Chapter 6

Exponential and Logarithmic Functions

6.5 Logarithmic and Exponential Equations

1. $$\begin{aligned} \log_4(x+2) &= \log_4 8 \\ x+2 &= 8 \\ x &= 6 \end{aligned}$$

2. $$\begin{aligned} \log_5(2x+3) &= \log_5 3 \\ 2x+3 &= 3 \\ 2x &= 0 \\ x &= 0 \end{aligned}$$

3. $$\begin{aligned} \tfrac{1}{2}\log_3 x &= 2\log_3 2 \\ \log_3 x^{\frac{1}{2}} &= \log_3 2^2 \\ x^{\frac{1}{2}} &= 4 \\ x &= 16 \end{aligned}$$

4. $$\begin{aligned} -2\log_4 x &= \log_4 9 \\ \log_4 x^{-2} &= \log_4 9 \\ x^{-2} &= 9 \end{aligned}$$
$$\frac{1}{x^2} = 9 \rightarrow x^2 = \frac{1}{9} \rightarrow x = \pm\frac{1}{3}$$
Since $\log_4(-\tfrac{1}{3})$ is undefined, the only solution is $x = \tfrac{1}{3}$.

5. $$\begin{aligned} 2\log_5 x &= 3\log_5 4 \\ \log_5 x^2 &= \log_5 4^3 \\ x^2 &= 64 \\ x &= \pm 8 \end{aligned}$$
Since $\log_5(-8)$ is undefined, the only solution is $x = 8$.

6. $$\begin{aligned} 3\log_2 x &= -\log_2 27 \\ \log_2 x^3 &= \log_2 27^{-1} \\ x^3 &= 27^{-1} \\ x^3 &= \frac{1}{27} \\ x &= \frac{1}{3} \end{aligned}$$

7. $$\begin{aligned} 3\log_2(x-1) + \log_2 4 &= 5 \\ \log_2(x-1)^3 + \log_2 4 &= 5 \\ \log_2 4(x-1)^3 &= 5 \\ 4(x-1)^3 &= 2^5 \\ (x-1)^3 &= \frac{32}{4} \\ (x-1)^3 &= 8 \\ x-1 &= 2 \\ x &= 3 \end{aligned}$$

8. $$\begin{aligned} 2\log_3(x+4) - \log_3 9 &= 2 \\ \log_3(x+4)^2 - \log_3 3^2 &= 2 \\ \log_3(x+4)^2 - 2 &= 2 \\ \log_3(x+4)^2 &= 4 \\ (x+4)^2 &= 3^4 \\ (x+4)^2 &= 81 \\ x+4 &= \pm 9 \\ x &= -4 \pm 9 \\ x = 5 &\text{ or } x = -13 \end{aligned}$$
Since $\log_3(-13+4)$ is undefined, the only solution is $x = 5$.

9. $\log x + \log(x+15) = 2$
$\log x(x+15) = 2$
$x(x+15) = 10^2$
$x^2 + 15x - 100 = 0$
$(x+20)(x-5) = 0$
$x = -20$ or $x = 5$
Since $\log(-20)$ is undefined, the only solution is $x = 5$.

10. $\log_4 x + \log_4(x-3) = 1$
$\log_4 x(x-3) = 1$
$x(x-3) = 4^1$
$x^2 - 3x - 4 = 0$
$(x+1)(x-4) = 0$
$x = -1$ or $x = 4$
Since $\log_4(-1)$ is undefined, the only solution is $x = 4$.

11. $\ln x + \ln(x+2) = 4$
$\ln x(x+2) = 4$
$x(x+2) = e^4$
$x^2 + 2x - e^4 = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-e^4)}}{2(1)} = \frac{-2 \pm \sqrt{4+4e^4}}{2}$$

$$= \frac{-2 \pm 2\sqrt{1+e^4}}{2} = -1 \pm \sqrt{1+e^4}$$

Since $\ln\left(-1-\sqrt{1+e^4}\right)$ is undefined, the only solution is $x = -1+\sqrt{1+e^4} \approx 6.456$.

