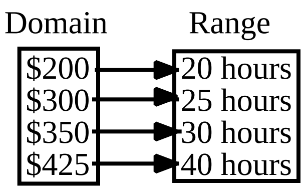
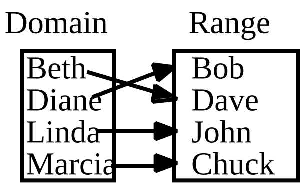
# Chapter 6

# Exponential and Logarithmic Functions

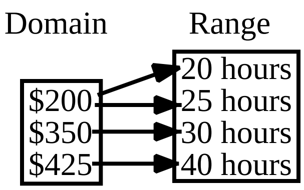
## 6.1 One-to-One Functions; Inverse Functions

1. (a) 

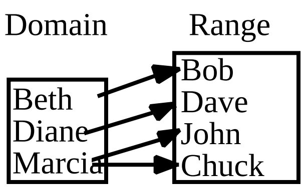

   (b) Inverse is a function.

2. (a) 


   (b) Inverse is a function.

3. (a) 


   (b) Inverse is not a function since $200 corresponds to two elements in the range.

4. (a) 


   (b) Inverse is not a function since Marcia corresponds to two elements in the range.

5. (a) $\{(6,2),(6,-3),(9,4),(10,1)\}$ (b) Inverse is not a function since 6 corresponds to 2 and –3.

6. (a) $\{(5,-2),(3,-1),(7,3),(12,4)\}$ (b) Inverse is a function.

7. (a) $\{(0,0),(1,1),(16,2),(81,3)\}$ (b) Inverse is a function.

8. (a) $\{(2,1),(8,2),(18,3),(32,4)\}$ (b) Inverse is a function.

9. Every horizontal line intersects the graph of $f$ at exactly one point. One-to-One.

10. Every horizontal line intersects the graph of $f$ at exactly one point. One-to-One.

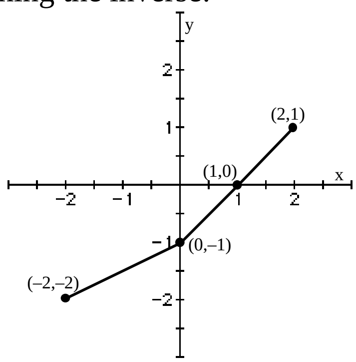
11. There are horizontal lines that intersect the graph of $f$ at more than one point. Not One-to-One.

12. There are horizontal lines that intersect the graph of $f$ at more than one point. Not One-to-One.

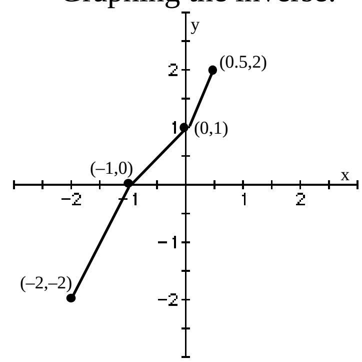
13. Every horizontal line intersects the graph of $f$ at exactly one point. One-to-One.

14. The horizontal line $y = 2$ intersects the graph of $f$ at every point. Not One-to-One.

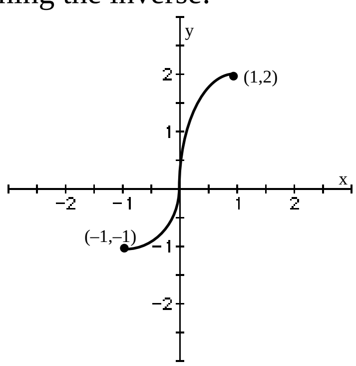
15. Graphing the inverse:



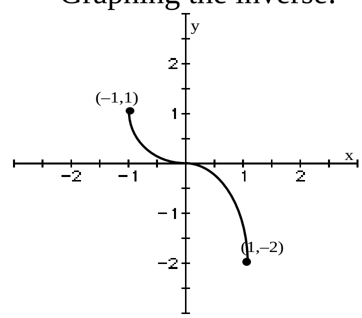

16. Graphing the inverse:



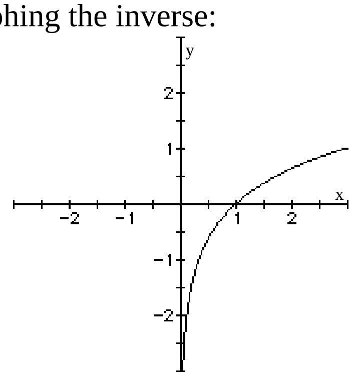

17. Graphing the inverse:



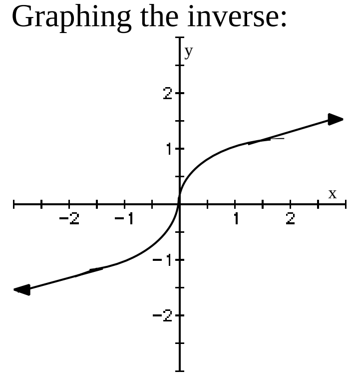

18. Graphing the inverse:




19. Graphing the inverse:




20. Graphing the inverse:



21. $f(x) = 3x + 4, \qquad g(x) = \frac{1}{3}(x-4)$

$$\begin{aligned} f(g(x)) &= f\left(\tfrac{1}{3}(x-4)\right) = 3\left(\tfrac{1}{3}(x-4)\right) + 4 \\ &= (x-4) + 4 = x \\ g(f(x)) &= g(3x+4) = \tfrac{1}{3}\left((3x+4) - 4\right) \\ &= \tfrac{1}{3}(3x) = x \end{aligned}$$

22. $f(x) = 3 - 2x, \qquad g(x) = -\frac{1}{2}(x-3)$

$$\begin{aligned} f(g(x)) &= f\left(-\tfrac{1}{2}(x-3)\right) = 3 - 2\left(-\tfrac{1}{2}(x-3)\right) \\ &= 3 + (x-3) = x \\ g(f(x)) &= g(3-2x) = -\tfrac{1}{2}\left((3-2x) - 3\right) \\ &= -\tfrac{1}{2}(-2x) = x \end{aligned}$$

23. $f(x) = 4x - 8, \qquad g(x) = \frac{x}{4} + 2$

$$\begin{aligned} f(g(x)) &= f\left(\frac{x}{4} + 2\right) = 4\left(\frac{x}{4} + 2\right) - 8 \\ &= (x+8) - 8 = x \\ g(f(x)) &= g(4x-8) = \frac{4x-8}{4} + 2 \\ &= x - 2 + 2 = x \end{aligned}$$

