Chapter 6

Exponential and Logarithmic Functions

6.7 Growth and Decay; Newton's Law; Logistic Models

1. $P(t) = 500e^{0.02t}$

Find t when $P = 1000$:
$$1000 = 500e^{0.02t}$$
$$2 = e^{0.02t}$$
$$\ln 2 = 0.02t$$
$$t = \frac{\ln 2}{0.02} \approx 34.7 \text{ days}$$

Find t when $P = 2000$:
$$2000 = 500e^{0.02t}$$
$$4 = e^{0.02t}$$
$$\ln 4 = 0.02t$$
$$t = \frac{\ln 4}{0.02} \approx 69.3 \text{ days}$$

2. $N(t) = 1000e^{0.01t}$

Find t when $N = 1500$:
$$1500 = 1000e^{0.01t}$$
$$1.5 = e^{0.01t}$$
$$\ln 1.5 = 0.01t$$
$$t = \frac{\ln 1.5}{0.01} \approx 40.55 \text{ hours}$$

Find t when $N = 2000$:
$$2000 = 1000e^{0.01t}$$
$$2 = e^{0.01t}$$
$$\ln 2 = 0.01t$$
$$t = \frac{\ln 2}{0.01} \approx 69.31 \text{ hours}$$

3. Find t when $A(t) = \frac{1}{2}A_0$:
$$\tfrac{1}{2}A_0 = A_0e^{-0.0244t}$$
$$\tfrac{1}{2} = e^{-0.0244t}$$
$$\ln\tfrac{1}{2} = -0.0244t$$
$$t = \frac{\ln\frac{1}{2}}{-0.0244} \approx 28.4 \text{ years}$$

4. Find t when $A(t) = \frac{1}{2}A_0$:
$$\tfrac{1}{2}A_0 = A_0e^{-0.087t}$$
$$\tfrac{1}{2} = e^{-0.087t}$$
$$\ln\tfrac{1}{2} = -0.087t$$
$$t = \frac{\ln\frac{1}{2}}{-0.087} \approx 7.97 \text{ days}$$

5. Use $N(t) = N_0e^{kt}$ and solve for k:
$$1800 = 1000e^{k(1)}$$
$$1.8 = e^{k}$$
$$k = \ln 1.8 \approx 0.5878$$

When $t = 3$:
$$N(3) = 1000e^{0.5878(3)} = 5832 \text{ mosquitos}$$

Find t when $N(t) = 10,000$:
$$10,000 = 1000e^{0.5878t}$$
$$10 = e^{0.5878t}$$
$$\ln 10 = 0.5878t$$
$$t = \frac{\ln 10}{0.5878} \approx 3.9 \text{ days}$$

6. Use $N(t) = N_0 e^{kt}$ and solve for k:

$$800 = 500e^{k(1)}$$
$$1.6 = e^k$$
$$k = \ln 1.6 \approx 0.4700$$

When $t = 5$:

$$N(5) = 500e^{0.4700(5)} = 5243 \text{ bacteria}$$

Find t when $N(t) = 20{,}000$:

$$20{,}000 = 500e^{0.4700t}$$
$$40 = e^{0.4700t}$$
$$\ln 40 = 0.4700t$$
$$t = \frac{\ln 40}{0.4700} \approx 7.85 \text{ hours}$$

7. Use $P(t) = P_0 e^{kt}$ and solve for k:

$$2P_0 = P_0 e^{k(1.5)}$$
$$2 = e^{1.5k}$$
$$\ln 2 = 1.5k$$
$$k = \frac{\ln 2}{1.5} \approx 0.4621$$

When $t = 2$:

$P(2) = 10{,}000e^{0.4621(2)} = 25{,}199$ is the population 2 years from now.

8. Use $P(t) = P_0 e^{kt}$ and solve for k:

$$800{,}000 = 900{,}000e^{k(2)}$$
$$\frac{8}{9} = e^{2k}$$
$$\ln\frac{8}{9} = 2k$$
$$k = \frac{\ln\frac{8}{9}}{2} \approx -0.05889$$

When $t = 4$:

$P(4) = 900{,}000e^{-0.05889(4)} = 711{,}115$ is the population in 1997.

