f(x)=(x-2)\left(4x^2+4x+1\right)=(x-2)(2x+1)(2x+1)$.

The zeros are 2 and $-\frac{1}{2}$ (multiplicity 2).

15. $f(x)=x^4-4x^3+9x^2-20x+20$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there are 4 or 2 or 0 positive real zeros.

$$f(-x)=(-x)^4-4(-x)^3+9(-x)^2-20(-x)+20$$
$$=x^4+4x^3+9x^2+20x+20$$

thus, there are no negative real zeros.

Step 3: Possible rational zeros:

$p=\pm1, \pm2, \pm4, \pm5, \pm10, \pm20;\quad q=\pm1;$

$\frac{p}{q}=\pm1, \pm2, \pm4, \pm5, \pm10, \pm20$

Step 4: Using the Bounds on Zeros Theorem:

$a_3=-4,\quad a_2=9,\quad a_1=-20,\quad a_0=20$

$\text{Max}\left\{1, |20|+|-20|+|9|+|-4|\right\}=\text{Max}\{1, 53\}=53$

$1+\text{Max}\left\{|20|, |-20|, |9|, |-4|\right\}=1+20=21$

The smaller of the two numbers is 21. Thus, every zero of f lies between –21 and 21

Step 5: Using synthetic division:

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 9 & -20 & 20 \\ & & 2 & -4 & 10 & -20 \\ \hline & 1 & -2 & 5 & -10 & 0 \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & -2 & 5 & -10 \\ & & 2 & 0 & 10 \\ \hline & 1 & 0 & 5 & 0 \end{array}$$

Since the remainder is 0, $x-2$ is a factor twice. The other factor is the quotient: x^2+5.

Thus, $f(x)=(x-2)(x-2)\left(x^2+5\right)=(x-2)^2\left(x^2+5\right)$.

The zero is 2 (multiplicity 2). ($x^2+5=0$ has no real solutions.)

16. $f(x)=x^4+6x^3+11x^2+12x+18$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there are no positive real zeros.

$$f(-x)=(-x)^4+6(-x)^3+11(-x)^2+12(-x)+18$$
$$=x^4-6x^3+11x^2-12x+18$$

thus, there are 4 or 2 or 0 negative real zeros.

Step 3: Possible rational zeros:

$p=\pm1, \pm2, \pm3, \pm6, \pm9, \pm18;\quad q=\pm1;$

$\frac{p}{q}=\pm1, \pm2, \pm3, \pm6, \pm9, \pm18$

Step 4: Using the Bounds on Zeros Theorem:

$a_3 = 6, \quad a_2 = 11, \quad a_1 = 12, \quad a_0 = 18$

$\text{Max}\left\{1, |18| + |12| + |11| + |6|\right\} = \text{Max}\left\{1, 47\right\} = 47$

$1 + \text{Max}\left\{ |18|, |12|, |11|, |6|\right\} = 1 + 18 = 19$

The smaller of the two numbers is 19. Thus, every zero of f lies between –19 and 19.

Step 5: Using synthetic division:

$$\begin{array}{r|rrrrr} -3 & 1 & 6 & 11 & 12 & 18 \\ & & -3 & -9 & -6 & -18 \\ \hline & 1 & 3 & 2 & 6 & 0 \end{array} \qquad \begin{array}{r|rrrr} -3 & 1 & 3 & 2 & 6 \\ & & -3 & 0 & -6 \\ \hline & 1 & 0 & 2 & 0 \end{array}$$

Since the remainder is 0, $x - (-3) = x + 3$ is a factor twice. The other factor is the quotient: $x^2 + 2$.

Thus, $f(x) = (x+3)(x+3)\left(x^2 + 2\right) = (x+3)^2\left(x^2 + 2\right)$.

The zero is –3 (multiplicity 2). ($x^2 + 2 = 0$ has no real solutions.)

17. $2x^4 + 2x^3 - 11x^2 + x - 6 = 0$

The solutions of the equation are the zeros of $f(x) = 2x^4 + 2x^3 - 11x^2 + x - 6$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there are 3 or 1 positive real zeros.

$f(-x) = 2(-x)^4 + 2(-x)^3 - 11(-x)^2 + (-x) - 6 = 2x^4 - 2x^3 - 11x^2 - x - 6$;

thus, there is 1 negative real zero.

Step 3: Possible rational zeros:

$$p = \pm 1, \pm 2, \pm 3, \pm 6; \quad q = \pm 1, \pm 2; \quad \frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$$

Step 4: Using the Bounds on Zeros Theorem:

$f(x) = 2\left(x^4 + x^3 - \tfrac{11}{2}x^2 + \tfrac{1}{2}x - 3\right)$

$a_3 = 1, \quad a_2 = -\tfrac{11}{2}, \quad a_1 = \tfrac{1}{2}, \quad a_0 = -3$

$\text{Max}\left\{1, |-3| + \left|\tfrac{1}{2}\right| + \left|-\tfrac{11}{2}\right| + |1|\right\} = \text{Max}\left\{1, 10\right\} = 10$

$1 + \text{Max}\left\{ |-3|, \left|\tfrac{1}{2}\right|, \left|-\tfrac{11}{2}\right|, |1|\right\} = 1 + \tfrac{11}{2} = \tfrac{13}{2} = 6.5$

The smaller of the two numbers is 6.5. Thus, every zero of f lies between –6.5 and 6.5.

Step 5: Using synthetic division:

$$\begin{array}{r|rrrrr} -3 & 2 & 2 & -11 & 1 & -6 \\ & & -6 & 12 & -3 & 6 \\ \hline & 2 & -4 & 1 & -2 & 0 \end{array} \qquad \begin{array}{r|rrrr} 2 & 2 & -4 & 1 & -2 \\ & & 4 & 0 & 2 \\ \hline & 2 & 0 & 1 & 0 \end{array}$$

Since the remainder is 0, $x + 3$ and $x - 2$ are factors. The other factor is the quotient: $2x^2 + 1$. The zeros are –3 and 2. ($2x^2 + 1 = 0$ has no real solutions.)

18. $3x^4 + 3x^3 - 17x^2 + x - 6 = 0$

The solutions of the equation are the zeros of $f(x) = 3x^4 + 3x^3 - 17x^2 + x - 6$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there are 3 or 1 positive real zeros.
$f(-x)=3(-x)^4+3(-x)^3-17(-x)^2+(-x)-6=3x^4-3x^3-17x^2-x-6$;
thus, there is 1 negative real zero.