24. $f(x) = 2x + 6, \qquad g(x) = \frac{1}{2}x - 3$

$$\begin{aligned} f(g(x)) &= f\left(\tfrac{1}{2}x - 3\right) = 2\left(\tfrac{1}{2}x - 3\right) + 6 \\ &= (x-6) + 6 = x \\ g(f(x)) &= g(2x+6) = \tfrac{1}{2}(2x+6) - 3 \\ &= x + 3 - 3 = x \end{aligned}$$

25. $f(x) = x^3 - 8, \qquad g(x) = \sqrt[3]{x+8}$

$$\begin{aligned} f(g(x)) &= f\left(\sqrt[3]{x+8}\right) = \left(\sqrt[3]{x+8}\right)^3 - 8 \\ &= (x+8) - 8 = x \\ g(f(x)) &= g(x^3 - 8) = \sqrt[3]{(x^3-8)+8} \\ &= \sqrt[3]{x^3} = x \end{aligned}$$

26. $f(x) = (x-2)^2,\ x \ge 2;\ g(x) = \sqrt{x} + 2,\ x \ge 0$

$$\begin{aligned} f(g(x)) &= f\left(\sqrt{x} + 2\right) = \left(\sqrt{x} + 2 - 2\right)^2 \\ &= \left(\sqrt{x}\right)^2 = x \\ g(f(x)) &= g\left((x-2)^2\right) = \sqrt{(x-2)^2} + 2 \\ &= x - 2 + 2 = x \end{aligned}$$

27. $f(x)=\frac{1}{x}, \qquad g(x)=\frac{1}{x}$

$$f(g(x))=f\left(\frac{1}{x}\right)=\frac{1}{\frac{1}{x}}=x$$

$$g(f(x))=g\left(\frac{1}{x}\right)=\frac{1}{\frac{1}{x}}=x$$

28. $f(x)=x, \qquad g(x)=x$

$$f(g(x))=f(x)=x$$
$$g(f(x))=g(x)=x$$

29. $f(x)=\frac{2x+3}{x+4}, \qquad g(x)=\frac{4x-3}{2-x}$

$$f(g(x))=f\left(\frac{4x-3}{2-x}\right)=\frac{2\left(\frac{4x-3}{2-x}\right)+3}{\left(\frac{4x-3}{2-x}\right)+4}=\frac{\frac{8x-6+6-3x}{2-x}}{\frac{4x-3+8-4x}{2-x}}=\frac{\frac{5x}{2-x}}{\frac{5}{2-x}}$$
$$=\frac{5x}{2-x}\ \frac{2-x}{5}=x$$

$$g(f(x))=g\left(\frac{2x+3}{x+4}\right)=\frac{4\left(\frac{2x+3}{x+4}\right)-3}{2-\left(\frac{2x+3}{x+4}\right)}=\frac{\frac{8x+12-3x-12}{x+4}}{\frac{2x+8-2x-3}{x+4}}=\frac{\frac{5x}{x+4}}{\frac{5}{x+4}}$$
$$=\frac{5x}{x+4}\ \frac{x+4}{5}=x$$

30. $f(x)=\frac{x-5}{2x+3}, \qquad g(x)=\frac{3x+5}{1-2x}$

$$f(g(x))=f\left(\frac{3x+5}{1-2x}\right)=\frac{\left(\frac{3x+5}{1-2x}\right)-5}{2\left(\frac{3x+5}{1-2x}\right)+3}=\frac{\frac{3x+5-5+10x}{1-2x}}{\frac{6x+10+3-6x}{1-2x}}=\frac{\frac{13x}{1-2x}}{\frac{13}{1-2x}}$$
$$=\frac{13x}{1-2x}\ \frac{1-2x}{13}=x$$

$$g(f(x))=g\left(\frac{x-5}{2x+3}\right)=\frac{3\left(\frac{x-5}{2x+3}\right)+5}{1-2\left(\frac{x-5}{2x+3}\right)}=\frac{\frac{3x-15+10x+15}{2x+3}}{\frac{2x+3-2x+10}{2x+3}}=\frac{\frac{13x}{2x+3}}{\frac{13}{2x+3}}$$
$$=\frac{13x}{2x+3}\ \frac{2x+3}{13}=x$$

31. $f(x) = 3x$

$y = 3x$

$x = 3y$ Inverse

$y = \frac{x}{3}$

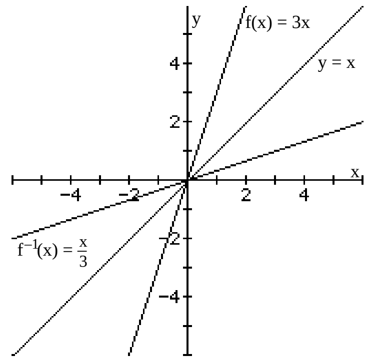
$f^{-1}(x) = \frac{x}{3}$

Verify: $f\left(f^{-1}(x)\right) = f\left(\frac{x}{3}\right) = 3\left(\frac{x}{3}\right) = x$

$f^{-1}\left(f(x)\right) = f^{-1}(3x) = \frac{3x}{3} = x$

Domain of $f$ = range of $f^{-1} = (-\infty, \infty)$

Range of $f$ = domain of $f^{-1} = (-\infty, \infty)$




32. $f(x) = -4x$

$y = -4x$

$x = -4y$ Inverse

$y = \frac{x}{-4}$

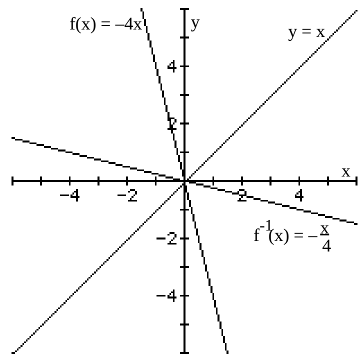
$f^{-1}(x) = -\frac{x}{4}$

Verify: $f\left(f^{-1}(x)\right) = f\left(-\frac{x}{4}\right) = -4\left(-\frac{x}{4}\right) = x$

