Chapter 7

Trigonometric Functions

7.2 Right Triangle Trigonometry

1. opposite = 5; adjacent = 12
   Find the hypotenuse:
$$5^2 + 12^2 = (\text{hypotenuse})^2$$
$$(\text{hypotenuse})^2 = 25 + 144 = 169$$
$$\text{hypotenuse} = 13$$
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{5}{13} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{12}{13} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{5}{12}$$
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{13}{5} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{13}{12} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{12}{5}$$

2. opposite = 3; adjacent = 4
   Find the hypotenuse:
$$3^2 + 4^2 = (\text{hypotenuse})^2$$
$$(\text{hypotenuse})^2 = 9 + 16 = 25$$
$$\text{hypotenuse} = 5$$
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{3}{5} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{3}{4}$$
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{3} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{4} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{4}{3}$$

3. opposite = 2; adjacent = 3
   Find the hypotenuse:
$$2^2 + 3^2 = (\text{hypotenuse})^2$$
$$(\text{hypotenuse})^2 = 4 + 9 = 13$$
$$\text{hypotenuse} = \sqrt{13}$$
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{2}{3}$$
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{13}}{2} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{13}}{3} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{2}$$

4. opposite = 3; adjacent = 3
Find the hypotenuse:

$$3^2 + 3^2 = (\text{hypotenuse})^2$$
$$(\text{hypotenuse})^2 = 9 + 9 = 18$$
$$\text{hypotenuse} = \sqrt{18} = 3\sqrt{2}$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{3}{3\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{3\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{3}{3} = 1$$
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{3\sqrt{2}}{3} = \sqrt{2} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{3\sqrt{2}}{3} = \sqrt{2} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{3} = 1$$

5. adjacent = 2; hypotenuse = 4
Find the opposite side:

$$(\text{opposite})^2 + 2^2 = 4^2$$
$$(\text{opposite})^2 = 16 - 4 = 12$$
$$\text{opposite} = \sqrt{12} = 2\sqrt{3}$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{2}{4} = \frac{1}{2} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{2\sqrt{3}}{2} = \sqrt{3}$$
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{4}{2\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{4}{2} = 2 \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{2}{2\sqrt{3}} = \frac{\sqrt{3}}{3}$$

6. opposite = 3; hypotenuse = 4
Find the adjacent side:

$$3^2 + (\text{adjacent})^2 = 4^2$$
$$(\text{adjacent})^2 = 16 - 9 = 7$$
$$\text{adjacent} = \sqrt{7}$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{3}{4} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{7}}{4} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{3}{\sqrt{7}} = \frac{3\sqrt{7}}{7}$$
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{4}{3} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{4}{\sqrt{7}} = \frac{4\sqrt{7}}{7} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{7}}{3}$$

7. opposite = $\sqrt{2}$; adjacent = 1
Find the hypotenuse:

$$\left(\sqrt{2}\right)^2 + 1^2 = (\text{hypotenuse})^2$$
$$(\text{hypotenuse})^2 = 2 + 1 = 3$$
$$\text{hypotenuse} = \sqrt{3}$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{3}}{1} = \sqrt{3} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

8. opposite = 2; adjacent = $\sqrt{3}$

Find the hypotenuse:

$$2^2 + \left(\sqrt{3}\right)^2 = (\text{hypotenuse})^2$$

$$(\text{hypotenuse})^2 = 4 + 3 = 7$$

$$\text{hypotenuse} = \sqrt{7}$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{2}{\sqrt{7}} = \frac{2\sqrt{7}}{7} \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{\sqrt{7}} = \frac{\sqrt{21}}{7} \quad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{7}}{2} \quad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{7}}{\sqrt{3}} = \frac{\sqrt{21}}{3} \quad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{3}}{2}$$

9. opposite = 1; hypotenuse = $\sqrt{5}$

Find the adjacent side:

$$1^2 + (\text{adjacent})^2 = \left(\sqrt{5}\right)^2$$

$$(\text{adjacent})^2 = 5 - 1 = 4$$

$$\text{adjacent} = 2$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \quad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{2}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{5}}{1} = \sqrt{5} \quad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{5}}{2} \quad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{2}{1} = 2$$

10. adjacent = 2; hypotenuse = $\sqrt{5}$

Find the opposite side:

$$(\text{opposite})^2 + 2^2 = \left(\sqrt{5}\right)^2$$

$$(\text{opposite})^2 = 5 - 4 = 1$$

$$\text{opposite} = 1$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \quad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{2}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{5}}{1} = \sqrt{5} \quad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{5}}{2} \quad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{2}{1} = 2$$

11. $\sin\theta = \frac{1}{2}$; $\cos\theta = \frac{\sqrt{3}}{2}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{2}\cdot\frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{1}{\left(\frac{\sqrt{3}}{3}\right)} = \frac{3}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

12. $\sin\theta = \frac{\sqrt{3}}{2}$; $\cos\theta = \frac{1}{2}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = \frac{\sqrt{3}}{2}\cdot\frac{2}{1} = \sqrt{3}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

