Chapter 6

Exponential and Logarithmic Functions

6.3 Logarithmic Functions

1. $9=3^2$ is equivalent to $2=\log_3 9$
2. $16=4^2$ is equivalent to $2=\log_4 16$
3. $a^2=1.6$ is equivalent to $2=\log_a 1.6$
4. $a^3=2.1$ is equivalent to $3=\log_a 2.1$
5. $1.1^2=M$ is equivalent to $2=\log_{1.1} M$
6. $2.2^3=N$ is equivalent to $3=\log_{2.2} N$
7. $2^x=7.2$ is equivalent to $x=\log_2 7.2$
8. $3^x=4.6$ is equivalent to $x=\log_3 4.6$
9. $x^{\sqrt{2}}=\pi$ is equivalent to $\sqrt{2}=\log_x \pi$
10. $x^\pi=e$ is equivalent to $\pi=\log_x e$
11. $e^x=8$ is equivalent to $x=\ln 8$
12. $e^{2.2}=M$ is equivalent to $2.2=\ln M$
13. $\log_2 8=3$ is equivalent to $2^3=8$
14. $\log_3\left(\frac{1}{9}\right)=-2$ is equivalent to $3^{-2}=\frac{1}{9}$
15. $\log_a 3=6$ is equivalent to $a^6=3$
16. $\log_b 4=2$ is equivalent to $b^2=4$
17. $\log_3 2=x$ is equivalent to $3^x=2$
18. $\log_2 6=x$ is equivalent to $2^x=6$
19. $\log_2 M=1.3$ is equivalent to $2^{1.3}=M$
20. $\log_3 N=2.1$ is equivalent to $3^{2.1}=N$
21. $\log_{\sqrt{2}}\pi=x$ is equivalent to $\left(\sqrt{2}\right)^x=\pi$
22. $\log_\pi x=\frac{1}{2}$ is equivalent to $\pi^{\frac{1}{2}}=x$
23. $\ln 4=x$ is equivalent to $e^x=4$
24. $\ln x=4$ is equivalent to $e^4=x$
25. $\log_2 1=0$ since $2^0=1$
26. $\log_8 8=1$ since $8^1=8$
27. $\log_5 25=2$ since $5^2=25$
28. $\log_3\left(\frac{1}{9}\right)=-2$ since $3^{-2}=\frac{1}{9}$
29. $\log_{\frac{1}{2}} 16=-4$ since $\left(\frac{1}{2}\right)^{-4}=2^4=16$
30. $\log_{\frac{1}{3}} 9=-2$ since $\left(\frac{1}{3}\right)^{-2}=3^2=9$
31. $\log_{10}\sqrt{10}=\frac{1}{2}$ since $10^{\frac{1}{2}}=\sqrt{10}$
32. $\log_5\sqrt[3]{25}=\frac{2}{3}$ since $5^{\frac{2}{3}}=25^{\frac{1}{3}}=\sqrt[3]{25}$

33. $\log_{\sqrt{2}} 4 = 4$ since $\left(\sqrt{2}\right)^4 = 4$

34. $\log_{\sqrt{3}} 9 = 4$ since $\left(\sqrt{3}\right)^4 = 9$

35. $\ln\sqrt{e} = \frac{1}{2}$ since $e^{\frac{1}{2}} = \sqrt{e}$

36. $\ln e^3 = 3$ since $e^3 = e^3$

37. The domain of $f(x) = \ln(x-3)$ is:
$$x - 3 > 0$$
$$x > 3$$
$$\{x \mid x > 3\}$$

38. The domain of $g(x) = \ln(x-1)$ is:
$$x - 1 > 0$$
$$x > 1$$
$$\{x \mid x > 1\}$$

39. The domain of $F(x) = \log_2 x^2$ is:
$$x^2 > 0$$
$$\{x \mid x \neq 0\}$$

40. The domain of $H(x) = \log_5 x^3$ is:
$$x^3 > 0$$
$$x > 0$$
$$\{x \mid x > 0\}$$

41. The domain of $h(x) = \log_{\frac{1}{2}}\left(x^2 - 2x + 1\right)$ is:
$$x^2 - 2x + 1 > 0$$
$$(x-1)^2 > 0$$
$$\{x \mid x \neq 1\}$$