$f^{-1}\left(f(x)\right) = f^{-1}(-4x) = -\frac{-4x}{4} = x$

Domain of $f$ = range of $f^{-1} = (-\infty, \infty)$

Range of $f$ = domain of $f^{-1} = (-\infty, \infty)$



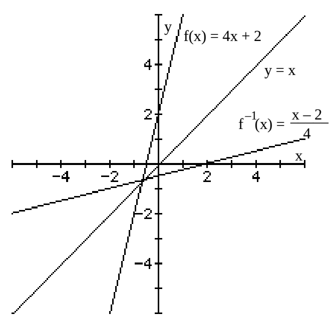

33. $f(x) = 4x + 2$

$y = 4x + 2$

$x = 4y + 2$ Inverse

$4y = x - 2$

$y = \frac{x-2}{4} \rightarrow f^{-1}(x) = \frac{x-2}{4}$




Verify: $f\left(f^{-1}(x)\right) = f\left(\frac{x-2}{4}\right) = 4\left(\frac{x-2}{4}\right) + 2 = x - 2 + 2 = x$

$f^{-1}\left(f(x)\right) = f^{-1}(4x + 2) = \frac{(4x+2)-2}{4} = \frac{4x}{4} = x$

Domain of $f$ = range of $f^{-1} = (-\infty, \infty)$

Range of $f$ = domain of $f^{-1} = (-\infty, \infty)$

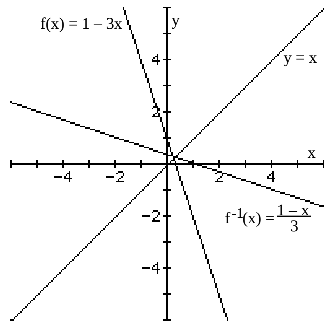
34. $f(x) = 1 - 3x$

$y = 1 - 3x$

$x = 1 - 3y$ Inverse

$3y = 1 - x$

$y = \dfrac{1-x}{3} \rightarrow f^{-1}(x) = \dfrac{1-x}{3}$




Verify: $f\left(f^{-1}(x)\right) = f\left(\dfrac{1-x}{3}\right) = 1 - 3\left(\dfrac{1-x}{3}\right) = 1 - (1-x) = x$

$f^{-1}\left(f(x)\right) = f^{-1}(1-3x) = \dfrac{1-(1-3x)}{3} = \dfrac{3x}{3} = x$

Domain of $f$ = range of $f^{-1}$ = $(-\infty, \infty)$

Range of $f$ = domain of $f^{-1}$ = $(-\infty, \infty)$

35. $f(x) = x^3 - 1$

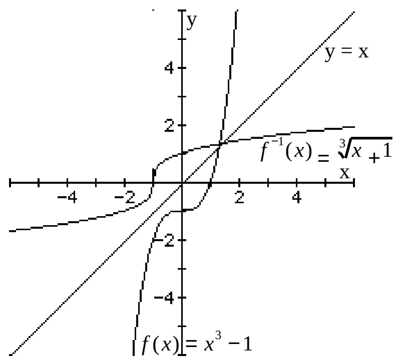
$y = x^3 - 1$

$x = y^3 - 1$ Inverse

$y^3 = x + 1$

$y = \sqrt[3]{x+1}$

$f^{-1}(x) = \sqrt[3]{x+1}$




Verify: $f\left(f^{-1}(x)\right) = f\left(\sqrt[3]{x+1}\right) = \left(\sqrt[3]{x+1}\right)^3 - 1 = x + 1 - 1 = x$

$f^{-1}\left(f(x)\right) = f^{-1}\left(x^3 - 1\right) = \sqrt[3]{\left(x^3-1\right)+1} = \sqrt[3]{x^3} = x$

Domain of $f$ = range of $f^{-1}$ = $(-\infty, \infty)$

Range of $f$ = domain of $f^{-1}$ = $(-\infty, \infty)$

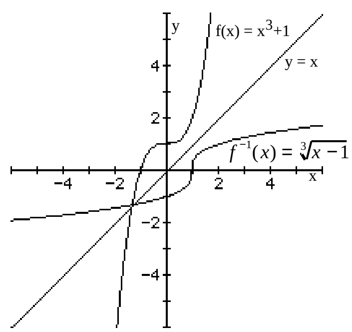
36. $f(x) = x^3 + 1$

$y = x^3 + 1$

$x = y^3 + 1$ Inverse

$y^3 = x - 1$

$y = \sqrt[3]{x-1} \rightarrow f^{-1}(x) = \sqrt[3]{x-1}$




Verify: $f\left(f^{-1}(x)\right) = f\left(\sqrt[3]{x-1}\right) = \left(\sqrt[3]{x-1}\right)^3 + 1 = x - 1 + 1 = x$

$f^{-1}\left(f(x)\right) = f^{-1}\left(x^3 + 1\right) = \sqrt[3]{\left(x^3+1\right)-1} = \sqrt[3]{x^3} = x$

Domain of $f$ = range of $f^{-1}$ = $(-\infty, \infty)$

Range of $f$ = domain of $f^{-1}$ = $(-\infty, \infty)$

37. $f(x) = x^2 + 4,\ \ x \ge 0$

$y = x^2 + 4 \quad x \ge 0$

$x = y^2 + 4 \quad y \ge 0$ Inverse

$y^2 = x - 4 \quad y \ge 0$

$y = \sqrt{x-4} \rightarrow f^{-1}(x) = \sqrt{x-4}$

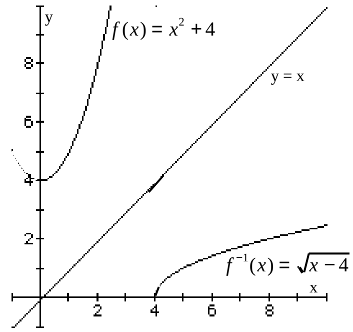
Verify:

$f\left(f^{-1}(x)\right) = f\left(\sqrt{x-4}\right) =$

$\left(\sqrt{x-4}\right)^2 + 4 = x - 4 + 4 = x$

$f^{-1}\left(f(x)\right) = f^{-1}\left(x^2+4\right) = \sqrt{\left(x^2+4\right)-4}$

$= \sqrt{x^2} = |x| = x,\ x \ge 0$




Domain of $f$ = range of $f^{-1}$ = [0, ∞)

Range of $f$ = domain of $f^{-1}$ = [4, ∞)

38. $f(x) = x^2 + 9,\ \ x \ge 0$

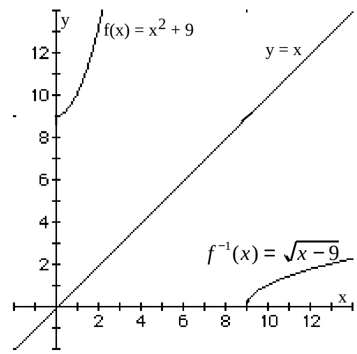
$y = x^2 + 9 \quad x \ge 0$

$x = y^2 + 9 \quad y \ge 0$ Inverse

$y^2 = x - 9 \quad y \ge 0$

$y = \sqrt{x-9}$

$f^{-1}(x) = \sqrt{x-9}$




Verify: $f\left(f^{-1}(x)\right) = f\left(\sqrt{x-9}\right) = \left(\sqrt{x-9}\right)^2 + 9 = x - 9 + 9 = x$