13. $\sin\theta = \frac{2}{3}$; $\cos\theta = \frac{\sqrt{5}}{3}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{2}{3}\right)}{\left(\frac{\sqrt{5}}{3}\right)} = \frac{2}{3}\cdot\frac{3}{\sqrt{5}} = \frac{2}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\left(\frac{2\sqrt{5}}{5}\right)} = \frac{5}{2\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{5\sqrt{5}}{10} = \frac{\sqrt{5}}{2}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\left(\frac{\sqrt{5}}{3}\right)} = \frac{3}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\left(\frac{2}{3}\right)} = \frac{3}{2}$$

14. $\sin\theta = \frac{1}{3}$; $\cos\theta = \frac{2\sqrt{2}}{3}$

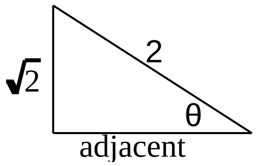
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2\sqrt{2}}{3}\right)} = \frac{1}{3}\cdot\frac{3}{2\sqrt{2}} = \frac{1}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\left(\frac{\sqrt{2}}{4}\right)} = \frac{4}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\left(\frac{2\sqrt{2}}{3}\right)} = \frac{3}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{4}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\left(\frac{1}{3}\right)} = 3$$

15. $\sin\theta = \dfrac{\sqrt{2}}{2}$ corresponds to the right triangle

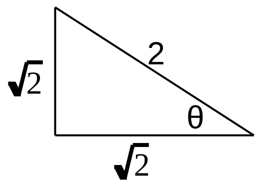


Using the Pythagorean Theorem:

$$(adjacent)^2 + \left(\sqrt{2}\right)^2 = 2^2$$

$$(adjacent)^2 + 2 = 4 \rightarrow (adjacent)^2 = 2 \rightarrow adjacent = \sqrt{2}$$

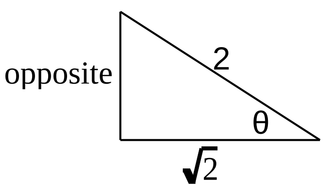
So the triangle becomes



$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\sqrt{2}}{2} = \dfrac{2\sqrt{2}}{2}$ $\quad \tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$ $\quad \cot\theta = \dfrac{\text{adj}}{\text{opp}} = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$

$\csc\theta = \dfrac{\text{hyp}}{\text{opp}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$ $\quad \sec\theta = \dfrac{\text{hyp}}{\text{adj}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

16. $\cos\theta = \dfrac{\sqrt{2}}{2}$ corresponds to the right triangle

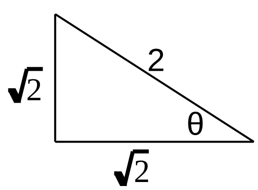


Using the Pythagorean Theorem:

$$(opposite)^2 + \left(\sqrt{2}\right)^2 = 2^2$$

$$(opposite)^2 + 2 = 4 \rightarrow (opposite)^2 = 2 \rightarrow opposite = \sqrt{2}$$

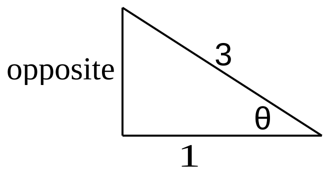
So the triangle becomes



$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{\sqrt{2}}{2} = \dfrac{2\sqrt{2}}{2}$ $\quad \tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$ $\quad \cot\theta = \dfrac{\text{adj}}{\text{opp}} = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$

$\csc\theta = \dfrac{\text{hyp}}{\text{opp}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$ $\quad \sec\theta = \dfrac{\text{hyp}}{\text{adj}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

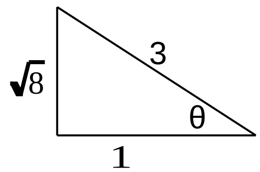
17. $\cos\theta = \dfrac{1}{3}$ corresponds to the right triangle



Using the Pythagorean Theorem:

$$(opposite)^2 + (1)^2 = 3^2$$
$$(opposite)^2 + 1 = 9 \rightarrow (opposite)^2 = 8 \rightarrow opposite = \sqrt{8}$$

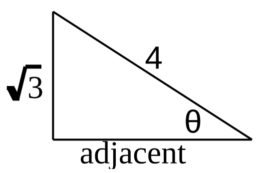
So the triangle becomes



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{8}}{1} = 2\sqrt{2} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{1}{\sqrt{8}}\frac{\sqrt{8}}{\sqrt{8}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{3}{\sqrt{8}}\frac{\sqrt{8}}{\sqrt{8}} = \frac{3\sqrt{8}}{8} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{3}{1} = 3 \qquad = \frac{\sqrt{8}}{8} = \frac{2\sqrt{2}}{8} = \frac{\sqrt{2}}{4}$$

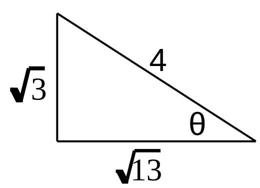
18. $\sin\theta = \dfrac{\sqrt{3}}{4}$ corresponds to the right triangle