Step 3: Possible rational zeros:

$$p=\pm 1,\ \pm 2,\ \pm 3,\ \pm 6;\quad q=\pm 1,\ \pm 3;\quad \frac{p}{q}=\pm 1,\ \pm 2,\ \pm 3,\ \pm 6,\ \pm\frac{1}{3},\ \pm\frac{2}{3}$$

Step 4: Using the Bounds on Zeros Theorem:

$$f(x)=3\left(x^4+x^3-\tfrac{17}{3}x^2+\tfrac{1}{3}x-2\right)$$
$$a_3=1,\quad a_2=-\tfrac{17}{3},\quad a_1=\tfrac{1}{3},\quad a_0=-2$$
$$\text{Max}\left\{1,\ |-2|+|\tfrac{1}{3}|+|-\tfrac{17}{3}|+|1|\right\}=\text{Max}\{1,\ 9\}=9$$
$$1+\text{Max}\left\{|-2|,|\tfrac{1}{3}|,|-\tfrac{17}{3}|,|1|\right\}=1+\tfrac{17}{3}=\tfrac{20}{3}$$

The smaller of the two numbers is 6.67. Thus, every zero of f lies between –6.67 and 6.67.

Step 5: Using synthetic division:

$$\begin{array}{r|rrrrr} -3 & 3 & 3 & -17 & 1 & -6 \\ & & -9 & 18 & -3 & 6 \\ \hline & 3 & -6 & 1 & -2 & 0 \end{array} \qquad \begin{array}{r|rrrr} 2 & 3 & -6 & 1 & -2 \\ & & 6 & 0 & 2 \\ \hline & 3 & 0 & 1 & 0 \end{array}$$

Since the remainder is 0, $x+3$ and $x-2$ are factors. The other factor is the quotient: $3x^2+1$.
The zeros are –3 and 2. ($3x^2+1=0$ has no real solutions.)

19. $2x^4+7x^3+x^2-7x-3=0$
The solutions of the equation are the zeros of $f(x)=2x^4+7x^3+x^2-7x-3$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there is 1 positive real zero.
$f(-x)=2(-x)^4+7(-x)^3+(-x)^2-7(-x)-3=2x^4-7x^3+x^2+7x-3$;
thus, there are 3 or 1 negative real zeros.

Step 3: Possible rational zeros:

$$p=\pm 1,\ \pm 3;\quad q=\pm 1,\ \pm 2;\quad \frac{p}{q}=\pm 1,\ \pm 3,\ \pm\frac{1}{2},\ \pm\frac{3}{2}$$

Step 4: Using the Bounds on Zeros Theorem:

$$f(x)=2\left(x^4+\tfrac{7}{2}x^3+\tfrac{1}{2}x^2-\tfrac{7}{2}x-\tfrac{3}{2}\right)$$
$$a_3=\tfrac{7}{2},\quad a_2=\tfrac{1}{2},\quad a_1=-\tfrac{7}{2},\quad a_0=-\tfrac{3}{2}$$
$$\text{Max}\left\{1,\ |-\tfrac{3}{2}|+|-\tfrac{7}{2}|+|\tfrac{1}{2}|+|\tfrac{7}{2}|\right\}=\text{Max}\{1,\ 9\}=9$$
$$1+\text{Max}\left\{|-\tfrac{3}{2}|,|-\tfrac{7}{2}|,|\tfrac{1}{2}|,|\tfrac{7}{2}|\right\}=1+\tfrac{7}{2}=\tfrac{9}{2}=4.5$$

The smaller of the two numbers is 4.5. Thus, every zero of f lies between –4.5 and 4.5.

Step 5: Using synthetic division:

$$\begin{array}{r|rrrrr} -3 & 2 & 7 & 1 & -7 & -3 \\ & & -6 & -3 & 6 & 3 \\ \hline & 2 & 1 & -2 & -1 & 0 \end{array} \qquad \begin{array}{r|rrrr} -1 & 2 & 1 & -2 & -1 \\ & & -2 & 1 & 1 \\ \hline & 2 & -1 & -1 & 0 \end{array}$$

Since the remainder is 0, $x+3$ and $x+1$ are factors. The other factor is the quotient: $2x^2-x-1$.

Thus, $f(x)=(x+3)(x+1)\left(2x^2-x-1\right)=(x+3)(x+1)(2x+1)(x-1)$.

The zeros are –3, –1, $\frac{-1}{2}$, and 1.

20. $2x^4+7x^3-5x^2-28x-12=0$

The solutions of the equation are the zeros of $f(x)=2x^4+7x^3-5x^2-28x-12$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there is 1 positive real zero.

$$\begin{aligned}f(-x)&=2(-x)^4+7(-x)^3-5(-x)^2-28(-x)-12\\&=2x^4-7x^3-5x^2+28x-12\end{aligned};$$

thus, there are 3 or 1 negative real zeros.

Step 3: Possible rational zeros:

$$p=\pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm6,\ \pm12;\quad q=\pm1,\ \pm2;$$

$$\frac{p}{q}=\pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm6,\ \pm12,\ \pm\frac{1}{2},\ \pm\frac{3}{2}$$

Step 4: Using the Bounds on Zeros Theorem:

$$f(x)=2\left(x^4+\tfrac{7}{2}x^3-\tfrac{5}{2}x^2-14x-6\right)$$

$$a_3=\tfrac{7}{2},\quad a_2=-\tfrac{5}{2},\quad a_1=-14,\quad a_0=-6$$

$$\text{Max}\left\{1,\ |-6|+|-14|+\left|-\tfrac{5}{2}\right|+\left|\tfrac{7}{2}\right|\right\}=\text{Max}\{1,\ 26\}=26$$

$$1+\text{Max}\left\{|-6|,|-14|,\left|-\tfrac{5}{2}\right|,\left|\tfrac{7}{2}\right|\right\}=1+14=15$$

The smaller of the two numbers is 15. Thus, every zero of f lies between –15 and 15.

Step 5: Using synthetic division:

$$\begin{array}{r|rrrrr} -3 & 2 & 7 & -5 & -28 & -12 \\ & & -6 & -3 & 24 & 12 \\ \hline & 2 & 1 & -8 & -4 & 0 \end{array}\qquad\qquad \begin{array}{r|rrrr} -2 & 2 & 1 & -8 & -4 \\ & & -4 & 6 & 4 \\ \hline & 2 & -3 & -2 & 0 \end{array}$$

Since the remainder is 0, $x+3$ and $x+2$ are factors. The other factor is the quotient: $2x^2-3x-2$.

Thus, $f(x)=(x+3)(x+2)\left(2x^2-3x-2\right)=(x+3)(x+2)(2x+1)(x-2)$.

The zeros are –3, –2, $\frac{-1}{2}$, and 2.

21. $f(x)=x^3-3x^2-6x+8$.