$f^{-1}\left(f(x)\right) = f^{-1}\left(x^2+9\right) = \sqrt{\left(x^2+9\right)-9} = \sqrt{x^2} = |x| = x,\ x \ge 0$

Domain of $f$ = range of $f^{-1}$ = [0, ∞)

Range of $f$ = domain of $f^{-1}$ = [9, ∞)

39. $f(x) = \dfrac{4}{x}$

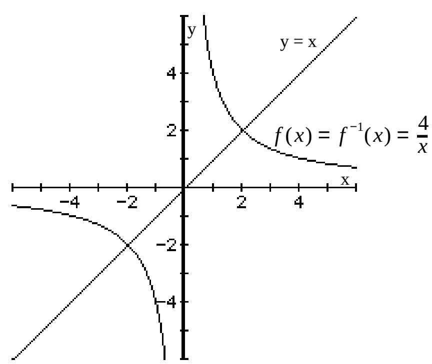
$y = \dfrac{4}{x}$

$x = \dfrac{4}{y}$ Inverse

$xy = 4$

$y = \dfrac{4}{x}$

$f^{-1}(x) = \dfrac{4}{x}$



Verify: $f\left(f^{-1}(x)\right) = f\left(\frac{4}{x}\right) = \frac{4}{\frac{4}{x}} = 4 \ \frac{x}{4} = x$

$f^{-1}\left(f(x)\right) = f^{-1}\left(\frac{4}{x}\right) = \frac{4}{\frac{4}{x}} = 4 \ \frac{x}{4} = x$

Domain of $f$ = range of $f^{-1}$ = all real numbers except 0
Range of $f$ = domain of $f^{-1}$ = all real numbers except 0

40. $f(x) = -\frac{3}{x}$

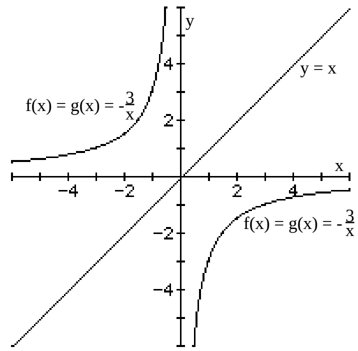
$y = -\frac{3}{x}$

$x = -\frac{3}{y}$ Inverse

$xy = -3$

$y = -\frac{3}{x}$

$f^{-1}(x) = -\frac{3}{x}$




Verify: $f\left(f^{-1}(x)\right) = f\left(-\frac{3}{x}\right) = -\frac{3}{-\frac{3}{x}} = 3 \ \frac{x}{3} = x$

$f^{-1}\left(f(x)\right) = f^{-1}\left(-\frac{3}{x}\right) = -\frac{3}{-\frac{3}{x}} = 3 \ \frac{x}{3} = x$

Domain of $f$ = range of $f^{-1}$ = all real numbers except 0
Range of $f$ = domain of $f^{-1}$ = all real numbers except 0

41. $f(x) = \frac{1}{x-2}$

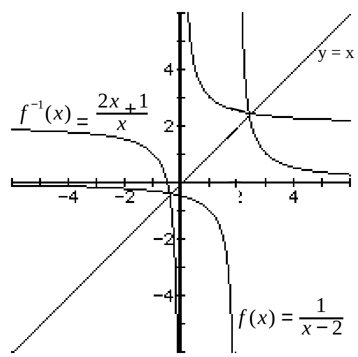
$y = \frac{1}{x-2} \rightarrow x = \frac{1}{y-2}$ Inverse

$x(y-2) = 1$

$xy - 2x = 1$

$xy = 2x + 1$

$y = \frac{2x+1}{x} \rightarrow f^{-1}(x) = \frac{2x+1}{x}$



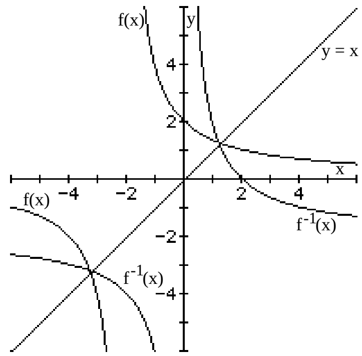

Verify: $f\left(f^{-1}(x)\right) = f\left(\frac{2x+1}{x}\right) = \frac{1}{\frac{2x+1}{x} - 2} = \frac{1}{\frac{2x+1-2x}{x}} = \frac{1}{\frac{1}{x}} = x$

$f^{-1}\left(f(x)\right) = f^{-1}\left(\frac{1}{x-2}\right) = \frac{2\left(\frac{1}{x-2}\right)+1}{\left(\frac{1}{x-2}\right)} = \frac{\frac{2+x-2}{x-2}}{\frac{1}{x-2}} = \frac{x}{x-2} \ \frac{x-2}{1} = x$

Domain of $f$ = range of $f^{-1}$ = all real numbers except 2
Range of $f$ = domain of $f^{-1}$ = all real numbers except 0

42. $f(x)=\dfrac{4}{x+2}$

$$y=\frac{4}{x+2}$$
$$x=\frac{4}{y+2} \quad \text{Inverse}$$
$$x(y+2)=4$$
$$xy+2x=4$$
$$xy=4-2x$$
$$y=\frac{4-2x}{x}$$
$$f^{-1}(x)=\frac{4-2x}{x}$$




Verify: $f\left(f^{-1}(x)\right)=f\left(\dfrac{4-2x}{x}\right)=\dfrac{4}{\dfrac{4-2x}{x}+2}=\dfrac{4}{\dfrac{4-2x+2x}{x}}=\dfrac{4}{\dfrac{4}{x}}=x$

$$f^{-1}\left(f(x)\right)=f^{-1}\left(\frac{4}{x+2}\right)=\frac{4-2\left(\frac{4}{x+2}\right)}{\left(\frac{4}{x+2}\right)}=\frac{\dfrac{4x+8-8}{x+2}}{\dfrac{4}{x+2}}=\frac{4x}{x+2}\ \frac{x+2}{4}=x$$