Using the Pythagorean Theorem:

$$(adjacent)^2 + \left(\sqrt{3}\right)^2 = 4^2$$
$$(adjacent)^2 + 3 = 16 \rightarrow (adjacent)^2 = 13 \rightarrow adjacent = \sqrt{13}$$

So the triangle becomes

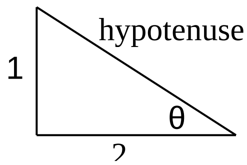


$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{13}}{4} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}}{\sqrt{13}}\frac{\sqrt{13}}{\sqrt{13}} = \frac{\sqrt{39}}{13}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{4}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}}{3} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{4}{\sqrt{13}}\frac{\sqrt{13}}{\sqrt{13}} = \frac{4\sqrt{13}}{13}$$

$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{13}}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{39}}{3}$$

19. $\tan\theta = \frac{1}{2}$ corresponds to the right triangle

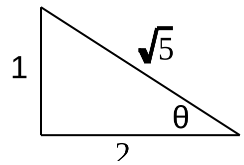


Using the Pythagorean Theorem:

$$(1)^2 + (2)^2 = (hypotenuse)^2$$

$$1 + 4 = (hypotenuse)^2 \rightarrow 5 = (hypotenuse)^2 \rightarrow hypotenuse = \sqrt{5}$$

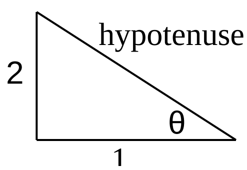
So the triangle becomes



$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{2}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{2}{1} = 2$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{5}}{1} = \sqrt{5} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{5}}{2}$$

20. $\cot\theta = \frac{1}{2}$ corresponds to the right triangle

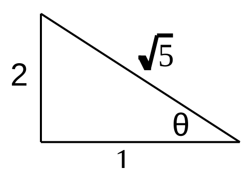


Using the Pythagorean Theorem:

$$(1)^2 + (2)^2 = (hypotenuse)^2$$

$$1 + 4 = (hypotenuse)^2 \rightarrow 5 = (hypotenuse)^2 \rightarrow hypotenuse = \sqrt{5}$$

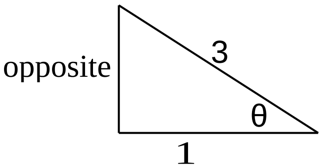
So the triangle becomes



$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5} \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{2}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{2}{1} = 2$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{5}}{2} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{5}}{1} = \sqrt{5}$$

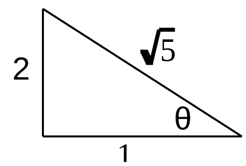
21. $\sec\theta = 3$ corresponds to the right triangle



Using the Pythagorean Theorem:

$$(opposite)^2 + (1)^2 = 3^2$$
$$(opposite)^2 + 1 = 9 \rightarrow (opposite)^2 = 8 \rightarrow opposite = \sqrt{8}$$

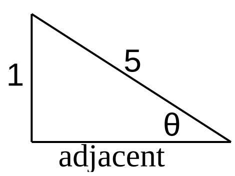
So the triangle becomes



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{8}}{1} = 2\sqrt{2} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{1}{\sqrt{8}}\,\frac{\sqrt{8}}{\sqrt{8}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{3}{\sqrt{8}}\,\frac{\sqrt{8}}{\sqrt{8}} = \frac{3\sqrt{8}}{8} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{3} \qquad = \frac{\sqrt{8}}{8} = \frac{2\sqrt{2}}{8} = \frac{\sqrt{2}}{4}$$

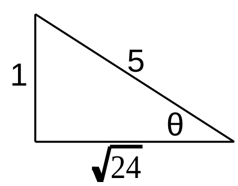
22. $\csc\theta = 5$ corresponds to the right triangle



Using the Pythagorean Theorem:

$$(adjacent)^2 + (1)^2 = 5^2$$
$$(adjacent)^2 + 1 = 25 \rightarrow (adjacent)^2 = 24 \rightarrow adjacent = \sqrt{24}$$

So the triangle becomes

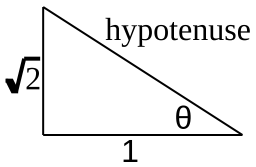


$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{24}}{5} = \frac{4\sqrt{6}}{5} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{24}}\,\frac{\sqrt{24}}{\sqrt{24}} = \frac{4\sqrt{6}}{24} = \frac{\sqrt{6}}{6}$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{1}{5} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{\sqrt{24}}\,\frac{\sqrt{24}}{\sqrt{24}} = \frac{20\sqrt{6}}{24} = \frac{5\sqrt{6}}{6}$$

$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{24}}{1} = 4\sqrt{6}$$

23. $\tan\theta = \sqrt{2}$ corresponds to the right triangle

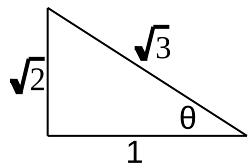


Using the Pythagorean Theorem:

$$(1)^2 + \left(\sqrt{2}\right)^2 = (\mathit{hypotenuse})^2$$

$$1 + 2 = (\mathit{hypotenuse})^2 \rightarrow 3 = (\mathit{hypotenuse})^2 \rightarrow \mathit{hypotenuse} = \sqrt{3}$$