12. $\ln(x+1) - \ln x = 2$
$\ln\left(\frac{x+1}{x}\right) = 2$
$\frac{x+1}{x} = e^2$
$x + 1 = e^2 x$
$e^2 x - x = 1$
$x\left(e^2 - 1\right) = 1$

$$x = \frac{1}{e^2 - 1}$$

13. $2^{2x} + 2^x - 12 = 0$
$\left(2^x\right)^2 + 2^x - 12 = 0$
$\left(2^x - 3\right)\left(2^x + 4\right) = 0$
$2^x - 3 = 0$ or $2^x + 4 = 0$
$2^x = 3$ or $2^x = -4$
$x = \log_2 3$ No solution
$x \approx 1.585$

14.
$$\begin{aligned}
3^{2x}+3^{x}-2&=0\\
\left(3^{x}\right)^{2}+3^{x}-2&=0\\
\left(3^{x}-1\right)\left(3^{x}+2\right)&=0
\end{aligned}$$
$$\begin{aligned}
3^{x}-1&=0 &&\text{or} & 3^{x}+2&=0\\
3^{x}&=1 &&\text{or} & 3^{x}&=-2\\
x&=0 && & &\text{No solution}
\end{aligned}$$

15.
$$\begin{aligned}
3^{2x}+3^{x+1}-4&=0\\
\left(3^{x}\right)^{2}+3\cdot 3^{x}-4&=0\\
\left(3^{x}-1\right)\left(3^{x}+4\right)&=0
\end{aligned}$$
$$\begin{aligned}
3^{x}-1&=0 &&\text{or} & 3^{x}+4&=0\\
3^{x}&=1 &&\text{or} & 3^{x}&=-4\\
x&=0 && & &\text{No solution}
\end{aligned}$$

16.
$$\begin{aligned}
2^{2x}+2^{x+2}-12&=0\\
\left(2^{x}\right)^{2}+2^{2}\cdot 2^{x}-12&=0\\
\left(2^{x}-2\right)\left(2^{x}+6\right)&=0
\end{aligned}$$
$$\begin{aligned}
2^{x}-2&=0 &&\text{or} & 2^{x}+6&=0\\
2^{x}&=2 &&\text{or} & 2^{x}&=-6\\
x&=1 && & &\text{No solution}
\end{aligned}$$

17.
$$\begin{aligned}
2^{x}&=10\\
\log\left(2^{x}\right)&=\log 10\\
x\log 2&=1\\
x&=\frac{1}{\log 2}\approx 3.322
\end{aligned}$$

18.
$$\begin{aligned}
3^{x}&=14\\
\log\left(3^{x}\right)&=\log 14\\
x\log 3&=\log 14\\
x&=\frac{\log 14}{\log 3}\approx 2.402
\end{aligned}$$

19.
$$\begin{aligned}
8^{-x}&=1.2\\
\log\left(8^{-x}\right)&=\log 1.2\\
-x\log 8&=\log 1.2\\
x&=\frac{\log 1.2}{-\log 8}\approx -0.088
\end{aligned}$$

20.
$$\begin{aligned}
2^{-x}&=1.5\\
\log\left(2^{-x}\right)&=\log 1.5\\
-x\log 2&=\log 1.5\\
x&=\frac{\log 1.5}{-\log 2}\approx -0.585
\end{aligned}$$

21.
$$\begin{aligned}
3^{1-2x}&=4^{x}\\
\log\left(3^{1-2x}\right)&=\log\left(4^{x}\right)\\
(1-2x)\log 3&=x\log 4\\
\log 3-2x\log 3&=x\log 4\\
\log 3&=x\log 4+2x\log 3\\
\log 3&=x(\log 4+2\log 3)\\
x&=\frac{\log 3}{\log 4+2\log 3}\approx 0.307
\end{aligned}$$