42. The domain of $G(x) = \log_{\frac{1}{2}}\left(x^2 - 1\right)$ is:
$$x^2 - 1 > 0$$
$$(x+1)(x-1) > 0$$
$$x < -1 \text{ or } x > 1$$
$$\{x \mid x < -1 \text{ or } x > 1\}$$

43. The domain of $f(x) = \ln\left(\frac{1}{x+1}\right)$ is:
$$\frac{1}{x+1} > 0$$
$$x + 1 > 0$$
$$x > -1$$
$$\{x \mid x > -1\}$$

44. The domain of $g(x) = \ln\left(\frac{1}{x-5}\right)$ is:
$$\frac{1}{x-5} > 0$$
$$x - 5 > 0$$
$$x > 5$$
$$\{x \mid x > 5\}$$

45. The domain of $g(x) = \log_5\left(\frac{x+1}{x}\right)$ requires that $\frac{x+1}{x} > 0$.

The expression is zero or undefined when $x = -1$ or $x = 0$.

Interval	Test Number	$f(x) = \frac{x+1}{x}$	Positive/Negative
$-\infty < x < -1$	–2	1/2	Positive
$-1 < x < 0$	–0.5	–1	Negative
$0 < x < \infty$	1	2	Positive

The domain is $\{x \mid x < -1 \text{ or } x > 0\}$

46. The domain of $h(x) = \log_3\left(\frac{x}{x-1}\right)$ requires that $\frac{x}{x-1} > 0$.
The expression is zero or undefined when $x = 0$ or $x = 1$.

Interval	Test Number	$f(x) = \frac{x}{x-1}$	Positive/Negative
$-\infty < x < 0$	–1	1/2	Positive
$0 < x < 1$	0.5	–1	Negative
$1 < x < \infty$	2	2	Positive

The domain is $\{x \mid x < 0 \text{ or } x > 1\}$.

47. $\ln\frac{5}{3} = 0.511$

48. $\frac{\ln 5}{3} \approx 0.536$

49. $\frac{\ln(10/3)}{0.04} = 30.099$

50. $\frac{\ln(2/3)}{-0.1} \approx 4.055$

51. For $f(x) = \log_a x$, find a so that $f(2) = \log_a 2 = 2$ or $a^2 = 2$ or $a = \sqrt{2}$.
(The base a must be positive by definition.)

52. For $f(x) = \log_a x$, find a so that $f\left(\frac{1}{2}\right) = \log_a\left(\frac{1}{2}\right) = -4$.

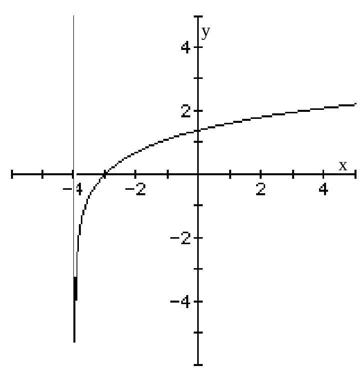
$$a^{-4} = \tfrac{1}{2} \quad \rightarrow \quad \frac{1}{a^4} = \frac{1}{2} \quad \rightarrow \quad a^4 = 2 \quad \rightarrow \quad a = \sqrt[4]{2}$$

(The base a must be positive by definition.)