9. Use $A = A_0 e^{kt}$ and solve for k:

$$\tfrac{1}{2}A_0 = A_0 e^{k(1690)}$$
$$\tfrac{1}{2} = e^{1690k}$$
$$\ln\tfrac{1}{2} = 1690k$$
$$k = \frac{\ln 0.5}{1690} \approx -0.00041$$

When $A_0 = 10$ and $t = 50$:

$$A = 10e^{-0.00041(50)} = 9.797 \text{ grams}$$

10. Use $A = A_0 e^{kt}$ and solve for k:

$$\tfrac{1}{2} A_0 = A_0 e^{k(1.3\times 10^9)}$$
$$\tfrac{1}{2} = e^{1.3\times 10^9 k}$$
$$\ln\tfrac{1}{2} = 1.3\times 10^9 k \to k = \frac{\ln 0.5}{1.3\times 10^9} \approx -5.3319\times 10^{-10}$$

When $A_0 = 10$ and $t = 100$: $A = 10e^{-5.3319\times 10^{-10}(100)} = 9.999999467$ grams

When $A_0 = 10$ and $t = 1000$: $A = 10e^{-5.3319\times 10^{-10}(1000)} = 9.999994668$ grams

11. Use $A = A_0 e^{kt}$ and solve for k:

$$\tfrac{1}{2} A_0 = A_0 e^{k(5600)}$$
$$\tfrac{1}{2} = e^{5600k}$$
$$\ln\tfrac{1}{2} = 5600k$$
$$k = \frac{\ln 0.5}{5600} \approx -0.000124$$

Solve for t when $A = 0.3A_0$:

$$0.3A_0 = A_0 e^{-0.000124t}$$
$$0.3 = e^{-0.000124t}$$
$$\ln 0.3 = -0.000124t$$
$$t = \frac{\ln 0.3}{-0.000124}$$
$$\approx 9709 \text{ years ago}$$

12. (a) Use $A = A_0 e^{kt}$ and solve for k:

$$\tfrac{1}{2} A_0 = A_0 e^{k(5600)}$$
$$\tfrac{1}{2} = e^{5600k}$$
$$\ln\tfrac{1}{2} = 5600k$$
$$k = \frac{\ln 0.5}{5600} \approx -0.000124$$

Solve for t when $A = 0.7A_0$:

$$0.7A_0 = A_0 e^{-0.000124t}$$
$$0.7 = e^{-0.000124t}$$
$$\ln 0.7 = -0.000124t$$
$$t = \frac{\ln 0.7}{-0.000124}$$
$$\approx 2876 \text{ years}$$

The fossil is 2876 years old.

13. (a) Using $u = T + (u_0 - T)e^{kt}$ where $t = 5$, $T = 70$, $u_0 = 450$, $u = 300$:

$$300 = 70 + (450 - 70)e^{k(5)}$$
$$230 = 380e^{5k}$$
$$0.6053 = e^{5k}$$
$$5k = \ln 0.6053$$
$$k = \frac{\ln 0.6053}{5} \approx -0.1004$$

$T = 70$, $u_0 = 450$, $u = 135$:

$$135 = 70 + (450 - 70)e^{-0.1004t}$$
$$65 = 380e^{-0.1004t}$$
$$0.17105 = e^{-0.1004t}$$
$$-0.1004t = \ln 0.17105$$
$$t = \frac{\ln 0.17105}{-0.1004} \approx 17.6 \text{ minutes}$$

The pizza will be cool enough to eat at 5:18 p.m.

(b) $T = 70$, $u_0 = 450$, $u = 160$

$$160 = 70 + (450 - 70)e^{-0.1004t}$$
$$90 = 380e^{-0.1004t}$$
$$\frac{90}{380} = e^{-0.1004t}$$
$$\ln\left(\frac{90}{380}\right) = \ln e^{-0.1004t} = -0.1004t$$
$$t = \frac{\ln\left(\frac{90}{380}\right)}{-0.1004} \approx 14.35 \text{ minutes}$$

The pizza will be 160˚F after about 14.3 minutes.

(c) As time passes the temperature gets closer to 70˚F.

14. (a) Using $u = T + (u_0 - T)e^{kt}$ where $t = 2$, $T = 38$, $u_0 = 72$, $u = 60$:

$$60 = 38 + (72 - 38)e^{k(2)}$$
$$22 = 34e^{2k}$$
$$0.6471 = e^{2k}$$
$$2k = \ln 0.6471$$
$$k = \frac{\ln 0.6471}{2} \approx -0.2176$$

$T = 38$, $u_0 = 72$, $t = 7$:

$$u = 38 + (72 - 38)e^{-0.2176(7)}$$
$$u = 38 + 34e^{-1.5232} \approx 45.4°\text{F}$$

After 7 minutes the thermometer reads 45.4˚F.