So the triangle becomes

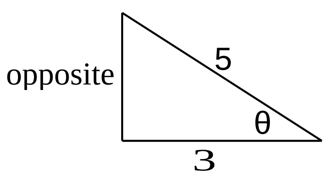


$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{3}}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

24. $\sec\theta = \frac{5}{3}$ corresponds to the right triangle

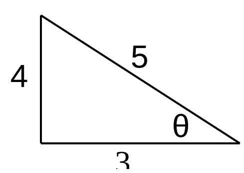


Using the Pythagorean Theorem:

$$(\mathit{opposite})^2 + (3)^2 = 5^2$$

$$(\mathit{opposite})^2 + 9 = 25 \rightarrow (\mathit{opposite})^2 = 16 \rightarrow \mathit{opposite} = 4$$

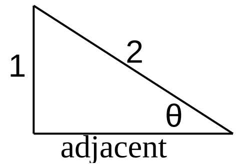
So the triangle becomes



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{4}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{4} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5}$$

25. $\csc\theta = 2$ corresponds to the right triangle

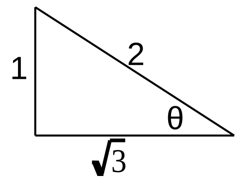


Using the Pythagorean Theorem:

$$(adjacent)^2 + (1)^2 = 2^2$$

$$(adjacent)^2 + 1 = 4 \rightarrow (adjacent)^2 = 3 \rightarrow adjacent = \sqrt{3}$$

So the triangle becomes

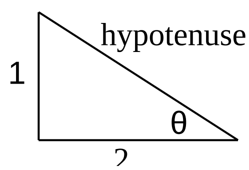


$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{2}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

26. $\cot\theta = 2$ corresponds to the right triangle

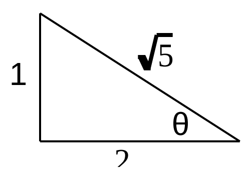


Using the Pythagorean Theorem:

$$(1)^2 + (2)^2 = (hypotenuse)^2$$

$$1 + 4 = (hypotenuse)^2 \rightarrow 5 = (hypotenuse)^2 \rightarrow hypotenuse = \sqrt{5}$$

So the triangle becomes



$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{2}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{2}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{5}}{1} = \sqrt{5} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{5}}{2}$$

27. $\sin^2(20^\circ) + \cos^2(20^\circ) = 1$, using the identity $\sin^2\theta + \cos^2\theta = 1$

28. $\sec^2(28^\circ) - \tan^2(28^\circ) = 1$, using the identity $\tan^2\theta + 1 = \sec^2\theta$

29. $\sin(80^\circ)\csc(80^\circ) = \sin(80^\circ)\left(\dfrac{1}{\sin(80^\circ)}\right) = 1$, using the identity $\csc\theta = \dfrac{1}{\sin\theta}$

30. $\tan(10^\circ)\cot(10^\circ) = \tan(10^\circ)\left(\dfrac{1}{\tan(10^\circ)}\right) = 1$, using the identity $\cot\theta = \dfrac{1}{\tan\theta}$

31. $\tan(50^\circ) - \dfrac{\sin(50^\circ)}{\cos(50^\circ)} = \dfrac{\sin(50^\circ)}{\cos(50^\circ)} - \dfrac{\sin(50^\circ)}{\cos(50^\circ)} = 0$, using the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

32. $\cot(25^\circ) - \dfrac{\cos(25^\circ)}{\sin(25^\circ)} = \dfrac{\cos(25^\circ)}{\sin(25^\circ)} - \dfrac{\cos(25^\circ)}{\sin(25^\circ)} = 0$, using the identity $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

33. $\sin(38^\circ) - \cos(52^\circ) = \sin(38^\circ) - \sin(90^\circ - 52^\circ) = \sin(38^\circ) - \sin(38^\circ) = 0$

34. $\tan(12^\circ) - \cot(78^\circ) = \tan(12^\circ) - \tan(90^\circ - 78^\circ) = \tan(12^\circ) - \tan(12^\circ) = 0$

35. $\dfrac{\cos(10^\circ)}{\sin(80^\circ)} = \dfrac{\sin(90^\circ - 10^\circ)}{\sin(80^\circ)} = \dfrac{\sin(80^\circ)}{\sin(80^\circ)} = 1$

36. $\dfrac{\cos(40^\circ)}{\sin(50^\circ)} = \dfrac{\sin(90^\circ - 40^\circ)}{\sin(50^\circ)} = \dfrac{\sin(50^\circ)}{\sin(50^\circ)} = 1$

37. $1 - \cos^2(20^\circ) - \cos^2(70^\circ) = \sin^2(20^\circ) - \sin^2(90^\circ - 70^\circ) = \sin^2(20^\circ) - \sin^2(20^\circ) = 0$

38. $1 + \tan^2(5^\circ) - \csc^2(85^\circ) = \sec^2(5^\circ) - \sec^2(90^\circ - 85^\circ) = \sec^2(5^\circ) - \sec^2(5^\circ) = 0$