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: By Descartes Rule of Signs, there are 2 or no positive real zeros.

$$\begin{aligned}f(-x)&=(-x)^3-3(-x)^2-6(-x)+8\\&=-x^3-3x^2+6x+8\end{aligned};$$

thus, there is 1 negative real zero.

Step 3: Possible rational zeros:

$$p=\pm1,\ \pm2,\ \pm4,\ \pm8;\quad q=\pm1$$

$$\frac{p}{q}=\pm1,\ \pm2,\ \pm4,\ \pm8$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrr} -1 & 1 & -3 & -6 & 8 \\ & & -1 & 4 & 2 \\ \hline & 1 & -4 & -2 & \{10\} \end{array} \quad \rightarrow x+1 \text{ is not a factor}$$

$$\begin{array}{r|rrrr} 1 & 1 & -3 & -6 & 8 \\ & & 1 & -2 & -8 \\ \hline & 1 & -2 & -8 & 0 \end{array} \quad \rightarrow x-1 \text{ is a factor}$$

Thus, $f(x)=(x-1)\left(x^2-2x-8\right)=(x-1)(x-4)(x+2)$.

The zeros are 1, 4, and - 2.

22. $f(x)=x^3-x^2-10x-8$.

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: By Descartes Rule of Signs, there is 1 positive real zero.

$$f(-x)=(-x)^3-(-x)^2-10(-x)-8$$
$$=-x^3+x^2+10x-8;$$

thus, there are 2 or no negative real zeros.

Step 3: Possible rational zeros:

$p=\pm1,\pm2,\pm4,\pm8;\quad q=\pm1$

$$\frac{p}{q}=\pm1,\pm2,\pm4,\pm8$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrr} -1 & 1 & -1 & -10 & -8 \\ & & -1 & 2 & 8 \\ \hline & 1 & -2 & -8 & 0 \end{array} \quad \rightarrow x+1 \text{ is a factor}$$

Thus, $f(x)=(x+1)\left(x^2-2x-8\right)=(x-1)(x-4)(x+2)$.

The zeros are - 1, 4, and - 2.

23. $f(x)=4x^3+4x^2-7x+2$.

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: By Descartes Rule of Signs, there are 2 or no positive real zeros.

$$f(-x)=4(-x)^3+4(-x)^2-7(-x)+2$$
$$=-4x^3+4x^2+7x+2;$$

thus, there is 1 negative real zero.

Step 3: Possible rational zeros:

$p=\pm1,\pm2;\quad q=\pm1,\pm2,\pm4;$

$$\frac{p}{q}=\pm1,\pm\frac{1}{2},\pm\frac{1}{4},\pm2$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrr} -1 & 4 & 4 & -7 & 2 \\ & & -4 & 0 & 7 \\ \hline & 4 & 0 & -7 & \{9\} \end{array} \quad \rightarrow x+1 \text{ is not a factor}$$

$$\begin{array}{r|rrrr} 1 & 4 & 4 & -7 & 2 \\ & & 4 & 8 & 1 \\ \hline & 4 & 8 & 1 & \{3\} \end{array} \quad \rightarrow x-1 \text{ is not a factor}$$

$$\begin{array}{r|rrrr} -2 & 4 & 4 & -7 & 2 \\ & & -8 & 8 & -2 \\ \hline & 4 & -4 & 1 & 0 \end{array} \quad \rightarrow x+2 \text{ is a factor}$$

Thus, $f(x)=(x+2)\left(4x^2-4x+1\right)=(x+2)(4x-2)\left(x-\frac{1}{2}\right)=4(x+2)\left(x-\frac{1}{2}\right)^2$.

The zeros are -2, and $\frac{1}{2}$ (with multiplicity 2).

24. $f(x)=4x^3-4x^2-7x-2$.

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: By Descartes Rule of Signs, there is 1 positive real zero.

$$\begin{aligned} f(-x) &= 4(-x)^3-4(-x)^2-7(-x)-2 \\ &= -4x^3-4x^2+7x-2 \end{aligned};$$

thus, there are 2 or no negative real zeros.

Step 3: Possible rational zeros:

$p=\pm 1,\pm 2;\quad q=\pm 1,\pm 2,\pm 4;$

$$\frac{p}{q}=\pm 1,\pm\frac{1}{2},\pm\frac{1}{4},\pm 2$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrr} 1 & 4 & -4 & -7 & -2 \\ & & 4 & 0 & -7 \\ \hline & 4 & 0 & -7 & \{-9\} \end{array} \quad \rightarrow x-1 \text{ is not a factor}$$

$$\begin{array}{r|rrrr} -1 & 4 & -4 & -7 & -2 \\ & & -4 & 8 & -1 \\ \hline & 4 & -8 & 1 & \{-3\} \end{array} \quad \rightarrow x+1 \text{ is not a factor}$$

$$\begin{array}{r|rrrr} 2 & 4 & -4 & -7 & -2 \\ & & 8 & 8 & 2 \\ \hline & 4 & 4 & 1 & 0 \end{array} \quad \rightarrow x-2 \text{ is a factor}$$

Thus, $f(x)=(x-2)\left(4x^2+4x+1\right)=(x-2)(2x+1)(2x+1)=(x-2)(2x+1)^2$.

The zeros are 2, and $-\frac{1}{2}$ (with multiplicity 2).

25. $f(x)=x^4-4x^3+9x^2-20x+20$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there are 4, 2 or no positive real zeros.

$$f(-x)=(-x)^4-4(-x)^3+9(-x)^2-20(-x)+20$$
$$=x^4+4x^3+9x^2+20x+20;$$

thus, there are no negative real zeros.

Step 3: Possible rational zeros:

$p=\pm1,\pm2,\pm4,\pm5,\pm10,\pm20;\quad q=\pm1;$

$$\frac{p}{q}=\pm1,\pm2,\pm4,\pm5,\pm10,\pm20$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrr} 1 & 1 & -4 & 9 & -20 & 20 \\ & & 1 & -3 & 6 & -14 \\ \hline & 1 & -3 & 6 & -14 & \{6\} \end{array}$$

$\rightarrow x-1$ is not a factor

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 9 & -20 & 20 \\ & & 2 & -4 & 10 & -20 \\ \hline & 2 & -2 & 5 & -10 & 0 \end{array}$$

$\rightarrow x-2$ is a factor

Thus, $f(x)=(x-2)\left(x^3-2x^2+5x-10\right)$.

We can factor $x^3-2x^2+5x-10$ by grouping

$$x^3-2x^2+5x-10=x^2(x-2)+5(x-2)=(x-2)\left(x^2+5\right)$$
$$=(x-2)\left(x+\sqrt{5}i\right)\left(x-\sqrt{5}i\right)$$
$$f(x)=(x-2)^2\left(x+\sqrt{5}i\right)\left(x-\sqrt{5}i\right)$$

The zeros are 2 (multiplicity 2), $\sqrt{5}i$, and $-\sqrt{5}i$.