(b) Find t when $u = 39°\text{F}$

$$39 = 38 + (72 - 38)e^{-0.2176t}$$
$$1 = 34e^{-0.2176t}$$
$$0.02941 = e^{-0.2176t}$$
$$-0.2176t = \ln 0.02941$$
$$t = \frac{\ln 0.02941}{-0.2176} \approx 16.2 \text{ minutes}$$

(c) Find t when $u = 45°\text{F}$

$$45 = 38 + (72 - 38)e^{-0.2176t}$$

$$7 = (34)e^{-0.2176t} \rightarrow \frac{7}{34} = e^{-0.2176t}$$

$$\ln\left(\frac{7}{34}\right) = \ln e^{-0.2176t} = -0.2176t \rightarrow t = \frac{\ln\left(\frac{7}{34}\right)}{-0.2176} \approx 7.26 \text{ minutes}$$

The thermometer will read 45˚F after about 7.3 minutes.

(d) As time passes the temperature gets closer to 38˚F.

15. Using $u = T + (u_0 - T)e^{kt}$ where $t = 3$, $T = 35$, $u_0 = 8$, $u = 15$:

$$15 = 35 + (8 - 35)e^{k(3)}$$
$$-20 = -27e^{3k}$$
$$0.74074 = e^{3k}$$
$$3k = \ln 0.74074$$
$$k = \frac{\ln 0.74074}{3} \approx -0.100035$$

At $t = 5$:

$$u = 35 + (8 - 35)e^{-0.100035(5)} = 18.63°\text{C}$$

At $t = 10$:

$$u = 35 + (8 - 35)e^{-0.100035(10)} = 25.1°\text{C}$$

16. Using $u = T + (u_0 - T)e^{kt}$ where $t = 10$,

$T = 70,\ u_0 = 28,\ u = 35$:

$$35 = 70 + (28 - 70)e^{k(10)}$$
$$-35 = -42e^{10k}$$
$$0.83333 = e^{10k}$$
$$10k = \ln 0.83333$$
$$k = \frac{\ln 0.83333}{10} \approx -0.01823$$

At $t = 30$:

$$u = 70 + (28 - 70)e^{-0.01823(30)} = 45.69^\circ \text{F}$$

Find time for temperature of 45˚F:

$$45 = 70 + (28 - 70)e^{-0.01823t}$$
$$-25 = -42e^{-0.01823t}$$
$$0.59524 = e^{-0.01823t}$$
$$\ln 0.59524 = -0.01823t$$
$$t = \frac{\ln 0.59524}{-0.01823} \approx 28.5 \text{ minutes}$$

17. Use $A = A_0 e^{kt}$ and solve for k:

$$15 = 25e^{k(10)}$$
$$0.6 = e^{10k}$$
$$\ln 0.6 = 10k$$
$$k = \frac{\ln 0.6}{10} \approx -0.0511$$

When $A_0 = 25$ and $t = 24$:

$$A = 25e^{-0.0511(24)} = 7.33 \text{ kilograms}$$

Find t when $A = \frac{1}{2}A_0$:

$$0.5 = 25e^{-0.0511t}$$
$$0.02 = e^{-0.0511t}$$
$$\ln 0.02 = -0.0511t$$
$$t = \frac{\ln 0.02}{-0.0511} \approx 76.6 \text{ hour}$$

18. Use $A = A_0 e^{kt}$ and solve for k:

$$10 = 40e^{k(2)}$$
$$0.25 = e^{2k}$$
$$\ln 0.25 = 2k$$
$$k = \frac{\ln 0.25}{2} \approx -0.6931$$

When $A_0 = 40$ and $t = 5$:

$$A = 40e^{-0.6931(5)} = 1.25 \text{ volts}$$

19. Use $A = A_0 e^{kt}$ and solve for k:

$$\tfrac{1}{2}A_0 = A_0 e^{k(8)}$$
$$0.5 = e^{8k}$$
$$\ln 0.5 = 8k$$
$$k = \frac{\ln 0.5}{8} \approx -0.0866$$

Find t when $A = 0.1A_0$:

$$0.1A_0 = A_0 e^{-0.0866t}$$
$$0.1 = e^{-0.0866t}$$
$$\ln 0.1 = -0.0866t$$
$$t = \frac{\ln 0.1}{-0.0866} \approx 26.6 \text{ days}$$

The farmers need to wait about 27 days before using the hay.