22.
$$\begin{aligned}
2^{x+1} &= 5^{1-2x}\\
\log\left(2^{x+1}\right) &= \log\left(5^{1-2x}\right)\\
(x+1)\log 2 &= (1-2x)\log 5\\
x\log 2+\log 2 &= \log 5 - 2x\log 5\\
x\log 2 + 2x\log 5 &= \log 5 - \log 2\\
x(\log 2 + 2\log 5) &= \log 5 - \log 2\\
x &= \frac{\log 5 - \log 2}{\log 2 + 2\log 5} \approx 0.234
\end{aligned}$$

23.
$$\begin{aligned}
\left(\tfrac{3}{5}\right)^{x} &= 7^{1-x}\\
\log\left(\left(\tfrac{3}{5}\right)^{x}\right) &= \log\left(7^{1-x}\right)\\
x\log\tfrac{3}{5} &= (1-x)\log 7\\
x(\log 3 - \log 5) &= \log 7 - x\log 7\\
x\log 3 - x\log 5 + x\log 7 &= \log 7\\
x(\log 3 - \log 5 + \log 7) &= \log 7\\
x &= \frac{\log 7}{\log 3 - \log 5 + \log 7} \approx 1.356
\end{aligned}$$

24.
$$\begin{aligned}
\left(\tfrac{4}{3}\right)^{1-x} &= 5^{x}\\
\log\left(\left(\tfrac{4}{3}\right)^{1-x}\right) &= \log\left(5^{x}\right)\\
(1-x)\log\tfrac{4}{3} &= x\log 5\\
\log\tfrac{4}{3} - x\log\tfrac{4}{3} &= x\log 5\\
x\log 5 + x\log\tfrac{4}{3} &= \log\tfrac{4}{3}\\
x(\log 5 + \log\tfrac{4}{3}) &= \log\tfrac{4}{3}\\
x &= \frac{\log\frac{4}{3}}{\log 5 + \log\frac{4}{3}} \approx 0.152
\end{aligned}$$

25.
$$\begin{aligned}
1.2^{x} &= (0.5)^{-x}\\
\log 1.2^{x} &= \log(0.5)^{-x}\\
x\log 1.2 &= -x\log 0.5\\
x\log 1.2 + x\log 0.5 &= 0\\
x(\log 1.2 + \log 0.5) &= 0\\
x &= 0
\end{aligned}$$

26.
$$\begin{aligned}
0.3^{1+x} &= 1.7^{2x-1}\\
\log 0.3^{1+x} &= \log 1.7^{2x-1}\\
(1+x)\log 0.3 &= (2x-1)\log 1.7\\
\log 0.3 + x\log 0.3 &= 2x\log 1.7 - \log 1.7\\
x\log 0.3 - 2x\log 1.7 &= -\log 1.7 - \log 0.3\\
x(\log 0.3 - 2\log 1.7) &= -\log 1.7 - \log 0.3\\
x &= \frac{-\log 1.7 - \log 0.3}{\log 0.3 - 2\log 1.7} \approx -0.297
\end{aligned}$$

27.
$$\begin{aligned}
\pi^{1-x} &= e^x\\
\ln \pi^{1-x} &= \ln e^x\\
(1-x)\ln \pi &= x\\
\ln \pi - x\ln \pi &= x\\
\ln \pi &= x + x\ln \pi\\
\ln \pi &= x(1+\ln \pi)\\
x &= \frac{\ln \pi}{1+\ln \pi} \approx 0.534
\end{aligned}$$

28.
$$\begin{aligned}
e^{x+3} &= \pi^x\\
\ln e^{x+3} &= \ln \pi^x\\
x+3 &= x\ln \pi\\
x - x\ln \pi &= -3\\
x(1-\ln \pi) &= -3\\
x &= \frac{-3}{1-\ln \pi} \approx 20.728
\end{aligned}$$