26. $f(x)=x^4+6x^3+11x^2+12x+18$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there are no positive real zeros.

$$f(-x)=(-x)^4+6(-x)^3+11(-x)^2+12(-x)+18$$
$$=x^4-6x^3+11x^2-12x+18;$$

thus, there are 4,2 or no negative real zeros.

Step 3: Possible rational zeros:

$$p = \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18; \quad q = \pm 1$$

$$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrr} 1 & 1 & 6 & 11 & 12 & 18 \\ & & 1 & 7 & 18 & 30 \\ \hline & 1 & 7 & 18 & 30 & \{48\} \end{array} \quad \rightarrow x-1 \text{ is not a factor}$$

$$\begin{array}{r|rrrrr} -1 & 1 & 6 & 11 & 12 & 18 \\ & & -1 & -5 & -6 & -6 \\ \hline & 1 & 5 & 6 & 6 & \{12\} \end{array} \quad \rightarrow x+1 \text{ is not a factor}$$

$$\begin{array}{r|rrrrr} 2 & 1 & 6 & 11 & 12 & 18 \\ & & 2 & 16 & 54 & 132 \\ \hline & 1 & 8 & 27 & 66 & \{150\} \end{array} \quad \rightarrow x-2 \text{ is not a factor}$$

$$\begin{array}{r|rrrrr} -2 & 1 & 6 & 11 & 12 & 18 \\ & & -2 & -8 & -6 & -12 \\ \hline & 1 & 4 & 3 & 6 & \{6\} \end{array} \quad \rightarrow x+2 \text{ is not a factor}$$

$$\begin{array}{r|rrrrr} 3 & 1 & 6 & 11 & 12 & 18 \\ & & 3 & 27 & 114 & 378 \\ \hline & 1 & 9 & 38 & 126 & \{396\} \end{array} \quad \rightarrow x+2 \text{ is not a factor}$$

$$\begin{array}{r|rrrrr} -3 & 1 & 6 & 11 & 12 & 18 \\ & & -3 & -9 & -6 & -18 \\ \hline & 1 & 3 & 2 & 6 & 0 \end{array} \quad \rightarrow x+3 \text{ is a factor}$$

Thus, $f(x) = (x+3)\left(x^3 + 3x^2 + 2x + 6\right)$.

We can factor $x^3 + 3x^2 + 2x + 6$ by grouping

$$x^3 + 3x^2 + 2x + 6 = x^2(x+3) + 2(x+3) = (x+3)\left(x^2+2\right)$$

$$= (x+3)\left(x+\sqrt{2}i\right)\left(x-\sqrt{2}i\right)$$

$$f(x) = (x+3)^2\left(x+\sqrt{2}i\right)\left(x-\sqrt{2}i\right)$$

The zeros are -3 (multiplicity 2), $\sqrt{2}i$, and $-\sqrt{2}i$.

27. $f(x) = 2x^4 + 2x^3 - 11x^2 + x - 6$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there are 3 or 1 positive real zeros.

$$f(-x) = 2(-x)^4 + 2(-x)^3 - 11(-x)^2 + (-x) - 6$$
$$= 2x^4 - 2x^3 - 11x^2 - x - 6;$$

thus, there is 1 negative real zero.

Step 3: Possible rational zeros:

$$p = \pm 1, \pm 2, \pm 3, \pm 6; \quad q = \pm 1, \pm 2;$$

$$\frac{p}{q} = \pm 1, \pm \frac{1}{2}, \pm 2, \pm 3, \pm \frac{3}{2}, \pm 6$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrr} -1 & 2 & 2 & -11 & 1 & -6 \\ & & -2 & 0 & 11 & -12 \\ \hline & 2 & 0 & -11 & 12 & \{-18\} \end{array} \quad \rightarrow x+1 \text{ is not a factor}$$

$$\begin{array}{r|rrrrr} 1 & 2 & 2 & -11 & 1 & -6 \\ & & 2 & 4 & -7 & -6 \\ \hline & 2 & 4 & -7 & -6 & \{-12\} \end{array} \quad \rightarrow x-1 \text{ is not a factor}$$

$$\begin{array}{r|rrrrr} -2 & 2 & 2 & -11 & 1 & -6 \\ & & -4 & 4 & 14 & -30 \\ \hline & 2 & -2 & -7 & 15 & \{-36\} \end{array} \quad \rightarrow x+2 \text{ is not a factor}$$

$$\begin{array}{r|rrrrr} 2 & 2 & 2 & -11 & 1 & -6 \\ & & 4 & 12 & 2 & 6 \\ \hline & 2 & 6 & 1 & 3 & 0 \end{array} \quad \rightarrow x-2 \text{ is a factor}$$

Thus, $f(x) = (x-2)\left(2x^3 + 6x^2 + x + 3\right)$.

We can factor $2x^3 + 6x^2 + x + 3$ by grouping

$$2x^3 + 6x^2 + x + 3 = 2x^2(x+3) + (x+3) = (x+3)\left(2x^2 + 1\right)$$
$$= (x+3)\left(\sqrt{2}x + i\right)\left(\sqrt{2}x - i\right)$$
$$f(x) = (x-2)(x+3)\left(\sqrt{2}x + i\right)\left(\sqrt{2}x - i\right)$$

The zeros are 2, -3, $-\frac{\sqrt{2}}{2}i$, and $\frac{\sqrt{2}}{2}i$.

28. $f(x) = 3x^4 + 3x^3 - 17x^2 + x - 6$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there are 3 or 1 positive real zeros.

$$f(-x) = 3(-x)^4 + 3(-x)^3 - 17(-x)^2 + (-x) - 6$$
$$= 3x^4 - 3x^3 - 17x^2 - x - 6;$$

thus, there is 1 negative real zero.

Step 3: Possible rational zeros:

$$p = \pm 1, \pm 2, \pm 3, \pm 6; \quad q = \pm 1, \pm 3;$$

$$\frac{p}{q} = \pm 1, \pm \frac{1}{3}, \pm 2, \pm \frac{2}{3}, \pm 3, \pm 6$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrr} 1 & 3 & 3 & -17 & 1 & -6 \\ & & 3 & 6 & -11 & -10 \\ \hline & 3 & 6 & -11 & -10 & \{-16\} \end{array} \quad \rightarrow x - 1 \text{ is not a factor}$$

$$\begin{array}{r|rrrrr} -1 & 3 & 3 & -17 & 1 & -6 \\ & & -3 & 0 & 17 & -18 \\ \hline & 3 & 0 & -17 & 18 & \{-24\} \end{array} \quad \rightarrow x + 1 \text{ is not a factor}$$

$$\begin{array}{r|rrrrr} 2 & 3 & 3 & -17 & 1 & -6 \\ & & 6 & 18 & 2 & 6 \\ \hline & 3 & 9 & 1 & 3 & 0 \end{array} \quad \rightarrow x - 2 \text{ is a factor}$$

Thus, $f(x) = (x-2)(3x^3 + 9x^2 + x + 3)$.