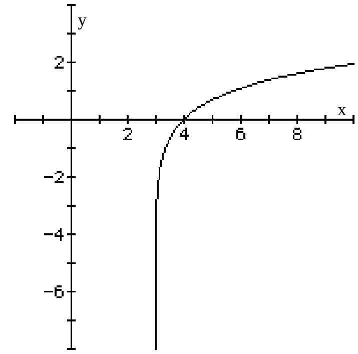
53. B 54. F 55. D 56. H

57. A 58. C 59. E 60. G

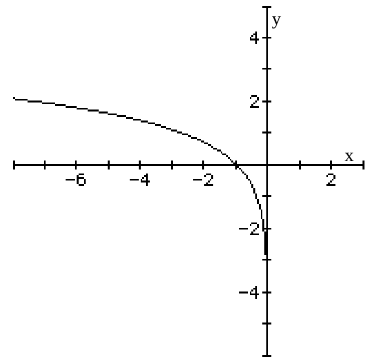
61. $f(x) = \ln(x + 4)$
Using the graph of $y = \ln x$, shift the graph 4 units to the left.
Domain: $(-4, \infty)$
Range: $(-\infty, \infty)$
Vertical Asymptote: $x = -4$



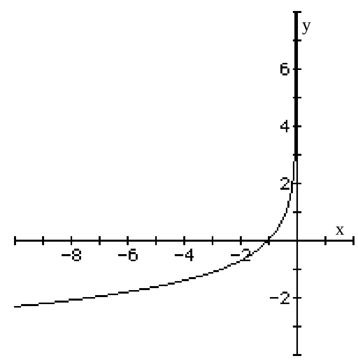
62. $f(x) = \ln(x - 3)$
Using the graph of $y = \ln x$, shift the graph 3 units to the right.
Domain: $(3, \infty)$
Range: $(-\infty, \infty)$
Vertical Asymptote: $x = 3$



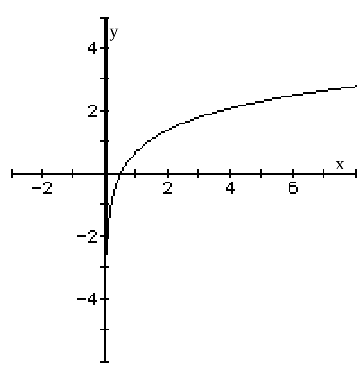
63. $f(x) = \ln(-x)$
Using the graph of $y = \ln x$, reflect the graph about the y-axis.
Domain: $(-\infty, 0)$
Range: $(-\infty, \infty)$
Vertical Asymptote: $x = 0$



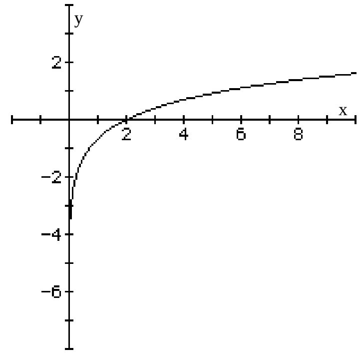
64. $f(x) = -\ln(-x)$
Using the graph of $y = \ln x$, reflect the graph about the y-axis, and reflect about the x-axis.
Domain: $(-\infty, 0)$
Range: $(-\infty, \infty)$
Vertical Asymptote: $x = 0$



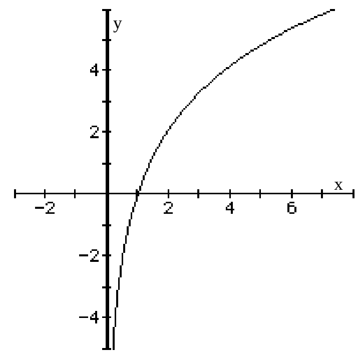
65. $g(x) = \ln(2x)$
Using the graph of $y = \ln x$, compress the graph horizontally by a factor of $\frac{1}{2}$.
Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Vertical Asymptote: $x = 0$



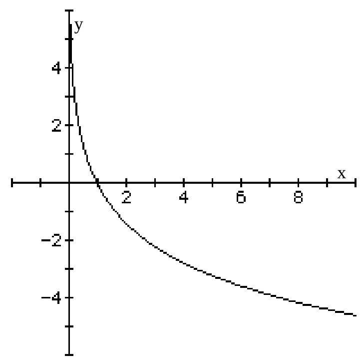
66. $h(x) = \ln\left(\frac{1}{2}x\right)$
Using the graph of $y = \ln x$, stretch the graph horizontally by a factor of 2.
Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Vertical Asymptote: $x = 0$