29.
$$\begin{aligned}
5\left(2^{3x}\right) &= 8\\
2^{3x} &= \tfrac{8}{5}\\
\log 2^{3x} &= \log\left(\tfrac{8}{5}\right)\\
3x\log 2 &= \log 8 - \log 5\\
x &= \frac{\log 8 - \log 5}{3\log 2} \approx 0.226
\end{aligned}$$

30.
$$\begin{aligned}
0.3\left(4^{0.2x}\right) &= 0.2\\
4^{0.2x} &= \tfrac{2}{3}\\
\log 4^{0.2x} &= \log\left(\tfrac{2}{3}\right)\\
0.2x\log 4 &= \log 2 - \log 3\\
x &= \frac{\log 2 - \log 3}{0.2\log 4} \approx -1.462
\end{aligned}$$

31. $\log_a(x-1) - \log_a(x+6) = \log_a(x-2) - \log_a(x+3)$

32.
$$\begin{aligned}
\log_a x + \log_a(x-2) &= \log_a(x+4)\\
\log_a\left(x(x-2)\right) &= \log_a(x+4)\\
x(x-2) &= x+4\\
x^2 - 2x &= x+4\\
x^2 - 3x - 4 &= 0\\
(x-4)(x+1) &= 0\\
x = 4 \text{ or } x &= -1
\end{aligned}$$

Since $\log_a(-1)$ is undefined, the only solution is $x = 4$.

33.
$$\begin{aligned}
\log_{\frac{1}{3}}(x^2+x) - \log_{\frac{1}{3}}(x^2-x) &= -1\\
\log_{\frac{1}{3}} \frac{x^2+x}{x^2-x} &= -1\\
\frac{x^2+x}{x^2-x} &= \left(\tfrac{1}{3}\right)^{-1}\\
\frac{x(x+1)}{x(x-1)} &= 3\\
x+1 &= 3(x-1)\\
x+1 &= 3x-3\\
-2x &= -4\\
x &= 2
\end{aligned}$$

34.
$$\begin{aligned}
\log_4(x^2-9) - \log_4(x+3) &= 3\\
\log_4 \frac{x^2-9}{x+3} &= 3\\
\frac{(x-3)(x+3)}{x+3} &= 4^3\\
x-3 &= 64\\
x &= 67
\end{aligned}$$

35.
$$\begin{aligned}
\log_2(x+1) - \log_4 x &= 1\\
\log_2(x+1) - \frac{\log_2 x}{\log_2 4} &= 1\\
\log_2(x+1) - \frac{\log_2 x}{2} &= 1\\
2\log_2(x+1) - \log_2 x &= 2\\
\log_2(x+1)^2 - \log_2 x &= 2\\
\log_2 \frac{(x+1)^2}{x} &= 2\\
\frac{(x+1)^2}{x} &= 2^2\\
x^2+2x+1 &= 4x\\
x^2-2x+1 &= 0\\
(x-1)^2 &= 0\\
x-1 &= 0\\
x &= 1
\end{aligned}$$

36.
$$\begin{aligned}
\log_2(3x+2) - \log_4 x &= 3\\
\log_2(3x+2) - \frac{\log_2 x}{\log_2 4} &= 3\\
\log_2(3x+2) - \frac{\log_2 x}{2} &= 3\\
2\log_2(3x+2) - \log_2 x &= 6\\
\log_2(3x+2)^2 - \log_2 x &= 6\\
\log_2 \frac{(3x+2)^2}{x} &= 6\\
\frac{(3x+2)^2}{x} &= 2^6\\
9x^2+12x+4 &= 64x\\
9x^2-52x+4 &= 0\\
x &= \frac{52 \pm \sqrt{(-52)^2-4(9)(4)}}{2(9)}\\
&= \frac{52 \pm \sqrt{2560}}{18} \approx 5.70 \text{ or } 0.08
\end{aligned}$$