Domain of $f$ = range of $f^{-1}$ = all real numbers except –2
Range of $f$ = domain of $f^{-1}$ = all real numbers except 0

43. $f(x)=\dfrac{2}{3+x}$

$$y=\frac{2}{3+x}$$
$$x=\frac{2}{3+y} \quad \text{Inverse}$$
$$x(3+y)=2$$
$$3x+xy=2$$
$$xy=2-3x$$
$$y=\frac{2-3x}{x}$$
$$f^{-1}(x)=\frac{2-3x}{x}$$

Domain of $f$ =
range of $f^{-1}$ = all real numbers except –3

Range of $f$ =
domain of $f^{-1}$ = all real numbers except 0

Verify: $f\left(f^{-1}(x)\right)=f\left(\dfrac{2-3x}{x}\right)=\dfrac{2}{3+\dfrac{2-3x}{x}}=\dfrac{2}{\dfrac{3x+2-3x}{x}}=\dfrac{2}{\dfrac{2}{x}}=2\ \dfrac{x}{2}=x$

$$f^{-1}\left(f(x)\right)=f^{-1}\left(\frac{2}{3+x}\right)=\frac{2-3\left(\frac{2}{3+x}\right)}{\left(\frac{2}{3+x}\right)}=\frac{\dfrac{6+2x-6}{3+x}}{\dfrac{2}{3+x}}=\frac{2x}{3+x}\ \frac{3+x}{2}=x$$

44. $f(x) = \dfrac{4}{2-x}$

$y = \dfrac{4}{2-x}$

$x = \dfrac{4}{2-y}$ Inverse

$x(2-y) = 4$

$2x - xy = 4$

$xy = 2x - 4$

$y = \dfrac{2x-4}{x}$

$f^{-1}(x) = \dfrac{2x-4}{x}$

Domain of $f$ = range of $f^{-1}$ = all real numbers except 2

Range of $f$ = domain of $f^{-1}$ = all real numbers except 0

Verify: $f\left(f^{-1}(x)\right) = f\left(\dfrac{2x-4}{x}\right) = \dfrac{4}{2-\dfrac{2x-4}{x}} = \dfrac{4}{\dfrac{2x-2x+4}{x}} = \dfrac{4}{\dfrac{4}{x}} = 4\ \dfrac{x}{4} = x$

$f^{-1}\left(f(x)\right) = f^{-1}\left(\dfrac{4}{2-x}\right) = \dfrac{2\left(\dfrac{4}{2-x}\right)-4}{\left(\dfrac{4}{2-x}\right)} = \dfrac{\dfrac{8-8+4x}{2-x}}{\dfrac{4}{2-x}} = \dfrac{4x}{2-x}\ \dfrac{2-x}{4} = x$

45. $f(x) = \dfrac{3x}{x+2}$

$y = \dfrac{3x}{x+2}$

$x = \dfrac{3y}{y+2}$ Inverse

$x\left(y+2\right) = 3y$

$xy + 2x = 3y$

$2x = 3y - xy$

$2x = y\left(3-x\right)$

$\dfrac{2x}{3-x} = y$

$f^{-1}(x) = \dfrac{2x}{3-x}$

Domain of $f$ = range of $f^{-1}$ = all real numbers except – 2

Range of $f$ = domain of $f^{-1}$ = all real numbers except 3

Verify:

$f\left(f^{-1}(x)\right) = f\left(\dfrac{2x}{3-x}\right) = \dfrac{3\left(\dfrac{2x}{3-x}\right)}{\left(\dfrac{2x}{3-x}\right)+2} = \dfrac{\left(\dfrac{6x}{3-x}\right)}{\left(\dfrac{2x+2(3-x)}{3-x}\right)} = \dfrac{\left(\dfrac{6x}{3-x}\right)}{\left(\dfrac{2x+6-2x}{3-x}\right)}$

$= \dfrac{\left(\dfrac{6x}{3-x}\right)}{\left(\dfrac{6}{3-x}\right)} = \left(\dfrac{6x}{3-x}\right)\left(\dfrac{3-x}{6}\right) = x$

$$f^{-1}\left(f(x)\right)=f^{-1}\left(\frac{3x}{x+2}\right)=\frac{2\left(\frac{3x}{x+2}\right)}{3-\left(\frac{3x}{x+2}\right)}=\frac{\left(\frac{6x}{x+2}\right)}{\left(\frac{3(x+2)-3x}{x+2}\right)}=\frac{\left(\frac{6x}{x+2}\right)}{\left(\frac{3x+6-3x}{x+2}\right)}$$

$$=\frac{\left(\frac{6x}{x+2}\right)}{\left(\frac{6}{x+2}\right)}=\left(\frac{6x}{x+2}\right)\left(\frac{x+2}{6}\right)=x$$

46. $f(x)=\dfrac{-2x}{x-1}$

$$y=\frac{-2x}{x-1}$$

$$x=\frac{-2y}{y-1} \quad \text{Inverse}$$

$$x(y-1)=-2y$$

$$xy-x=-2y$$

$$xy+2y=x$$

$$y(x+2)=x$$

$$y=\frac{x}{x+2}$$

$$f^{-1}(x)=\frac{x}{x+2}$$

Domain of $f$ = range of $f^{-1}$ = all real numbers except 1

Range of $f$ = domain of $f^{-1}$ = all real numbers except - 2

Verify:

$$f\left(f^{-1}(x)\right)=f\left(\frac{x}{x+2}\right)=\frac{-2\left(\frac{x}{x+2}\right)}{\left(\frac{x}{x+2}\right)-1}=\frac{\left(\frac{-2x}{x+2}\right)}{\left(\frac{x-1(x+2)}{x+2}\right)}=\frac{\left(\frac{-2x}{x+2}\right)}{\left(\frac{x-x-2}{x+2}\right)}$$

$$=\frac{\left(\frac{-2x}{x+2}\right)}{\left(\frac{-2}{x+2}\right)}=\left(\frac{-2x}{x+2}\right)\left(\frac{x+2}{-2}\right)=x$$

$$f^{-1}\left(f(x)\right) = f^{-1}\left(\frac{-2x}{x-1}\right) = \frac{\left(\frac{-2x}{x-1}\right)}{\left(\frac{-2x}{x-1}+2\right)} = \frac{\left(\frac{-2x}{x-1}\right)}{\left(\frac{-2x+2\left(x-1\right)}{x-1}\right)} = \frac{\left(\frac{-2x}{x-1}\right)}{\left(\frac{-2x+2x-2}{x-1}\right)}$$

$$= \frac{\left(\frac{-2x}{x-1}\right)}{\left(\frac{-2}{x-1}\right)} = \left(\frac{-2x}{x-1}\right)\left(\frac{x-1}{-2}\right) = x$$