39. $\tan(20^\circ) - \dfrac{\cos(70^\circ)}{\cos(20^\circ)} = \tan(20^\circ) - \dfrac{\sin(90^\circ - 70^\circ)}{\cos(20^\circ)} = \tan(20^\circ) - \dfrac{\sin(20^\circ)}{\cos(20^\circ)} = \tan(20^\circ) - \tan(20^\circ) = 0$

40. $\cot(40^\circ) - \dfrac{\sin(50^\circ)}{\sin(40^\circ)} = \cot(40^\circ) - \dfrac{\cos(90^\circ - 50^\circ)}{\sin(40^\circ)} = \cot(40^\circ) - \dfrac{\cos(40^\circ)}{\sin(40^\circ)} = \cot(40^\circ) - \cot(40^\circ) = 0$

41. $\tan(35^\circ)\sec(55^\circ)\cos(35^\circ) = \left(\dfrac{\sin(35^\circ)}{\cos(35^\circ)}\right)\sec(55^\circ)\cos(35^\circ) = \sin(35^\circ)\sec(55^\circ)$

$= \sin(35^\circ)\csc(90^\circ - 55^\circ) = \sin(35^\circ)\csc(35^\circ) = \sin(35^\circ)\left(\dfrac{1}{\sin(35^\circ)}\right) = 1$

42. $\cot(25^\circ)\csc(65^\circ)\sin(25^\circ)$

$= \left(\dfrac{\cos(25^\circ)}{\sin(25^\circ)}\right)\csc(65^\circ)\sin(25^\circ) = \cos(25^\circ)\csc(65^\circ) = \cos(25^\circ)\sec(90^\circ - 65^\circ)$

$= \cos(25^\circ)\sec(25^\circ) = \cos(25^\circ)\left(\dfrac{1}{\cos(25^\circ)}\right) = 1$

43. $\cos(35^\circ)\sin(55^\circ)+\cos(55^\circ)\sin(35^\circ)=\sin(55^\circ+35^\circ)=\sin(90^\circ)=1$

44. $\sec(35^\circ)\csc(55^\circ)-\tan(35^\circ)\cot(55^\circ)=\sec(35^\circ)\sec(35^\circ)-\tan(35^\circ)\tan(35^\circ)$
$=\sec^2 35^\circ-\tan^2 35^\circ=1$

45. Given: $\sin(30^\circ)=\frac{1}{2}$

(a) $\cos(60^\circ)=\sin(90^\circ-60^\circ)=\sin(30^\circ)=\frac{1}{2}$

(b) $\cos^2(30^\circ)=1-\sin^2(30^\circ)=1-\left(\frac{1}{2}\right)^2=1-\frac{1}{4}=\frac{3}{4}$

(c) $\csc\left(\frac{\pi}{6}\right)=\csc(30^\circ)=\frac{1}{\sin(30^\circ)}=\frac{1}{\left(\frac{1}{2}\right)}=2$

(d) $\sec\left(\frac{\pi}{3}\right)=\sec(60^\circ)=\csc(90^\circ-60^\circ)=\csc(30^\circ)=2$

46. Given: $\sin 60^\circ=\frac{\sqrt{3}}{2}$

(a) $\cos(30^\circ)=\sin(90^\circ-30^\circ)=\sin(60^\circ)=\frac{\sqrt{3}}{2}$

(b) $\cos^2(60^\circ)=1-\sin^2(60^\circ)=1-\left(\frac{\sqrt{3}}{2}\right)^2=1-\frac{3}{4}=\frac{1}{4}$

(c) $\sec\left(\frac{\pi}{6}\right)=\sec(30^\circ)=\frac{1}{\cos(30^\circ)}=\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}=\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$

(d) $\csc\left(\frac{\pi}{3}\right)=\csc(60^\circ)=\sec(90^\circ-60^\circ)=\sec(30^\circ)=\frac{2\sqrt{3}}{3}$

47. Given: $\tan\theta=4$

(a) $\sec^2\theta=1+\tan^2\theta=1+4^2=1+16=17$

(b) $\cot\theta=\frac{1}{\tan\theta}=\frac{1}{4}$

(c) $\cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta=4$

(d) $\csc^2\theta=1+\cot^2\theta=1+\frac{1}{\tan^2\theta}=1+\frac{1}{4^2}=1+\frac{1}{16}=\frac{17}{16}$

48. Given: $\sec\theta = 3$

(a) $\cos\theta = \dfrac{1}{\sec\theta} = \dfrac{1}{3}$

(b) $\tan^2\theta = \sec^2\theta - 1 = 3^2 - 1 = 9 - 1 = 8$

(c) $\csc(90^{o} - \theta) = \sec\theta = 3$

(d) $\sin^2\theta = 1 - \cos^2\theta = 1 - \dfrac{1}{\sec^2\theta} = 1 - \dfrac{1}{3^2} = 1 - \dfrac{1}{9} = \dfrac{8}{9}$