20. Using $u = T + (u_0 - T)e^{kt}$ where $t = 2$, $T = 325$, $u_0 = 75$, $u = 100$:

$$100 = 325 + (75 - 325)e^{k(2)}$$
$$-225 = -250e^{2k}$$
$$0.9 = e^{2k}$$
$$2k = \ln 0.9$$
$$k = \frac{\ln 0.9}{2} \approx -0.05268$$

Find time for temperature of 175˚F:

$$175 = 325 + (75 - 325)e^{-0.05268t}$$
$$-150 = -250e^{-0.05268t}$$
$$0.6 = e^{-0.05268t}$$
$$\ln 0.6 = -0.05268t$$
$$t = \frac{\ln 0.6}{-0.05268} \approx 9.7 \text{ hours}$$

The hotel may serve their guests about 9.7 hours after noon or at 9:42 p.m.

21. (a) $P(0) = \dfrac{0.9}{1 + 6e^{-0.32(0)}} = \dfrac{0.9}{1 + 6 \cdot 1} = \dfrac{0.9}{7} = 0.1286$

(b) The maximum proportion is the carrying capacity, 0.9.

(c)
$$0.8 = \frac{0.9}{1 + 6e^{-0.32t}}$$
$$0.8\left(1 + 6e^{-0.32t}\right) = 0.9$$
$$1 + 6e^{-0.32t} = 1.125$$
$$6e^{-0.32t} = 0.125$$
$$e^{-0.32t} = 0.020833$$
$$-0.32t = \ln(0.020833)$$
$$t = \frac{\ln(0.020833)}{-0.32} \approx 12.1$$

80% of households will own VCR's in 1996 (t = 12).

22. (a) $P(0) = \dfrac{0.9}{1+3.5e^{-0.339(0)}} = \dfrac{0.9}{1+3.5 \quad 1} = \dfrac{0.9}{4.5} = 0.2$

(b) The maximum proportion is the carrying capacity, 0.9.

(c)
$$\begin{aligned} 0.75 &= \frac{0.9}{1+3.5e^{-0.339t}} \\ 0.75\left(1+3.5e^{-0.339t}\right) &= 0.9 \\ 1+3.5e^{-0.339t} &= 1.2 \\ 3.5e^{-0.339t} &= 0.2 \\ e^{-0.339t} &= 0.05714 \\ -0.339t &= \ln(0.05714) \\ t &= \frac{\ln(0.05714)}{-0.339} \approx 8.4 \text{ months} \end{aligned}$$

23. (a) As $t \to \infty$, $e^{-0.439t} \to 0$. Thus, $P(t) \to 1000$. The carrying capacity is 1000.

(b) $P(0) = \dfrac{1000}{1+32.33e^{-0.439(0)}} = \dfrac{1000}{33.33} = 30$

(c)
$$\begin{aligned} 800 &= \frac{1000}{1+32.33e^{-0.439t}} \\ 800\left(1+32.33e^{-0.439t}\right) &= 1000 \\ 1+32.33e^{-0.439t} &= 1.25 \\ 32.33e^{-0.439t} &= 0.25 \\ e^{-0.439t} &= 0.007733 \\ -0.439t &= \ln(0.007733) \\ t &= \frac{\ln(0.007733)}{-0.439} \approx 11.076 \text{ hours} \end{aligned}$$

24. (a) As $t \to \infty$, $e^{-0.162t} \to 0$. Thus, $P(t) \to 500$. The carrying capacity is 500.

(b) $P(20) = \dfrac{500}{1+83.33e^{-0.162(20)}} = \dfrac{500}{4.2635} \approx 117.27$ or 117

(c)
$$\begin{aligned} 300 &= \frac{500}{1+83.33e^{-0.162t}} \\ 300\left(1+83.33e^{-0.162t}\right) &= 500 \\ 1+83.33e^{-0.162t} &= 1.6667 \\ 83.33e^{-0.162t} &= 0.6667 \\ e^{-0.162t} &= 0.0080 \\ -0.162t &= \ln(0.0080) \\ t &= \frac{\ln(0.0080)}{-0.162} \approx 29.8 \text{ years} \end{aligned}$$