We can factor $3x^3 + 9x^2 + x + 3$ by grouping

$$3x^3 + 9x^2 + x + 3 = 3x^2(x+3) + (x+3) = (x+3)(3x^2+1)$$
$$= (x+3)(\sqrt{3}x + i)(\sqrt{3}x - i)$$
$$f(x) = (x-2)(x+3)(\sqrt{3}x + i)(\sqrt{3}x - i)$$

The zeros are 2, -3, $-\frac{\sqrt{3}}{3}i$, and $\frac{\sqrt{3}}{3}i$.

29. $f(x) = 2x^4 + 7x^3 + x^2 - 7x - 3$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there is 1 positive real zero.

$$f(-x) = 2(-x)^4 + 7(-x)^3 + (-x)^2 - 7(-x) - 3$$
$$= 2x^4 - 7x^3 + x^2 + 7x - 3;$$

thus, there are 3 or 1 negative real zeros.

Step 3: Possible rational zeros:

$p = \pm 1, \pm 3; \quad q = \pm 1, \pm 2;$

$$\frac{p}{q} = \pm 1, \pm\frac{1}{2}, \pm 3, \pm\frac{3}{2}$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrr} 1 & 2 & 7 & 1 & -7 & -3 \\ & & 2 & 9 & 10 & 3 \\ \hline & 2 & 9 & 10 & 3 & 0 \end{array} \rightarrow x-1 \text{ is a factor}$$

Thus, $f(x) = (x-1)\left(2x^3 + 9x^2 + 10x + 3\right)$.

We now concentrate on $2x^3 + 9x^2 + 10x + 3$.

$g(x) = 2x^3 + 9x^2 + 10x + 3$ has the same possible rational roots as f. However, since we have already found the only positive real zero for f, so we only need to look at the possible negative zeros

$$\frac{p}{q} = -1, -\frac{1}{2}, -3, -\frac{3}{2}$$

$$\begin{array}{r|rrrr} -1 & 2 & 9 & 10 & 3 \\ & & -2 & -7 & -3 \\ \hline & 2 & 7 & 3 & 0 \end{array} \rightarrow x+1 \text{ is a factor}$$

$2x^3 + 9x^2 + 10x + 3 = (x+1)\left(2x^2 + 7x + 3\right)$

$= (x+1)(2x+1)(x+3)$

$f(x) = (x-1)(x+1)(2x+1)(x+3)$ The zeros are 1, -1, $-\frac{1}{2}$, and -3.

30. $f(x) = 2x^4 + 7x^3 - 5x^2 - 28x - 12$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: By Descartes Rule of Signs, there is 1 positive real zero.

$f(-x) = 2(-x)^4 + 7(-x)^3 - 5(-x)^2 - 28(-x) - 12$

$= 2x^4 - 7x^3 - 5x^2 + 28x - 12$;

thus, there are 3 or 1 negative real zeros.

Step 3: Possible rational zeros:

$p = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12; \quad q = \pm 1, \pm 2;$

$$\frac{p}{q} = \pm 1, \pm\frac{1}{2}, \pm 3, \pm 4, \pm\frac{3}{2}, \pm 6, \pm 12$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrr} 1 & 2 & 7 & -5 & -28 & -12 \\ & & 2 & 9 & 4 & -24 \\ \hline & 2 & 9 & 4 & -24 & \{-36\} \end{array} \rightarrow x-1 \text{ is not a factor}$$

$$\begin{array}{r|rrrrr} -1 & 2 & 7 & -5 & -28 & -12 \\ & & -2 & -5 & 10 & 18 \\ \hline & 2 & 5 & -10 & -18 & \{6\} \end{array} \rightarrow x+1 \text{ is not a factor}$$

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -5 & -28 & -12 \\ & & 4 & 22 & 34 & 12 \\ \hline & 2 & 11 & 17 & 6 & 0 \end{array} \rightarrow x-2 \text{ is a factor}$$

Thus, $f(x) = (x-2)\left(2x^3 + 11x^2 + 17x + 6\right)$.

We now concentrate on $2x^3 + 9x^2 + 10x + 3$.

$g(x) = 2x^3 + 11x^2 + 17x + 6$ has the same possible rational roots as f. However, since we have already found the only positive real zero for f, so we only need to look at the possible negative zeros

$$\frac{p}{q} = -1, -\frac{1}{2}, -3, -4, -\frac{3}{2}, -6, -12$$

$$\begin{array}{r|rrrr} -2 & 2 & 11 & 17 & 6 \\ & & -4 & -14 & 6 \\ \hline & 2 & 7 & 3 & 0 \end{array} \rightarrow x+2 \text{ is a factor}$$

$2x^3 + 11x^2 + 17x + 6 = (x+2)\left(2x^2 + 7x + 3\right)$

$= (x+2)(2x+1)(x+3)$

$f(x) = (x-2)(x+2)(2x+1)(x+3)$

The zeros are 2, -2, $-\frac{1}{2}$, and -3.

31. $f(x) = x^3 - x^2 - 4x + 2$

$a_2 = -1, \quad a_1 = -4, \quad a_0 = 2$

$\text{Max}\left\{1, |2| + |-4| + |-1|\right\} = \text{Max}\left\{1, 7\right\} = 7$

$1 + \text{Max}\left\{|2|, |-4|, |-1|\right\} = 1 + 4 = 5$

The smaller of the two numbers is 5. Thus, every zero of f lies between –5 and 5.

32. $f(x) = x^3 + x^2 - 10x - 5$

$a_2 = 1, \quad a_1 = -10, \quad a_0 = -5$

$\text{Max}\left\{1, |-5| + |-10| + |1|\right\} = \text{Max}\left\{1, 16\right\} = 16$

$1 + \text{Max}\left\{|-5|, |-10|, |1|\right\} = 1 + 10 = 11$

The smaller of the two numbers is 11. Thus, every zero of f lies between –11 and 11.