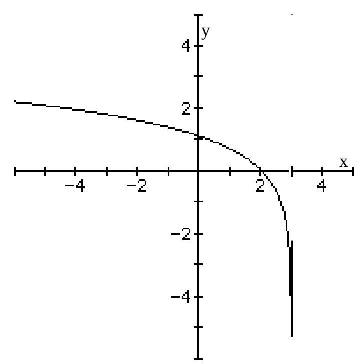
67. $f(x) = 3\ln x$
Using the graph of $y = \ln x$, stretch the graph vertically by a factor of 3.
Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Vertical Asymptote: $x = 0$



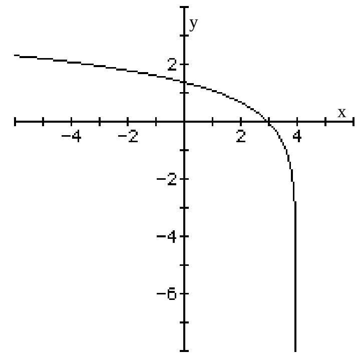
68. $f(x) = -2\ln x$
Using the graph of $y = \ln x$, stretch the graph vertically by a factor of 2, and reflect about the x-axis.
Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Vertical Asymptote: $x = 0$



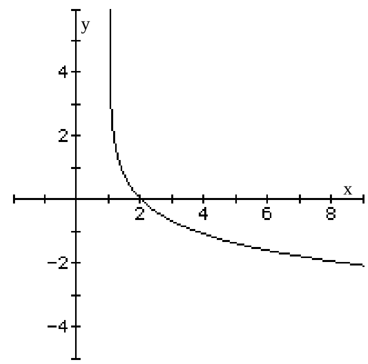
69. $g(x) = \ln(3 - x) = \ln(-(x - 3))$
Using the graph of $y = \ln x$, reflect the graph about the y-axis, and shift 3 units to the right.
Domain: $(-\infty, 3)$
Range: $(-\infty, \infty)$
Vertical Asymptote: $x = 3$



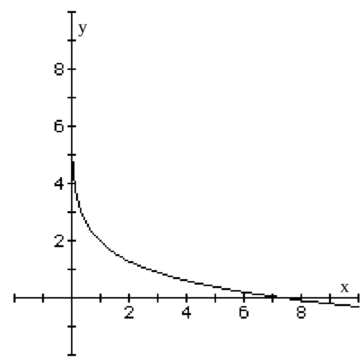
70. $h(x) = \ln(4 - x) = \ln\big(-(x - 4)\big)$
Using the graph of $y = \ln x$, reflect the graph about the y-axis, and shift 4 units to the right.
Domain: $(-\infty, 4)$
Range: $(-\infty, \infty)$
Vertical Asymptote: $x = 4$



71. $f(x) = -\ln(x - 1)$
Using the graph of $y = \ln x$, shift the graph 1 unit to the right, and reflect about the x-axis.
Domain: $(1, \infty)$
Range: $(-\infty, \infty)$
Vertical Asymptote: $x = 1$



72. $f(x) = 2 - \ln x$
Using the graph of $y = \ln x$, reflect the graph about the x-axis, and shift 2 units up.
Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Vertical Asymptote: $x = 0$



73. $\log_3 x = 2$
$x = 3^2$
$x = 9$

74. $\log_5 x = 3$
$x = 5^3$
$x = 125$

75. $\log_2(2x + 1) = 3$
$2x + 1 = 2^3$
$2x + 1 = 8$
$2x = 7$
$x = \frac{7}{2}$

76. $\log_3(3x - 2) = 2$
$3x - 2 = 3^2$
$3x - 2 = 9$
$3x = 11$
$x = \frac{11}{3}$

77. $\log_x 4 = 2$
$x^2 = 4$
$x = 2$ ($x \neq -2$, base is positive)