37.
$$\begin{aligned}
\log_{16} x + \log_4 x + \log_2 x &= 7\\
\frac{\log_2 x}{\log_2 16} + \frac{\log_2 x}{\log_2 4} + \log_2 x &= 7\\
\frac{\log_2 x}{4} + \frac{\log_2 x}{2} + \log_2 x &= 7\\
\log_2 x + 2\log_2 x + 4\log_2 x &= 28\\
7\log_2 x &= 28\\
\log_2 x &= 4\\
x &= 2^4 = 16
\end{aligned}$$

38.
$$\begin{aligned}
\log_9 x + 3\log_3 x &= 14\\
\frac{\log_3 x}{\log_3 9} + 3\log_3 x &= 14\\
\frac{\log_3 x}{2} + 3\log_3 x &= 14\\
\frac{7}{2}\log_3 x &= 14\\
\log_3 x &= 4\\
x &= 3^4 = 81
\end{aligned}$$

39.
$$\begin{aligned}
\left(\sqrt[3]{2}\right)^{2-x} &= 2^{x^2}\\
\left(2^{\frac{1}{3}}\right)^{2-x} &= 2^{x^2}\\
2^{\frac{1}{3}(2-x)} &= 2^{x^2}\\
\tfrac{1}{3}(2-x) &= x^2\\
2-x &= 3x^2\\
3x^2+x-2 &= 0\\
(3x-2)(x+1) &= 0\\
x = \tfrac{2}{3} \quad &\text{or} \quad x = -1
\end{aligned}$$

40.
$$\begin{aligned}
\log_2 x^{\log_2 x} &= 4\\
\log_2 x \, \log_2 x &= 4\\
\left(\log_2 x\right)^2 &= 4\\
\log_2 x = -2 \quad &\text{or} \quad \log_2 x = 2\\
x = 2^{-2} \quad &\text{or} \quad x = 2^2\\
x = \tfrac{1}{4} \quad &\text{or} \quad x = 4
\end{aligned}$$

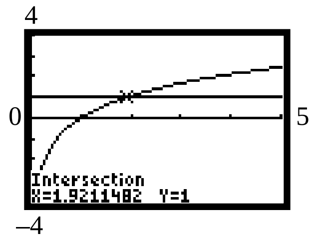
41.
$$\frac{e^x + e^{-x}}{2} = 1$$
$$e^x + e^{-x} = 2$$
$$e^x\left(e^x + e^{-x}\right) = 2e^x$$
$$e^{2x} + 1 = 2e^x$$
$$(e^x)^2 - 2e^x + 1 = 0$$
$$\left(e^x - 1\right)^2 = 0$$
$$e^x - 1 = 0$$
$$e^x = 1$$
$$x = 0$$

42.
$$\frac{e^x + e^{-x}}{2} = 3$$
$$e^x + e^{-x} = 6$$
$$e^x\left(e^x + e^{-x}\right) = 6e^x$$
$$e^{2x} + 1 = 6e^x$$
$$(e^x)^2 - 6e^x + 1 = 0$$
$$e^x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(1)}}{2(1)} = \frac{6 \pm \sqrt{32}}{2}$$
$$= \frac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}$$
$x = \ln\left(3 + 2\sqrt{2}\right)$ or $x = \ln\left(3 - 2\sqrt{2}\right)$

43.
$$\frac{e^x - e^{-x}}{2} = 2$$
$$e^x - e^{-x} = 4$$
$$e^x\left(e^x - e^{-x}\right) = 4e^x$$
$$e^{2x} - 1 = 4e^x$$
$$(e^x)^2 - 4e^x - 1 = 0$$
$$e^x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)} = \frac{4 \pm \sqrt{20}}{2}$$
$$= \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}$$
$$x = \ln\left(2 + \sqrt{5}\right)$$
$\ln\left(2 - \sqrt{5}\right)$ is undefined; it is not a solution.