33. $f(x)=2x^3-7x^2-10x+35=2\left(x^3-\frac{7}{2}x^2-5x+\frac{35}{2}\right)$
$a_2=-\frac{7}{2},\quad a_1=-5,\quad a_0=\frac{35}{2}$
$\text{Max}\left\{1,\left|\frac{35}{2}\right|+\left|-5\right|+\left|-\frac{7}{2}\right|\right\}=\text{Max}\left\{1,\ 26\right\}=26$
$1+\text{Max}\left\{\left|\frac{35}{2}\right|,\left|-5\right|,\left|-\frac{7}{2}\right|\right\}=1+\frac{35}{2}=\frac{37}{2}=18.5$
The smaller of the two numbers is 18.5. Thus, every zero of f lies between –18.5 and 18.5.

34. $f(x)=3x^3-7x^2-6x+14=3\left(x^3-\frac{7}{3}x^2-2x+\frac{14}{3}\right)$
$a_2=-\frac{7}{3},\quad a_1=-2,\quad a_0=\frac{14}{3}$
$\text{Max}\left\{1,\left|\frac{14}{3}\right|+\left|-2\right|+\left|-\frac{7}{3}\right|\right\}=\text{Max}\left\{1,\ 9\right\}=9$
$1+\text{Max}\left\{\left|\frac{14}{3}\right|,\left|-2\right|,\left|-\frac{7}{3}\right|\right\}=1+\frac{14}{3}=\frac{17}{3}=5.67$
The smaller of the two numbers is 5.67. Thus, every zero of f lies between –5.67 and 5.67.

35. $f(x)=3x^3-x-1;\quad [0,1]$
$f(0)=-1<0$ and $f(1)=1>0$
Since one is positive and one is negative, there is a zero in the interval.

36. $f(x)=2x^3-x^2-3;\quad [1,\ 2]$
$f(1)=-2<0$ and $f(2)=9>0$
Since one is positive and one is negative, there is a zero in the interval.

37. $f(x)=8x^4-4x^3-2x-1;\quad [0,1]$
$f(0)=-1<0$ and $f(1)=1>0$
Since one is positive and one is negative, there is a zero in the interval.

38. $f(x)=3x^4+4x^3-8x-2;\quad [1,\ 2]$
$f(1)=-3<0$ and $f(2)=62>0$
Since one is positive and one is negative, there is a zero in the interval.

39. $f(x)=x^3-x-2$
$f(1)=-2;\quad f(2)=4,$ so by the Intermediate Value Theorem, f has a zero on the interval [1,2].
Subdivide the interval [1,2] into 10 equal subintervals:

[1,1.1]; [1.1,1.2]; [1.2,1.3]; [1.3,1.4]; [1.4,1.5]; [1.5,1.6]; [1.6,1.7]; [1.7,1.8]; [1.8,1.9]; [1.9,2]

$f(1)=-2;f(1.1)=-1.769$
$f(1.1)=-1.769;f(1.2)=-1.472$
$f(1.2)=-1.472;f(1.3)=-1.103$
$f(1.3)=-1.103;f(1.4)=-0.656$
$f(1.4)=-0.656;\ f(1.5)=-0.125$
$f(1.5)=-0.125;f(1.6)=0.496$ so f has a real zero on the interval [1.5,1.6].
Subdivide the interval [1.5,1.6] into 10 equal subintervals:

[1.5,1.51]; [1.51,1.52]; [1.52,1.53]; [1.53,1.54]; [1.54,1.55]; [1.55,1.56];[1.56,1.57]; [1.57,1.58]; [1.58,1.59]; [1.59,1.6]

$f(1.5) = -0.125; f(1.51) = -0.0670$
$f(1.51) = -0.0670; f(1.52) = -0.0082$
$f(1.52) = -0.0082; f(1.53) = 0.0516$ so f has a real zero on the interval [1.52,1.53], therefore $r = 1.52$, correct to 2 decimal places.

40. $f(x) = 2x^3 - x^2 - 3$
$f(1) = -2;\ f(2) = 9$, so by the Intermediate Value Theorem, f has a zero on the interval [1,2].

Subdivide the interval [1,2] into 10 equal subintervals:

[1,1.1]; [1.1,1.2]; [1.2,1.3]; [1.3,1.4]; [1.4,1.5]; [1.5,1.6]; [1.6,1.7]; [1.7,1.8]; [1.8,1.9]; [1.9,2]

$f(1) = -2; f(1.1) = -1.548$
$f(1.1) = -1.548; f(1.2) = -0.984$
$f(1.2) = -0.984; f(1.3) = -0.296$
$f(1.3) = -0.296; f(1.4) = 0.528$ so f has a real zero on the interval [1.3,1.4].

Subdivide the interval [1.3,1.4] into 10 equal subintervals:

[1.3,1.31]; [1.31,1.32]; [1.32,1.33]; [1.33,1.34]; [1.34,1.35]; [1.35,1.36];[1.36,1.37]; [1.37,1.38]; [1.38,1.39]; [1.39,1.4]

$f(1.3) = -0.296;\ f(1.31) = -0.2200$
$f(1.31) = -0.2200; f(1.32) = -0.1425$
$f(1.32) = -0.1425; f(1.33) = -0.0636$
$f(1.33) = -0.0636; f(1.34) = 0.0166$ so f has a real zero on the interval [1.33,01.34], therefore $r = 1.33$, correct to 2 decimal places.

41. $f(x) = 8x^4 - 4x^3 - 2x - 1$
$f(0) = -1;\ f(1) = 1$, so by the Intermediate Value Theorem, f has a zero on the interval [0,1].

Subdivide the interval [0,1] into 10 equal subintervals:

[0,0.1]; [0.1,0.2]; [0.2,0.3]; [0.3,0.4]; [0.4,0.5]; [0.5,0.6]; [0.6,0.7]; [0.7,0.8]; [0.8,0.9]; [0.9,1]

$f(0) = -1; f(0.1) = -1.2032$
$f(0.1) = -1.2032; f(0.2) = -1.4192$

$f(0.2) = -1.4192; f(0.3) = -1.6432$
$f(0.3) = -1.6432; f(0.4) = -1.8512$
$f(0.4) = -1.8512; f(0.5) = -2$
$f(0.5) = -2; f(0.6) = -2.0272$
$f(0.6) = -2.0272; f(0.7) = -1.8512$
$f(0.7) = -1.8512; f(0.8) = -1.3712$
$f(0.8) = -1.3712; f(0.9) = -0.4672$
$f(0.9) = -0.4672; f(1) = 1$ so f has a real zero on the interval [0.9,1].