78. $\log_x\left(\frac{1}{8}\right) = 3$
$x^3 = \frac{1}{8}$
$x = \frac{1}{2}$

79. $\ln e^x = 5$
$e^x = e^5$
$x = 5$

80. $\ln e^{-2x} = 8$
$e^{-2x} = e^8$
$-2x = 8$
$x = -4$

81. $\log_4 64 = x$
$4^x = 64$
$4^x = 4^3$
$x = 3$

82. $\log_5 625 = x$
$5^x = 625$
$5^x = 5^4$
$x = 4$

83. $\log_3 243 = 2x + 1$
$3^{2x+1} = 243$
$3^{2x+1} = 3^5$
$2x + 1 = 5$
$2x = 4$
$x = 2$

84. $\log_6 36 = 5x + 3$
$6^{5x+3} = 36$
$6^{5x+3} = 6^2$
$5x + 3 = 2$
$5x = -1$
$x = -\frac{1}{5}$

85. $e^{3x} = 10$
$3x = \ln 10$
$x = \dfrac{\ln 10}{3}$

86. $e^{-2x} = \frac{1}{3}$
$-2x = \ln\frac{1}{3}$
$x = \dfrac{\ln\frac{1}{3}}{-2}$

87. $e^{2x+5} = 8$
$2x + 5 = \ln 8$
$2x = -5 + \ln 8$
$x = \dfrac{-5 + \ln 8}{2}$

88. $e^{-2x+1} = 13$
$-2x + 1 = \ln 13$
$-2x = -1 + \ln 13$
$x = \dfrac{-1 + \ln 13}{-2}$

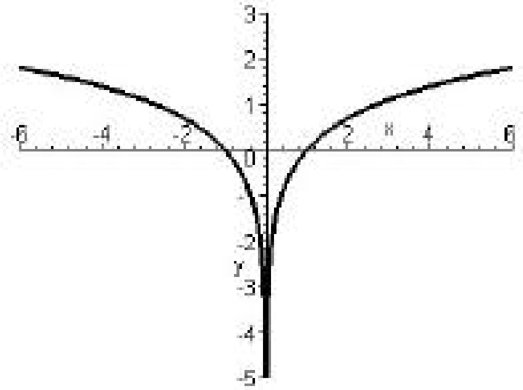
89. $\log_3\left(x^2 + 1\right) = 2$
$x^2 + 1 = 3^2$
$x^2 + 1 = 9$
$x^2 = 8$
$x = \pm\sqrt{8} = \pm 2\sqrt{2}$

90. $\log_5\left(x^2 + x + 4\right) = 2$
$x^2 + x + 4 = 5^2$
$x^2 + x + 4 = 25$
$x^2 + x - 21 = 0$
$x = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-21)}}{2(1)} = \dfrac{-1 \pm \sqrt{85}}{2}$
$x = \dfrac{-1 - \sqrt{85}}{2}$ or $\dfrac{-1 + \sqrt{85}}{2}$

91. $\log_2 8^x = -3$
$8^x = 2^{-3}$
$8^x = \frac{1}{8}$
$8^x = 8^{-1}$
$x = -1$

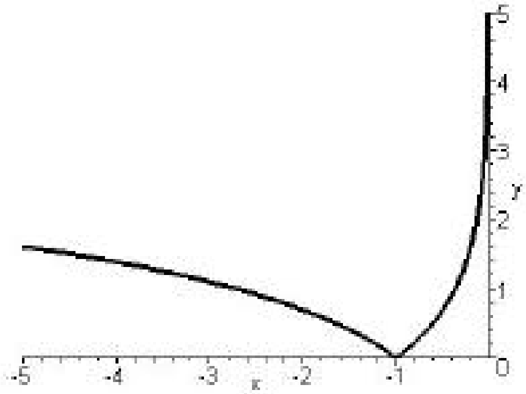
92. $\log_3 3^x = -1$
$3^x = 3^{-1}$
$x = -1$

93. $f(x) = \begin{cases} \ln(-x) & \text{if } x < 0 \\ \ln x & \text{if } x > 0 \end{cases}$