47. $f(x) = \dfrac{2x}{3x-1}$

$$y = \frac{2x}{3x-1}$$

$$x = \frac{2y}{3y-1} \quad \text{Inverse}$$

$$x(3y-1) = 2y$$

$$3xy - x = 2y$$

$$3xy - 2y = x$$

$$y(3x-2) = x$$

$$y = \frac{x}{3x-2}$$

$$f^{-1}(x) = \frac{x}{3x-2}$$

Domain of $f$ =
range of $f^{-1}$ = all real numbers except $\dfrac{1}{3}$

Range of $f$ =
domain of $f^{-1}$ = all real numbers except $\dfrac{2}{3}$

Verify:

$$f\left(f^{-1}(x)\right) = f\left(\frac{x}{3x-2}\right) = \frac{2\left(\frac{x}{3x-2}\right)}{3\left(\frac{x}{3x-2}\right)-1} = \frac{\left(\frac{2x}{3x-2}\right)}{\left(\frac{3x-1\left(3x-2\right)}{3x-2}\right)} = \frac{\left(\frac{2x}{3x-2}\right)}{\left(\frac{3x-3x+2}{3x-2}\right)} = \frac{2x}{3x-2} \cdot \frac{3x-2}{2} = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}\left(\frac{2x}{3x-1}\right) = \frac{\left(\frac{2x}{3x-1}\right)}{3\left(\frac{2x}{3x-1}\right)-2} = \frac{\left(\frac{2x}{3x-1}\right)}{\left(\frac{6x-2\left(3x-1\right)}{3x-1}\right)} = \frac{\left(\frac{2x}{3x-1}\right)}{\left(\frac{6x-6x+2}{3x-1}\right)} = \frac{2x}{3x-1} \cdot \frac{3x-1}{2} = x$$

48. $f(x)=\dfrac{3x+1}{-x}$

$$y=\frac{3x+1}{-x}$$

$$x=\frac{3y+1}{-y} \quad \text{Inverse}$$

$$-xy=3y+1$$

$$-xy-3y=1$$

$$y(-x-3)=1$$

$$y=\frac{1}{-x-3}$$

$$f^{-1}(x)=\frac{1}{-x-3}$$

Domain of $f$ = range of $f^{-1}$ = all real numbers except 0

Range of $f$ = domain of $f^{-1}$ = all real numbers except - 3

Verify: $$f\left(f^{-1}(x)\right)=f\left(\frac{1}{-x-3}\right)=\frac{3\left(\frac{1}{-x-3}\right)+1}{-\left(\frac{1}{-x-3}\right)}=\frac{\frac{3-x-3}{-x-3}}{\frac{-1}{-x-3}}=\frac{\frac{-x}{-x-3}}{\frac{-1}{-x-3}}=\frac{-x}{-x-3}\ \frac{-x-3}{-1}=x$$

$$f^{-1}\left(f(x)\right)=f^{-1}\left(\frac{3x+1}{x}\right)=\frac{1}{-\left(\frac{3x+1}{-x}\right)-3}=\frac{1}{\left(\frac{3x+1-3x}{x}\right)}=1\ \frac{x}{1}=x$$

49. $f(x)=\dfrac{3x+4}{2x-3}$

$$y=\frac{3x+4}{2x-3}$$

$$x=\frac{3y+4}{2y-3} \quad \text{Inverse}$$

$$x(2y-3)=3y+4$$

$$2xy-3x=3y+4$$

$$2xy-3y=3x+4$$

$$y(2x-3)=3x+4$$

$$y=\frac{3x+4}{2x-3}$$

$$f^{-1}(x)=\frac{3x+4}{2x-3}$$

Domain of $f$ = range of $f^{-1}$ = all real numbers except $\frac{3}{2}$

Range of $f$ = domain of $f^{-1}$ = all real numbers except $\frac{3}{2}$

Verify: $$f\left(f^{-1}(x)\right)=f\left(\frac{3x+4}{2x-3}\right)=\frac{3\left(\frac{3x+4}{2x-3}\right)+4}{2\left(\frac{3x+4}{2x-3}\right)-3}=\frac{\frac{9x+12+8x-12}{2x-3}}{\frac{6x+8-6x+9}{2x-3}}=\frac{\frac{17x}{2x-3}}{\frac{17}{2x-3}}$$

$$=\frac{17x}{2x-3}\ \frac{2x-3}{17}=x$$

$$f^{-1}\big(f(x)\big) = f^{-1}\left(\frac{3x+4}{2x-3}\right) = \frac{3\left(\frac{3x+4}{2x-3}\right)+4}{2\left(\frac{3x+4}{2x-3}\right)-3} = \frac{\frac{9x+12+8x-12}{2x-3}}{\frac{6x+8-6x+9}{2x-3}} = \frac{\frac{17x}{2x-3}}{\frac{17}{2x-3}}$$

$$= \frac{17x}{2x-3} \cdot \frac{2x-3}{17} = x$$

50. $f(x) = \dfrac{2x-3}{x+4}$

$$\begin{aligned}
y &= \frac{2x-3}{x+4} \\
x &= \frac{2y-3}{y+4} \quad \text{Inverse} \\
x(y+4) &= 2y-3 \\
xy+4x &= 2y-3 \\
xy-2y &= -4x-3 \\
y(x-2) &= -4x-3 \\
y &= \frac{-4x-3}{x-2} \\
f^{-1}(x) &= \frac{-4x-3}{x-2}
\end{aligned}$$