Subdivide the interval [1.5,1.6] into 10 equal subintervals:

[0.9,0.91]; [0.91,0.92]; [0.92,0.93]; [0.93,0.94]; [0.94,0.95]; [0.95,0.96];[0.96,0.97]; [0.97,0.98]; [0.98,0.99]; [0.99,1]

$f(0.9) = -0.4672; f(0.91) = -0.3483$
$f(0.91) = -0.3483; f(0.92) = -0.2236$
$f(0.92) = -0.2236; f(0.93) = -0.0930$
$f(0.93) = -0.0930; f(0.94) = 0.0437$

so f has a real zero on the interval [0.93,0.94], therefore $r = 0.93$, correct to 2 decimal places.

42. $f(x) = 3x^4 + 4x^3 - 8x - 2$
$f(1) = -3;\ f(2) = 62$, so by the Intermediate Value Theorem, f has a zero on the interval [1,2].

Subdivide the interval [1,2] into 10 equal subintervals:

[1,1.1]; [1.1,1.2]; [1.2,1.3]; [1.3,1.4]; [1.4,1.5]; [1.5,1.6]; [1.6,1.7]; [1.7,1.8]; [1.8,1.9]; [1.9,2]

$f(1) = -3; f(1.1) = -1.0837$
$f(1.1) = -1.0837; f(1.2) = 1.5328$ so f has a real zero on the interval [1.1,1.2].
Subdivide the interval [1.1,1.2] into 10 equal subintervals:

[1.1,1.11]; [1.11,1.12]; [1.12,1.13]; [1.13,1.14]; [1.14,1.15]; [1.15,1.16];[1.16,1.17]; [1.17,1.18]; [1.18,1.19]; [1.19,1.2]

$f(1.1) = -1.0837; f(1.11) = -0.8553$
$f(1.11) = -0.8553; f(1.12) = -0.6197$
$f(1.12) = -0.6197; f(1.13) = -0.3770$
$f(1.13) = -0.3770; f(1.14) = -0.1269$
$f(1.14) = -0.1269; f(1.15) = 0.1305$ so f has a real zero on the interval [1.14,1.15], therefore $r = 1.14$, correct to 2 decimal places.

43. $(6+3i)-(2-4i) = (6-2)+(3-(-4))i = 4+7i$

44. $(8-3i)+(-6+2i) = (8-6)+(-3+2)i = 2-i$

45. $4(3-i)+3(-5+2i) = 12-4i-15+6i = -3+2i$

46. $2(1+i)-3(2-3i) = 2+2i-6+9i = -4+11i$

47. $\dfrac{3}{3+i} = \dfrac{3}{3+i}\ \dfrac{3-i}{3-i} = \dfrac{9-3i}{9-3i+3i-i^2} = \dfrac{9-3i}{10} = \dfrac{9}{10}-\dfrac{3}{10}i$

48. $\dfrac{4}{2-i} = \dfrac{4}{2-i}\ \dfrac{2+i}{2+i} = \dfrac{8+4i}{4+2i-2i-i^2} = \dfrac{8+4i}{5} = \dfrac{8}{5}+\dfrac{4}{5}i$

49. $i^{50} = i^{48}\ i^2 = (i^4)^{12}\ i^2 = 1^{12}(-1) = -1$

50. $i^{29} = i^{28}\ i = (i^4)^7\ i = 1^7\ i = i$

51. $(2+3i)^3 = (2+3i)^2(2+3i) = \left(4+12i+9i^2\right)(2+3i) = (-5+12i)(2+3i)$
$= -10-15i+24i+36i^2 = -46+9i$

52. $(3-2i)^3 = (3-2i)^2(3-2i) = \left(9-12i+4i^2\right)(3-2i) = (5-12i)(3-2i)$
$= 15-10i-36i+24i^2 = -9-46i$

53. Since complex zeros appear in conjugate pairs, $4-i$, the conjugate of $4+i$, is the remaining zero of f.

54. Since complex zeros appear in conjugate pairs, $3-4i$, the conjugate of $3+4i$, is the remaining zero of f.

55. Since complex zeros appear in conjugate pairs, $-i$, the conjugate of i, and $1-i$, the conjugate of $1+i$, are the remaining zeros of f.

56. Since complex zeros appear in conjugate pairs, $1-i$, the conjugate of $1+i$, is the remaining zero of f.

57. $x^2+x+1=0$
$a=1,\ b=1,\ c=1,\quad b^2-4ac = 1^2-4(1)(1) = 1-4 = -3$
$x = \dfrac{-1\pm\sqrt{-3}}{2(1)} = \dfrac{-1\pm\sqrt{3}\,i}{2} = \dfrac{-1}{2}\pm\dfrac{\sqrt{3}}{2}i$
The solution set is $\left\{\dfrac{-1}{2}-\dfrac{\sqrt{3}}{2}i,\ \dfrac{-1}{2}+\dfrac{\sqrt{3}}{2}i\right\}$.

58. $x^2 - x + 1 = 0$

$a = 1,\ b = -1,\ c = 1, \quad b^2 - 4ac = (-1)^2 - 4(1)(1) = 1 - 4 = -3$

$x = \dfrac{-(-1) \pm \sqrt{-3}}{2(1)} = \dfrac{1 \pm \sqrt{3}\,i}{2} = \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$

The solution set is $\left\{ \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i,\ \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i \right\}$.

59. $2x^2 + x - 2 = 0$

$a = 2, b = 1, c = -2, \quad b^2 - 4ac = 1^2 - 4(2)(-2) = 1 + 16 = 17$

$x = \dfrac{-1 \pm \sqrt{17}}{2(2)} = \dfrac{-1 \pm \sqrt{17}}{4}$ The solution set is $\left\{ \dfrac{-1 - \sqrt{17}}{4},\ \dfrac{-1 + \sqrt{17}}{4} \right\}$.

60. $3x^2 - 2x - 1 = 0$

$(3x + 1)(x - 1) = 0$

$x = -\frac{1}{3}$ or $x = 1$

The solution set is $\left\{ -\frac{1}{3},\ 1 \right\}$.

61. $x^2 + 3 = x$

$x^2 - x + 3 = 0$

$a = 1,\ b = -1,\ c = 3, \quad b^2 - 4ac = (-1)^2 - 4(1)(3) = 1 - 12 = -11$

$x = \dfrac{-(-1) \pm \sqrt{-11}}{2(1)} = \dfrac{1 \pm \sqrt{11}\,i}{2} = \dfrac{1}{2} \pm \dfrac{\sqrt{11}}{2}i$

The solution set is $\left\{ \dfrac{1}{2} - \dfrac{\sqrt{11}}{2}i,\ \dfrac{1}{2} + \dfrac{\sqrt{11}}{2}i \right\}$.