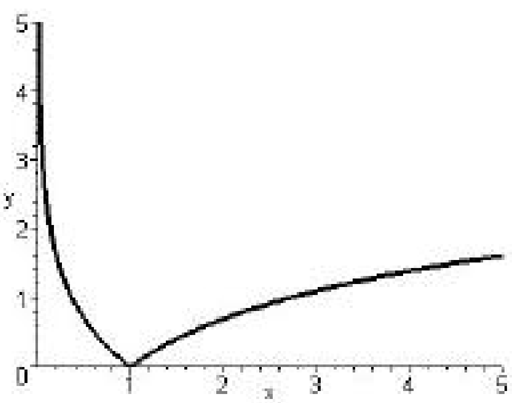
Domain: $(-\infty, 0)$ $(0, \infty)$
Range: $(-\infty, \infty)$
x-intercept: (- 1, 0), (1, 0)
vertical asymptote: $x = 0$

94. $f(x) = \begin{cases} \ln(-x) & \text{if } x \le -1 \\ -\ln(-x) & \text{if } -1 < x < 0 \end{cases}$



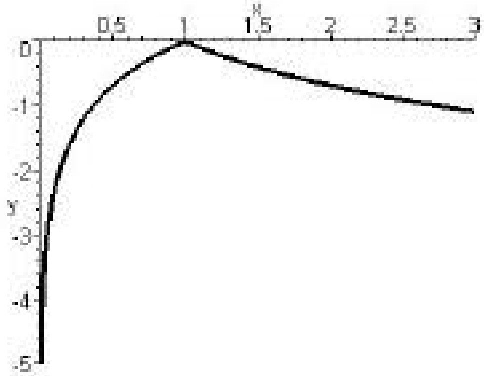
Domain: $(-\infty, 0)$
Range: $[0, \infty)$
x-intercept (- 1, 0)
vertical asymptote: $x = 0$

95. $f(x) = \begin{cases} -\ln x & \text{if } 0 < x < 1 \\ \ln x & \text{if } x \ge 1 \end{cases}$



Domain: $(0, \infty)$
Range: $[0, \infty)$
x-intercept: (1, 0)
vertical asymptote: $x = 0$

96. $f(x)=\begin{cases}\ln x & \text{if } 0<x<1\\ -\ln x & \text{if } x\ge 1\end{cases}$



Domain: (0, ∞)
Range: $(-\infty, 0]$
x-intercept: (1, 0)
vertical asymptote: $x=0$

97. $P=100e^{-0.1n}$

(a) $50=100e^{-0.1n}$
$0.5=e^{-0.1n}$
$\ln 0.5=-0.1n$
$n=\dfrac{\ln 0.5}{-0.1}$
$n\approx 6.93$
7 panes of glass are needed.

(b) $25=100e^{-0.1n}$
$0.25=e^{-0.1n}$
$\ln 0.25=-0.1n$
$n=\dfrac{\ln 0.25}{-0.1}$
$n\approx 13.86$
14 panes of glass are needed.

98. $pH=-\log_{10}\left[H^+\right]$

(a) $pH=-\log_{10}[0.0000001]=-(-7)=7$

(b) $4.2=-\log_{10}\left[H^+\right] \rightarrow -4.2=\log_{10}\left[H^+\right] \rightarrow \left[H^+\right]=10^{-4.2}$
$=6.31\times 10^{-5}=0.0000631$

99. $w=50e^{-0.004d}$

(a) $30=50e^{-0.004d}$
$0.6=e^{-0.004d}$
$\ln 0.6=-0.004d$
$d=\dfrac{\ln 0.6}{-0.004}$
$d\approx 127.7$
Approximately 128 days.

(b) $5=50e^{-0.004d}$
$0.1=e^{-0.004d}$
$\ln 0.1=-0.004d$
$d=\dfrac{\ln 0.1}{-0.004}$
$d\approx 575.6$
Approximately 576 days.

100. $A=A_0e^{-0.35n}$

(a) $50=100e^{-0.35n}$
$0.5=e^{-0.35n}$
$\ln 0.5=-0.35n$
$n=\dfrac{\ln 0.5}{-0.35}\approx 1.98$ days
Approximately 2 days.