Domain of $f$ = range of $f^{-1}$ = all real numbers except –4

Range of $f$ = domain of $f^{-1}$ = all real numbers except 2

Verify:

$$f\big(f^{-1}(x)\big) = f\left(\frac{-4x-3}{x-2}\right) = \frac{2\left(\frac{-4x-3}{x-2}\right)-3}{\left(\frac{-4x-3}{x-2}\right)+4} = \frac{\frac{-8x-6-3x+6}{x-2}}{\frac{-4x-3+4x-8}{x-2}} = \frac{\frac{-11x}{x-2}}{\frac{-11}{x-2}}$$

$$= \frac{-11x}{x-2} \cdot \frac{x-2}{-11} = x$$

$$f^{-1}\big(f(x)\big) = f^{-1}\left(\frac{2x-3}{x+4}\right) = \frac{-4\left(\frac{2x-3}{x+4}\right)-3}{\left(\frac{2x-3}{x+4}\right)-2} = \frac{\frac{-8x+12-3x-12}{x+4}}{\frac{2x-3-2x-8}{x+4}} = \frac{\frac{-11x}{x+4}}{\frac{-11}{x+4}}$$

$$= \frac{-11x}{x+4} \cdot \frac{x+4}{-11} = x$$

51. $f(x) = \dfrac{2x+3}{x+2}$

$$\begin{aligned}
y &= \frac{2x+3}{x+2} \\
x &= \frac{2y+3}{y+2} \quad \text{Inverse} \\
x(y+2) &= 2y+3 \\
xy+2x &= 2y+3 \\
xy-2y &= -2x+3 \\
y(x-2) &= -2x+3 \\
y &= \frac{-2x+3}{x-2} \\
f^{-1}(x) &= \frac{-2x+3}{x-2}
\end{aligned}$$

Domain of $f$ = range of $f^{-1}$ = all real numbers except –2

Range of $f$ = domain of $f^{-1}$ = all real numbers except 2

Verify: $$f\left(f^{-1}(x)\right) = f\left(\frac{-2x+3}{x-2}\right) = \frac{2\left(\frac{-2x+3}{x-2}\right)+3}{\left(\frac{-2x+3}{x-2}\right)+2} = \frac{\frac{-4x+6+3x-6}{x-2}}{\frac{-2x+3+2x-4}{x-2}} = \frac{\frac{-x}{x-2}}{\frac{-1}{x-2}}$$

$$= \frac{-x}{x-2}\ \frac{x-2}{-1} = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}\left(\frac{2x+3}{x+2}\right) = \frac{-2\left(\frac{2x+3}{x+2}\right)+3}{\left(\frac{2x+3}{x+2}\right)-2} = \frac{\frac{-4x-6+3x+6}{x+2}}{\frac{2x+3-2x-4}{x+2}} = \frac{\frac{-x}{x+2}}{\frac{-1}{x+2}}$$

$$= \frac{-x}{x+2}\ \frac{x+2}{-1} = x$$

52. $f(x) = \dfrac{-3x-4}{x-2}$

$$y = \frac{-3x-4}{x-2}$$

$$x = \frac{-3y-4}{y-2} \quad \text{Inverse}$$

$$x(y-2) = -3y-4$$

$$xy-2x = -3y-4$$

$$xy+3y = 2x-4$$

$$y(x+3) = 2x-4$$

$$y = \frac{2x-4}{x+3}$$

$$f^{-1}(x) = \frac{2x-4}{x+3}$$

Domain of $f$ = range of $f^{-1}$ = all real numbers except 2

Range of $f$ = domain of $f^{-1}$ = all real numbers except –3

Verify: $$f\left(f^{-1}(x)\right) = f\left(\frac{2x-4}{x+3}\right) = \frac{-3\left(\frac{2x-4}{x+3}\right)-4}{\left(\frac{2x-4}{x+3}\right)-2} = \frac{\frac{-6x+12-4x-12}{x+3}}{\frac{2x-4-2x-6}{x+3}} = \frac{\frac{-10x}{x+3}}{\frac{-10}{x+3}}$$

$$= \frac{-10x}{x+3}\ \frac{x+3}{-10} = x$$

$$f^{-1}\left(f(x)\right) = f^{-1}\left(\frac{-3x-4}{x-2}\right) = \frac{2\left(\frac{-3x-4}{x-2}\right)-4}{\left(\frac{-3x-4}{x-2}\right)+3} = \frac{\frac{-6x-8-4x+8}{x-2}}{\frac{-3x-4+3x-6}{x-2}} = \frac{\frac{-10x}{x-2}}{\frac{-10}{x-2}}$$

$$= \frac{-10x}{x-2}\ \frac{x-2}{-10} = x$$

53. $f(x)=\dfrac{x^2-4}{2x^2}, x>0$

$$y=\frac{x^2-4}{2x^2}, x>0$$

$$x=\frac{y^2-4}{2y^2}, y>0 \quad \text{Inverse}$$

$$2xy^2=y^2-4, y>0$$

$$2xy^2-y^2=-4, y>0$$

$$y^2(2x-1)=-4, y>0$$

$$y^2=\frac{-4}{2x-1} \rightarrow y=\sqrt{\frac{-4}{2x-1}}$$

$$f^{-1}(x)=\sqrt{\frac{-4}{2x-1}}, x>0$$

Domain of $f$ = range of $f^{-1}$ = $(0,\infty)$

Range of $f$ = domain of = $f^{-1}=\left(0,\frac{1}{2}\right)$

Verify:

$$f\left(f^{-1}(x)\right)=f\left(\sqrt{\frac{-4}{2x-1}}\right)=\frac{\left(\sqrt{\frac{-4}{2x-1}}\right)^2-4}{2\left(\sqrt{\frac{-4}{2x-1}}\right)^2}=\frac{\left(\frac{-4}{2x-1}-4\right)}{2\left(\frac{-4}{2x-1}\right)}=\frac{\left(\frac{-4-4(2x-1)}{2x-1}\right)}{\left(\frac{-8}{2x-1}\right)}=\frac{\left(\frac{-4-8x+4}{2x-1}\right)}{\left(\frac{-8}{2x-1}\right)}$$

$$=\frac{\left(\frac{-8x}{2x-1}\right)}{\left(\frac{-8}{2x-1}\right)}=\left(\frac{-8x}{2x-1}\right)\left(\frac{2x-1}{-8}\right)=x$$

$$f^{-1}\left(f(x)\right)=f^{-1}\left(\frac{x^2-4}{2x^2}\right)=\sqrt{\frac{-4}{2\left(\frac{x^2-4}{2x^2}\right)-1}}=\sqrt{\frac{-4}{\left(\frac{x^2-4}{x^2}\right)-1}}=\sqrt{\frac{-4}{\left(\frac{x^2-4-x^2}{x^2}\right)}}=\sqrt{\frac{-4}{\left(\frac{-4}{x^2}\right)}}$$