49. Given: $\csc\theta = 4$

(a) $\sin\theta = \dfrac{1}{\csc\theta} = \dfrac{1}{4}$

(b) $\cot^2\theta = \csc^2\theta - 1 = 4^2 - 1 = 16 - 1 = 15$

(c) $\sec(90^{o} - \theta) = \csc\theta = 4$

(d) $\sec^2\theta = 1 + \tan^2\theta = 1 + \dfrac{1}{\cot^2\theta} = 1 + \dfrac{1}{\csc^2\theta - 1} = 1 + \dfrac{1}{4^2 - 1} = 1 + \dfrac{1}{15} = \dfrac{16}{15}$

50. Given: $\cot\theta = 2$

(a) $\tan\theta = \dfrac{1}{\cot\theta} = \dfrac{1}{2}$

(b) $\csc^2\theta = \cot^2\theta + 1 = 2^2 + 1 = 4 + 1 = 5$

(c) $\tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta = 2$

(d) $\sec^2\theta = 1 + \tan^2\theta = 1 + \dfrac{1}{\cot^2\theta} = 1 + \dfrac{1}{2^2} = 1 + \dfrac{1}{4} = \dfrac{5}{4}$

51. Given: $\sin(38^{o}) \approx 0.62$

(a) $\cos(38^{o}) \approx ?$

$\sin^2(38^{o}) + \cos^2(38^{o}) = 1$

$\cos^2(38^{o}) = 1 - \sin^2(38^{o}) \rightarrow \cos(38^{o}) = \sqrt{1 - \sin^2(38^{o})} \approx \sqrt{1 - (0.62)^2} \approx 0.785$

(b) $\tan(38^{o}) \approx ?$

$\tan(38^{o}) = \dfrac{\sin(38^{o})}{\cos(38^{o})} \approx \dfrac{0.62}{0.785} \approx 0.79$

(c) $\cot(38^{o}) \approx ?$

$\cot(38^{o}) = \dfrac{\cos(38^{o})}{\sin(38^{o})} \approx \dfrac{0.785}{0.62} \approx 1.266$

(d) $\sec(38^{o}) \approx ?$

$\sec(38^{o}) = \dfrac{1}{\cos(38^{o})} \approx \dfrac{1}{0.785} \approx 1.274$

(e) $\csc(38^{o}) \approx ?$

$\csc(38^{o}) = \dfrac{1}{\sin(38^{o})} \approx \dfrac{1}{0.62} \approx 1.613$

(f) $\sin(52)^{o} \approx ?$

$\sin(52^{o}) = \cos(90^{o} - 52^{o})$

$= \cos(38^{o}) \approx 0.785$

(g) $\cos(52^{o}) \approx ?$

$\cos(52^{o}) = \sin(90^{o} - 52^{o})$

$= \sin(38^{o}) \approx 0.62$

(h) $\tan\left(52^{o}\right) \approx ?$

$\tan\left(52^{o}\right) = \cot\left(90^{o} - 52^{o}\right) = \cot\left(38^{o}\right) \approx 1.266$

52. Given: $\cos\left(21^{o}\right) \approx 0.93$

(a) $\sin\left(21^{o}\right) \approx ?$

$\sin^{2}\left(21^{o}\right) + \cos^{2}\left(21^{o}\right) = 1$

$\sin^{2}\left(21^{o}\right) = 1 - \cos^{2}\left(21^{o}\right) \rightarrow \sin\left(21^{o}\right) = \sqrt{1 - \cos^{2}\left(21^{o}\right)} \approx \sqrt{1 - \left(0.93\right)^{2}} \approx 0.368$

(b) $\tan\left(21^{o}\right) \approx ?$

$\tan\left(21^{o}\right) = \dfrac{\sin\left(21^{o}\right)}{\cos\left(21^{o}\right)} \approx \dfrac{0.368}{0.93} = 0.396$

(c) $\cot\left(21^{o}\right) \approx ?$

$\cot\left(21^{o}\right) = \dfrac{\cos\left(21^{o}\right)}{\sin\left(21^{o}\right)} \approx \dfrac{0.93}{0.368} = 2.527$

(d) $\sec\left(21^{o}\right) \approx ?$

$\sec\left(21^{o}\right) = \dfrac{1}{\cos\left(21^{o}\right)} \approx \dfrac{1}{0.93} \approx 1.075$

(e) $\csc\left(21^{o}\right) \approx ?$

$\csc\left(21^{o}\right) = \dfrac{1}{\sin\left(21^{o}\right)} \approx \dfrac{1}{0.368} \approx 2.717$

(f) $\sin\left(69^{o}\right) \approx ?$

$\sin\left(69^{o}\right) = \cos\left(90^{o} - 69^{o}\right)$

$= \cos\left(21^{o}\right) \approx 0.93$

(g) $\cos\left(69^{o}\right) \approx ?$

$\cos\left(69^{o}\right) = \sin\left(90^{o} - 69^{o}\right)$

$= \sin\left(21^{o}\right) \approx 0.368$

(h) $\tan\left(69^{o}\right) \approx ?$

$\tan\left(69^{o}\right) = \cot\left(90^{o} - 69^{o}\right) = \cot\left(21^{o}\right) \approx 2.527$