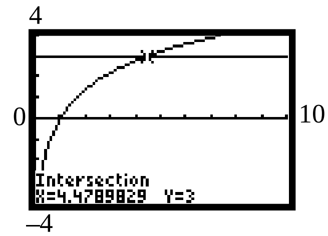
44.
$$\frac{e^x - e^{-x}}{2} = -2$$
$$e^x - e^{-x} = -4$$
$$e^x\left(e^x - e^{-x}\right) = -4e^x$$
$$e^{2x} - 1 = -4e^x$$
$$(e^x)^2 + 4e^x - 1 = 0$$
$$e^x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-1)}}{2(1)} = \frac{-4 \pm \sqrt{20}}{2}$$
$$= \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$
$$x = \ln\left(-2 + \sqrt{5}\right)$$
$\ln\left(-2 - \sqrt{5}\right)$ is undefined; it is not a solution.

45. Using INTERSECT to solve:
$y_1 = \ln(x)/\ln(5) + \ln(x)/\ln(3)$
$y_2 = 1$



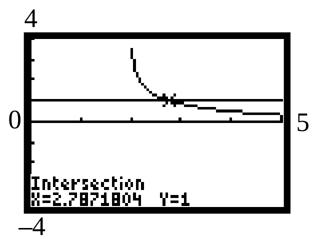
The solution is 1.92.

46. Using INTERSECT to solve:
$y_1 = \ln(x)/\ln(2) + \ln(x)/\ln(6)$
$y_2 = 3$



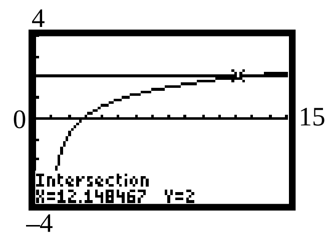
The solution is 4.48.

47. Using INTERSECT to solve:
$y_1 = \ln(x+1)/\ln(5) - \ln(x-2)/\ln(4)$
$y_2 = 1$



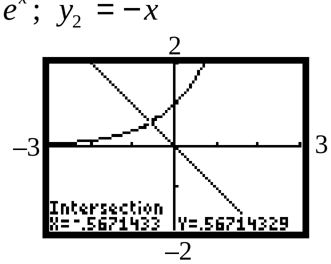
The solution is 2.79.

48. Using INTERSECT to solve:
$y_1 = \ln(x-1)/\ln(2) - \ln(x+2)/\ln(6)$
$y_2 = 2$



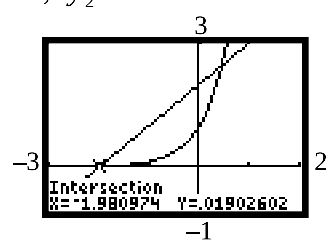
The solution is 12.15.

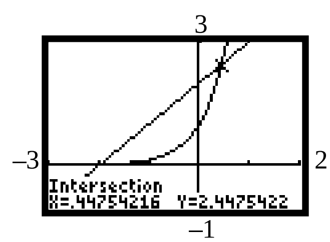
49. Using INTERSECT to solve:
$y_1 = e^x; \; y_2 = -x$



The solution is –0.57.

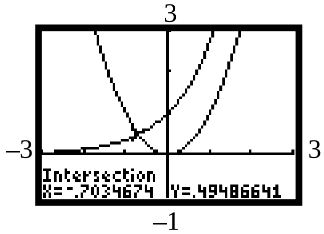
50. Using INTERSECT to solve:
$y_1 = e^{2x}; \; y_2 = x+2$





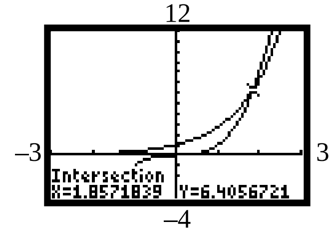
The solutions are –1.98 and 0.45.