(b) $10=100e^{-0.35n}$
$0.1=e^{-0.35n}$
$\ln 0.1=-0.35n$
$n=\dfrac{\ln 0.1}{-0.35}\approx 6.58$ days
Approximately 6.6 days.

101. $F(t) = 1 - e^{-0.1t}$

(a)
$$\begin{aligned} 0.5 &= 1 - e^{-0.1t} \\ -0.5 &= -e^{-0.1t} \\ 0.5 &= e^{-0.1t} \\ \ln 0.5 &= -0.1t \\ t &= \frac{\ln 0.5}{-0.1} \\ n &\approx 6.93 \end{aligned}$$
Approximately 7 minutes.

(b)
$$\begin{aligned} 0.8 &= 1 - e^{-0.1t} \\ -0.2 &= -e^{-0.1t} \\ 0.2 &= e^{-0.1t} \\ \ln 0.2 &= -0.1t \\ t &= \frac{\ln 0.2}{-0.1} \\ n &\approx 16.09 \end{aligned}$$
Approximately 16 minutes.

(c) It is impossible for the probability to reach 100% because $e^{-0.1t}$ will never equal zero.

102. $F(t) = 1 - e^{-0.15t}$

(a)
$$\begin{aligned} 0.50 &= 1 - e^{-0.15t} \\ -0.5 &= -e^{-0.15t} \\ 0.5 &= e^{-0.15t} \\ \ln 0.5 &= -0.15t \\ t &= \frac{\ln 0.5}{-0.15} \approx 4.62 \text{ minutes} \end{aligned}$$
Approximately 5 minutes.

(b)
$$\begin{aligned} 0.80 &= 1 - e^{-0.15t} \\ -0.2 &= -e^{-0.15t} \\ 0.2 &= e^{-0.15t} \\ \ln 0.2 &= -0.15t \\ t &= \frac{\ln 0.2}{-0.15} \approx 10.73 \text{ minutes} \end{aligned}$$
Approximately 11 minutes.

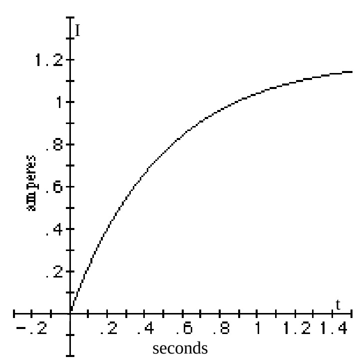
103. $D = 5e^{-0.4h}$
$$\begin{aligned} 2 &= 5e^{-0.4h} \\ 0.4 &= e^{-0.4h} \\ \ln 0.4 &= -0.4h \\ h &= \frac{\ln 0.4}{-0.4} \\ h &\approx 2.29 \text{ hours} \end{aligned}$$

104. $N = P\left(1 - e^{-0.15d}\right)$
$$\begin{aligned} 450 &= 1000\left(1 - e^{-0.15d}\right) \\ 0.45 &= 1 - e^{-0.15d} \\ -0.55 &= -e^{-0.15d} \\ 0.55 &= e^{-0.15d} \\ \ln 0.55 &= -0.15d \\ d &= \frac{\ln 0.55}{-0.15} \approx 3.99 \text{ days} \end{aligned}$$

105. $I = \dfrac{E}{R}\left[1 - e^{-\left(\frac{R}{L}\right)t}\right]$

0.5 ampere:
$$\begin{aligned} 0.5 &= \frac{12}{10}\left[1 - e^{-\left(\frac{10}{5}\right)t}\right] \\ 0.4167 &= 1 - e^{-2t} \\ e^{-2t} &= 0.5833 \\ -2t &= \ln 0.5833 \\ t &= \frac{\ln 0.5833}{-2} \\ t &= 0.2695 \text{ seconds} \end{aligned}$$

1.0 ampere:
$$\begin{aligned} 1.0 &= \frac{12}{10}\left[1 - e^{-\left(\frac{10}{5}\right)t}\right] \\ 0.8333 &= 1 - e^{-2t} \\ e^{-2t} &= 0.1667 \\ -2t &= \ln 0.1667 \\ t &= \frac{\ln 0.1667}{-2} \\ t &= 0.8958 \text{ seconds} \end{aligned}$$