$$=\sqrt{\frac{-4}{1}\cdot\frac{x^2}{-4}}=\sqrt{x^2}=x$$

54. $f(x)=\dfrac{x^2+3}{3x^2},\ x>0$

Domain of $f$ = range of $f^{-1}$ = $(0,\infty)$

$$y=\frac{x^2+3}{3x^2},\ x>0$$

$$x=\frac{y^2+3}{3y^2},\ x>0 \quad \text{Inverse}$$

$$3xy^2=y^2+3,\ x>0$$

$$3xy^2-y^2=3,\ x>0$$

$$y^2(3x-1)=3,\ x>0$$

$$y^2=\frac{3}{3x-1}\ \rightarrow\ y=\sqrt{\frac{3}{3x-1}}$$

$$f^{-1}(x)=\sqrt{\frac{3}{3x-1}},\ x>0$$

Range of $f$ = domain of = $f^{-1}=\left(\frac{1}{3},0\right)$

Verify:

$$f\left(f^{-1}(x)\right)=f\left(\sqrt{\frac{3}{3x-1}}\right)=\frac{\left(\sqrt{\frac{3}{3x-1}}\right)^2+3}{3\left(\sqrt{\frac{3}{3x-1}}\right)^2}=\frac{\left(\frac{3}{3x-1}+3\right)}{3\left(\frac{3}{3x-1}\right)}=\frac{\left(\frac{3+3(3x-1)}{3x-1}\right)}{\left(\frac{9}{3x-1}\right)}=\frac{\left(\frac{3+9x-3}{3x-1}\right)}{\left(\frac{9}{3x-1}\right)}$$

$$=\frac{\left(\frac{9x}{3x-1}\right)}{\left(\frac{9}{3x-1}\right)}=\left(\frac{9x}{3x-1}\right)\left(\frac{3x-1}{9}\right)=x$$

$$f^{-1}\left(f(x)\right)=f^{-1}\left(\frac{x^2+3}{3x^2}\right)=\sqrt{\frac{3}{3\left(\frac{x^2+3}{3x^2}\right)-1}}=\sqrt{\frac{3}{\left(\frac{x^2+3}{x^2}\right)-1}}=\sqrt{\frac{3}{\left(\frac{x^2+3-x^2}{x^2}\right)}}=\sqrt{\frac{3}{\left(\frac{3}{x^2}\right)}}$$

$$=\sqrt{\frac{3}{1}\cdot\frac{x^2}{3}}=\sqrt{x^2}=x$$

55. $f(x)=mx+b,\quad m\neq 0$

$$y=mx+b$$

$$x=my+b \quad \text{Inverse}$$

$$x-b=my$$

$$y=\frac{x-b}{m}$$

$$f^{-1}(x)=\frac{x-b}{m},\quad m\neq 0$$

56. $f(x)=\sqrt{r^2-x^2},\quad 0\le x\le r$

$$y=\sqrt{r^2-x^2}$$

$$x=\sqrt{r^2-y^2} \quad \text{Inverse}$$

$$x^2=r^2-y^2$$

$$y^2=r^2-x^2$$

$$y=\sqrt{r^2-x^2}$$

$$f^{-1}(x)=\sqrt{r^2-x^2},\quad 0\le x\le r$$

57. $f^{-1}$ lies in quadrant I. Whenever $(a,b)$ is on $f$, then $(b,a)$ is on $f^{-1}$. Since both coordinates of $(a,b)$ are positive, both coordinates of $(b,a)$ are positive and it is in quadrant I.

58. $f^{-1}$ lies in quadrant IV. Whenever $(a,b)$ is on $f$, then $(b,a)$ is on $f^{-1}$. Since $a$ is negative and $b$ is positive, $(b,a)$ must be a point in quadrant IV.

59. $f(x) = |x|, x \geq 0$ is one-to-one. Thus, $f(x) = x, x \geq 0$ and $f^{-1}(x) = x, x \geq 0$.

60. $f(x) = x^4, x \geq 0$ is one-to-one.

$$y = x^4 \quad x \geq 0$$
$$x = y^4 \quad \text{Inverse}$$
$$y = \sqrt[4]{x}$$
$$f^{-1}(x) = \sqrt[4]{x}$$

61. $f(x) = \frac{9}{5}x + 32 \qquad g(x) = \frac{5}{9}(x-32)$

$$f(g(x)) = f\left(\tfrac{5}{9}(x-32)\right) = \tfrac{9}{5}\left(\tfrac{5}{9}(x-32)\right) + 32 = x - 32 + 32 = x$$
$$g(f(x)) = g\left(\tfrac{9}{5}x + 32\right) = \tfrac{5}{9}\left(\tfrac{9}{5}x + 32 - 32\right) = \tfrac{5}{9}\left(\tfrac{9}{5}x\right) = x$$

62.
$$p(x) = 300 - 50x$$
$$p = 300 - 50x$$
$$50x = 300 - p$$
$$x = \frac{300-p}{50}$$
$$x(p) = \frac{300-p}{50}$$

63. $T(l) = 2\pi\sqrt{\frac{l}{g}}, \quad g \approx 32.2$

$$T = 2\pi\sqrt{\frac{l}{g}} \rightarrow \frac{T}{2\pi} = \sqrt{\frac{l}{g}}$$
$$\rightarrow \frac{T^2}{4\pi^2} = \frac{l}{g} \rightarrow l = \frac{gT^2}{4\pi^2}$$
$$l(T) = \frac{gT^2}{4\pi^2}$$

64. $f(x) = \dfrac{ax+b}{cx+d}$

(a) domain of $f$ all real numbers except $\dfrac{-d}{c}$.

(b)
$$y = \frac{ax+b}{cx+d}$$
$$x = \frac{ay+b}{cy+d} \quad \text{Inverse}$$
$$x(cy+d) = ay+b$$
$$cxy + dx = ay + b$$
$$cxy - ay = b - dx$$
$$y(cx-a) = b - dx \rightarrow y = \frac{b-dx}{cx-a} \rightarrow f^{-1}(x) = \frac{-dx+b}{cx-a}$$

(c) range of $f$ all real numbers except $\frac{a}{c}$.

(d)

$$f = f^{-1} \quad \text{if } \frac{ax+b}{cx+d} = \frac{-dx+b}{cx-a}$$

This is true if $a = -d$.

65. An even function cannot be one-to-one. When a function is even, $f(-x) = f(x)$. Thus, both $x$ and $-x$ produce the same $y$ value.

66. An odd function may not be one-to-one. Consider a function such as $f(x) = x^3 - x$.

67. If the graph of a function and its inverse intersect, they must intersect at a point on the line $y = x$. However, the graphs do not have to intersect.

68. Yes, consider the function $f(x) = \frac{1}{x}$. In general the graph of $f$ must have symmetry across the line $y = x$.

69. Answers will vary.

70. $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ -x, & \text{if } x \text{ is irrational} \end{cases}$