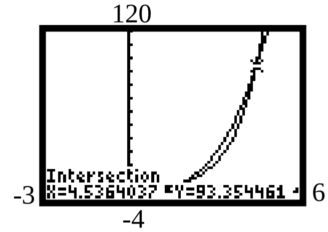
51. Using INTERSECT to solve:
$y_1 = e^x$; $y_2 = x^2$



The solution is –0.70.

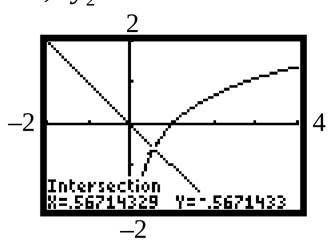
52. Using INTERSECT to solve:
$y_1 = e^x$; $y_2 = x^3$





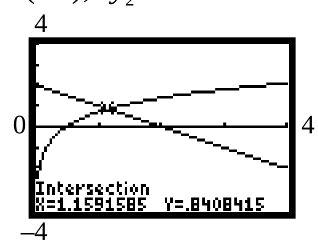
The solutions are 1.86 and 4.54.

53. Using INTERSECT to solve:
$y_1 = \ln x$; $y_2 = -x$



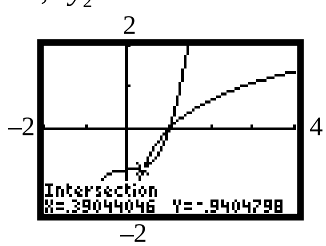
The solution is 0.57.

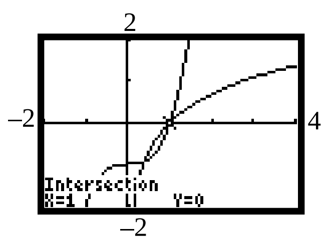
54. Using INTERSECT to solve:
$y_1 = \ln(2x)$; $y_2 = -x + 2$



The solution is 1.16.

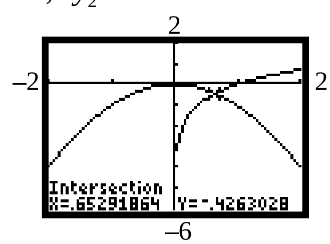
55. Using INTERSECT to solve:
$y_1 = \ln x$; $y_2 = x^3 - 1$





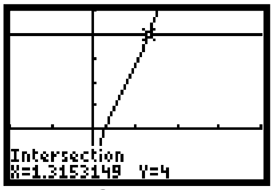
The solutions are 0.39, 1.00.

56. Using INTERSECT to solve:
$y_1 = \ln x$; $y_2 = -x^2$



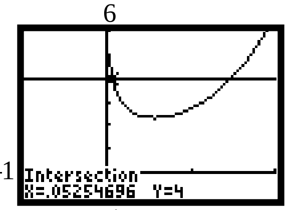
The solution is 0.65.

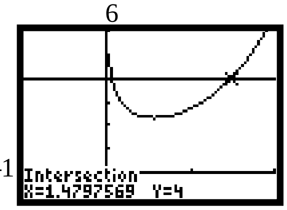
57. Using INTERSECT to solve:
$y_1 = e^x + \ln x;\ y_2 = 4$



The solution is 1.32.

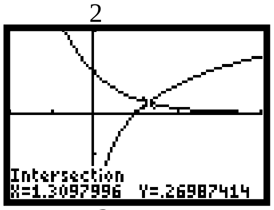
58. Using INTERSECT to solve:
$y_1 = e^x - \ln x;\ y_2 = 4$





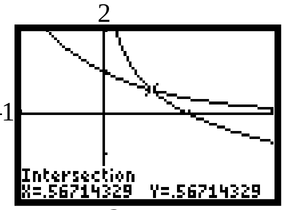
The solutions are 0.05 and 1.48.

59. Using INTERSECT to solve:
$y_1 = e^{-x};\ y_2 = \ln x$



The solution is 1.31.

60. Using INTERSECT to solve:
$y_1 = e^{-x};\ y_2 = -\ln x$



The solution is 0.57.