53. Given: $\sin\theta = 0.3$

$\sin\theta + \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta + \sin\theta = 0.3 + 0.3 = 0.6$

54. Given: $\tan\theta = 4$

$\tan\theta + \tan\left(\frac{\pi}{2} - \theta\right) = \tan\theta + \cot\theta = \tan\theta + \dfrac{1}{\tan\theta} = 4 + \dfrac{1}{4} = \dfrac{17}{4}$

55. $\sin\theta = \cos\left(2\theta + 30^{o}\right) = \cos\left(90^{o} - \left(60^{o} - 2\theta\right)\right) = \sin\left(60^{o} - 2\theta\right)$

The equation $\sin\theta = \sin\left(60^{o} - 2\theta\right)$ will be true when

$$\theta = 60^{o} - 2\theta \rightarrow 3\theta = 60^{o} \rightarrow \theta = \frac{60^{o}}{3} = 20^{o}.$$

56. $\tan\theta = \cot(\theta + 45^\circ) = \cot(90^0 - (45^\circ - \theta)) = \tan(45^\circ - \theta)$

The equation $\tan\theta = \tan(45^\circ - \theta)$ will be true when

$$\theta = 45^\circ - \theta \rightarrow 2\theta = 45^\circ \rightarrow \theta = \frac{45^\circ}{2} = 22.5^\circ.$$

57. (a) $T = \dfrac{1500}{300} + \dfrac{500}{100} = 5 + 5 = 10$ minutes

(b) $T = \dfrac{500}{100} + \dfrac{1500}{100} = 5 + 15 = 20$ minutes

(c) $\tan\theta = \dfrac{500}{x} \quad \rightarrow \quad x = \dfrac{500}{\tan\theta}$

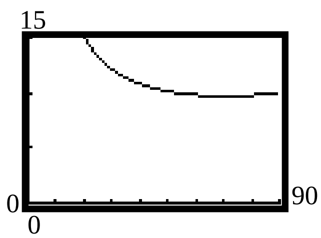
$\sin\theta = \dfrac{500}{\text{distance in sand}} \quad \rightarrow \quad \text{distance in sand} = \dfrac{500}{\sin\theta}$

$$T = \frac{1500 - x}{300} + \frac{\text{distance in sand}}{100} = \frac{1500 - \left(\dfrac{500}{\tan\theta}\right)}{300} + \frac{\left(\dfrac{500}{\sin\theta}\right)}{100} = 5 - \frac{5}{3\tan\theta} + \frac{5}{\sin\theta}$$

(d) 1000 feet along the paved path leaves an additional 500 feet in the direction of the path, so the angle of the path across the sand is 45˚.

$$T = 5 - \frac{5}{3\tan(45^\circ)} + \frac{5}{\sin(45^\circ)} = 5 - \frac{5}{3 \cdot 1} + \frac{5}{\left(\dfrac{\sqrt{2}}{2}\right)} = 5 - \frac{5}{3} + \frac{10}{\sqrt{2}} \approx 10.4 \text{ minutes}$$

(e) Graph:



The time is least when the angle is approximately 70.53˚. The least time is approximately 9.71 minutes. The value of x is:

$$x = \frac{500}{\tan(70.53^\circ)} \approx 176.8 \text{ feet}$$

58. Consider the length of the ladder in two sections, x, the portion across the hall that is 3 feet wide and y, the portion across that hall that is 4 feet wide. Then,

$\cos\theta = \dfrac{3}{x} \quad \rightarrow \quad x = \dfrac{3}{\cos\theta}$ and $\sin\theta = \dfrac{4}{y} \quad \rightarrow \quad y = \dfrac{4}{\sin\theta}$

$L = x + y = \dfrac{3}{\cos\theta} + \dfrac{4}{\sin\theta}$

59. (a) $|OA| = |OC| = 1$; $\quad OAC = OCA$;

$OAC + OAC + 180^\circ - \theta = 180^\circ$

$$2(OAC) = \theta \rightarrow OAC = \frac{\theta}{2}$$

(b) $\sin\theta = \dfrac{|CD|}{|OC|} = |CD|$ $\cos\theta = \dfrac{|OD|}{|OC|} = |OD|$

(c) $\tan\dfrac{\theta}{2} = \dfrac{|CD|}{|AD|} = \dfrac{|CD|}{1+|OD|} = \dfrac{\sin\theta}{1+\cos\theta}$

60. Let h be the height of the triangle and b be the base of the triangle.

$$\sin\theta = \frac{h}{a} \;\rightarrow\; h = a\sin\theta \qquad \cos\theta = \frac{\left(\frac{1}{2}b\right)}{a} \;\rightarrow\; b = 2a\cos\theta$$

$$A = \frac{1}{2}bh = \frac{1}{2}(2a\cos\theta)(a\sin\theta) = a^2\sin\theta\cos\theta$$

61. $h = x\dfrac{h}{x} = x\tan\theta;\quad h = (1-x)\dfrac{h}{1-x} = (1-x)\tan n\theta$

$$x\tan\theta = (1-x)\tan n\theta$$

$$x\tan\theta = \tan n\theta - x\tan n\theta \rightarrow x(\tan\theta + \tan n\theta) = \tan n\theta$$