62. $2x^2 + 1 = 2x$

$2x^2 - 2x + 1 = 0$

$a = 2,\ b = -2,\ c = 1, \quad b^2 - 4ac = (-2)^2 - 4(2)(1) = 4 - 8 = -4$

$x = \dfrac{-(-2) \pm \sqrt{-4}}{2(2)} = \dfrac{2 \pm 2i}{4} = \dfrac{1}{2} \pm \dfrac{1}{2}i$

The solution set is $\left\{ \dfrac{1}{2} - \dfrac{1}{2}i,\ \dfrac{1}{2} + \dfrac{1}{2}i \right\}$.

63. $x(1 - x) = 6$

$-x^2 + x - 6 = 0$

$a = -1,\ b = 1,\ c = -6, \quad b^2 - 4ac = 1^2 - 4(-1)(-6) = 1 - 24 = -23$

$x = \dfrac{-1 \pm \sqrt{-23}}{2(-1)} = \dfrac{-1 \pm \sqrt{23}\,i}{-2} = \dfrac{1}{2} \pm \dfrac{\sqrt{23}}{2}i$

The solution set is $\left\{ \dfrac{1}{2} - \dfrac{\sqrt{23}}{2}i,\ \dfrac{1}{2} + \dfrac{\sqrt{23}}{2}i \right\}$.

64. $x(1+x) = 2$
$x^2 + x - 2 = 0$
$(x+2)(x-1) = 0$
$x = -2$ or $x = 1$

The solution set is $\{-2, 1\}$.

65. $x^4 + 2x^2 - 8 = 0$
$\left(x^2 + 4\right)\left(x^2 - 2\right) = 0$
$x^2 + 4 = 0$ or $x^2 - 2 = 0$
$x^2 = -4$ or $x^2 = 2$
$x = \pm 2i$ or $x = \pm\sqrt{2}$

The solution set is $\{-2i, 2i, -\sqrt{2}, \sqrt{2}\}$.

66. $x^4 + 8x^2 - 9 = 0$
$\left(x^2 + 9\right)\left(x^2 - 1\right) = 0$
$\left(x^2 + 9\right)(x-1)(x+1) = 0$
$x^2 + 9 = 0$ or $x - 1 = 0$ or $x + 1 = 0$
$x^2 = -9$ or $x = 1$ or $x = -1$
$x = \pm 3i$

The solution set is $\{-3i, 3i, -1, 1\}$.

67. $x^3 - x^2 - 8x + 12 = 0$
The solutions of the equation are the zeros of the function $f(x) = x^3 - x^2 - 8x + 12$.
Step 1: $f(x)$ has 3 complex zeros.
Step 2: By Descartes Rule of Signs, there are 2 or 0 positive real zeros.
$f(-x) = (-x)^3 - (-x)^2 - 8(-x) + 12 = -x^3 - x^2 + 8x + 12$; thus, there is 1 negative real zero.
Step 3: Possible rational zeros:
$p = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12; \quad q = \pm 1; \quad \frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$
Step 4: Using synthetic division:

$$\begin{array}{r|rrrr} 2 & 1 & -1 & -8 & 12 \\ & & 2 & 2 & -12 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

Since the remainder is 0, $x - 2$ is a factor. The other factor is the quotient:
$x^2 + x - 6 = (x+3)(x-2)$.
The complex zeros are –3, 2 (multiplicity 2).

68. $x^3 - 3x^2 - 4x + 12 = 0$
$x^2(x-3) - 4(x-3) = 0$
$(x-3)\left(x^2 - 4\right) = 0$
$(x-3)(x-2)(x+2) = 0$
$x = 3$ or $x = 2$ or $x = -2$

The solution set is $\{-2, 2, 3\}$

69. $3x^4 - 4x^3 + 4x^2 - 4x + 1 = 0$

The solutions of the equation are the zeros of the function

$$f(x) = 3x^4 - 4x^3 + 4x^2 - 4x + 1$$

Step 1: $f(x)$ has 4 complex zeros.

Step 2: By Descartes Rule of Signs, there are 4 or 2 or 0 positive real zeros.

$$f(-x) = 3(-x)^4 - 4(-x)^3 + 4(-x)^2 - 4(-x) + 1$$
$$= 3x^4 + 4x^3 + 4x^2 + 4x + 1$$

thus, there are no negative real zeros.

Step 3: Possible rational zeros:

$$p = \pm 1; \quad q = \pm 1, \pm 3; \quad \frac{p}{q} = \pm 1, \pm \frac{1}{3}$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrr} 1 & 3 & -4 & 4 & -4 & 1 \\ & & 3 & -1 & 3 & -1 \\ \hline & 3 & -1 & 3 & -1 & 0 \end{array} \qquad \begin{array}{r|rrrr} \frac{1}{3} & 3 & -1 & 3 & -1 \\ & & 1 & 0 & 1 \\ \hline & 3 & 0 & 3 & 0 \end{array}$$

Since the remainder is 0, $x - 1$ and $x - \frac{1}{3}$ are factors. The other factor is the quotient: $3x^2 + 3 = 3\left(x^2 + 1\right)$.

Solving $x^2 + 1 = 0$:

$$x^2 = -1$$
$$x = \pm i$$

The complex zeros are $1, \frac{1}{3}, -i, i$.

70. $x^4 + 4x^3 + 2x^2 - 8x - 8 = 0$

The solutions of the equation are the zeros of the function

$$f(x) = x^4 + 4x^3 + 2x^2 - 8x - 8$$

Step 1: $f(x)$ has 4 complex zeros.

Step 2: By Descartes Rule of Signs, there is 1 positive real zero.

$$f(-x) = (-x)^4 + 4(-x)^3 + 2(-x)^2 - 8(-x) - 8$$
$$= x^4 - 4x^3 + 2x^2 + 8x - 8$$

thus, there are 3 or 1 negative real zeros.

Step 3: Possible rational zeros:

$$p = \pm 1, \pm 2, \pm 4, \pm 8; \quad q = \pm 1; \quad \frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 8$$

Step 4: Using synthetic division:

$$\begin{array}{r|rrrrr} -2 & 1 & 4 & 2 & -8 & -8 \\ & & -2 & -4 & 4 & 8 \\ \hline & 1 & 2 & -2 & -4 & 0 \end{array} \qquad \begin{array}{r|rrrr} -2 & 1 & 2 & -2 & -4 \\ & & -2 & 0 & 4 \\ \hline & 1 & 0 & -2 & 0 \end{array}$$

Since the remainder is 0, $x + 2$ is a factor twice. The other factor is the quotient: $x^2 - 2$.

Solving $x^2 - 2 = 0$:

$$x^2 = 2$$
$$x = \pm\sqrt{2}$$

The solutions are -2 (multiplicity 2), $-\sqrt{2}, \sqrt{2}$.