106. $L(t) = A\left(1 - e^{-k\,t}\right)$

(a) $20 = 200\left(1 - e^{-k(5)}\right)$

$0.1 = 1 - e^{-5k}$

$e^{-5k} = 0.9$

$-5k = \ln 0.9$

$k = \dfrac{\ln 0.9}{-5} \approx 0.021$

(b) $L(10) = 200\left(1 - e^{-0.021(10)}\right) = 200\left(1 - e^{-0.21}\right) = 200\left(1 - 0.811\right) \approx 38$ words

(c) $L(15) = 200\left(1 - e^{-0.021(15)}\right) = 200\left(1 - e^{-0.315}\right) = 200\left(1 - 0.730\right) \approx 54$ words

(d) $180 = 200\left(1 - e^{-0.021\,t}\right)$

$0.9 = 1 - e^{-0.021\,t}$

$e^{-0.021\,t} = 0.1$

$-0.021t = \ln 0.1$

$t = \dfrac{\ln 0.1}{-0.021} \approx 109.65 \approx 110$ days

107. $R = 3e^{kx}$

(a)

$10 = 3e^{k(0.06)}$

$\dfrac{10}{3} = e^{k(0.06)}$

$\ln\left(\dfrac{10}{3}\right) = k(0.06)$

$k = \dfrac{\ln\left(\dfrac{10}{3}\right)}{0.06} \approx 20.07$

(b)

$R = 3e^{(20.07)(0.17)}$

$R = 3e^{3.4119} \approx 90.97\%$

(c)

$100 = 3e^{(20.07)x}$

$\dfrac{100}{3} = e^{(20.07)x}$

$\ln\left(\dfrac{100}{3}\right) = (20.07)x$

$x = \dfrac{\ln\left(\dfrac{100}{3}\right)}{(20.07)} \approx 0.1747$

(d)

$15 = 3e^{(20.07)(x)}$

$5 = e^{(20.07)(x)}$

$\ln(5) = (20.07)(x)$

$x = \dfrac{\ln(5)}{20.07} \approx 0.080$

(e) Answers will vary.

108. Answers will vary.

109. New = Old$\left(e^{Rt}\right)$

Age	Depreciation rate		Age	Depreciation rate
1	$38000 = 36600e^{R\,(1)}$ $\dfrac{38000}{36600} = e^{R}$ $\ln\left(\dfrac{38000}{36600}\right) = R$ $\rightarrow\ R \approx 0.03754 = 3.8\%$		2	$38000 = 32400e^{R\,(2)}$ $\dfrac{38000}{32400} = e^{2R}$ $\ln\left(\dfrac{38000}{32400}\right) = 2R$ $\dfrac{\ln\left(\dfrac{38000}{32400}\right)}{2} = R$ $\rightarrow\ R \approx 0.07971 = 8\%$

Age	Depreciation rate		Age	Depreciation rate
3	$38000 = 28750e^{R\,(3)}$ $\dfrac{38000}{28750} = e^{3R}$ $\ln\left(\dfrac{38000}{28750}\right) = 3R$ $\dfrac{\ln\left(\dfrac{38000}{28750}\right)}{3} = R$ $\rightarrow\ R \approx 0.0930 = 9.3\%$		4	$38000 = 25400e^{R\,(4)}$ $\dfrac{38000}{25400} = e^{4R}$ $\ln\left(\dfrac{38000}{25400}\right) = 4R$ $\dfrac{\ln\left(\dfrac{38000}{28750}\right)}{4} = R$ $\rightarrow\ R \approx 0.1007 = 10.1\%$

Age	Depreciation rate
5	$38000 = 21200e^{R\,(5)}$ $\dfrac{38000}{21200} = e^{5R}$ $\ln\left(\dfrac{38000}{21200}\right) = 5R$ $\dfrac{\ln\left(\dfrac{38000}{21200}\right)}{5} = R \rightarrow R \approx 0.1167$