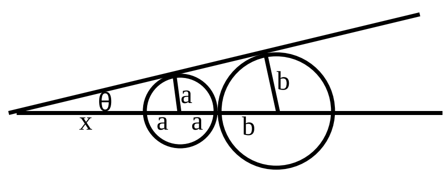
$$x = \frac{\tan n\theta}{\tan\theta + \tan n\theta}$$

62. $\sin\theta = \dfrac{a}{x+a} = \dfrac{b}{x+2a+b}$

$$xb + ab = xa + 2a^2 + ab$$

$$x(b-a) = 2a^2$$

$$x = \frac{2a^2}{b-a}$$



$$\sin\theta = \frac{a}{x+a} = \frac{a}{\frac{2a^2}{b-a}+a} = \frac{a}{\frac{2a^2+ab-a^2}{b-a}} = \frac{a(b-a)}{a^2+ab} = \frac{a(b-a)}{a(b+a)} = \frac{b-a}{b+a}$$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{b-a}{b+a}\right)^2} = \sqrt{1-\frac{b^2-2ab+a^2}{b^2+2ab+a^2}}$$

$$= \sqrt{\frac{b^2+2ab+a^2-b^2+2ab-a^2}{b^2+2ab+a^2}} = \sqrt{\frac{4ab}{(a+b)^2}} = \frac{2\sqrt{ab}}{a+b} = \frac{\sqrt{ab}}{\left(\frac{a+b}{2}\right)}$$

63. (a) Area $\Delta OAC = \dfrac{1}{2}|OC|\,|AC| = \dfrac{1}{2}\dfrac{|OC|}{1}\dfrac{|AC|}{1} = \dfrac{1}{2}\cos\alpha\sin\alpha = \dfrac{1}{2}\sin\alpha\cos\alpha$

(b) Area $\Delta OCB = \dfrac{1}{2}|OC|\,|BC| = \dfrac{1}{2}|OB|^2\dfrac{|OC|}{|OB|}\dfrac{|BC|}{|OB|} = \dfrac{1}{2}|OB|^2\cos\beta\sin\beta$

$$= \frac{1}{2}|OB|^2\sin\beta\cos\beta$$

(c) Area $\Delta OAB = \dfrac{1}{2}|BD|\,|OA| = \dfrac{1}{2}|BD|\,1 = \dfrac{1}{2}|OB|\dfrac{|BD|}{|OB|} = \dfrac{1}{2}|OB|\sin(\alpha+\beta)$

(d) $$\frac{\cos\alpha}{\cos\beta} = \frac{\dfrac{|OC|}{|OA|}}{\dfrac{|OC|}{|OB|}} = \frac{|OC|}{1}\ \frac{|OB|}{|OC|} = |OB|$$

(e) $$\text{Area } \Delta OAB = \text{Area } \Delta OAC + \text{Area } \Delta OCB$$

$$\frac{1}{2}|OB|\sin(\alpha+\beta) = \frac{1}{2}\sin\alpha\cos\alpha + \frac{1}{2}|OB|^2\sin\beta\cos\beta$$

$$\frac{\cos\alpha}{\cos\beta}\sin(\alpha+\beta) = \sin\alpha\cos\alpha + \frac{\cos^2\alpha}{\cos^2\beta}\sin\beta\cos\beta$$

$$\sin(\alpha+\beta) = \frac{\cos\beta}{\cos\alpha}\sin\alpha\cos\alpha + \frac{\cos\alpha}{\cos\beta}\sin\beta\cos\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

64. (a) $$\text{Area of } \Delta OBC = \frac{1}{2}\ 1\ 1\ \sin\theta = \frac{1}{2}\sin\theta$$

(b) $$\text{Area of } \Delta OBD = \frac{1}{2}\ 1\ \tan\theta = \frac{1}{2}\tan\theta = \frac{\sin\theta}{2\cos\theta}$$

(c) $$\text{Area } \Delta OBC < \text{ Area arc } OBC < \text{ Area } \Delta OBD$$

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{\sin\theta}{2\cos\theta} \rightarrow \frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\sin\theta}{\sin\theta\cos\theta}$$

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

65. $$\sin\alpha = \frac{\sin\alpha}{\cos\alpha}\ \cos\alpha = \tan\alpha\cos\alpha = \cos\beta\cos\alpha = \cos\beta\tan\beta = \cos\beta\ \frac{\sin\beta}{\cos\beta} = \sin\beta$$

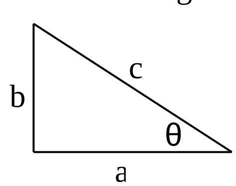
$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin^2\alpha + \tan^2\beta = 1 \rightarrow \sin^2\alpha + \frac{\sin^2\beta}{\cos^2\beta} = 1 \rightarrow \sin^2\alpha + \frac{\sin^2\alpha}{1-\sin^2\alpha} = 1$$

$$\left(1-\sin^2\alpha\right)\left(\sin^2\alpha + \frac{\sin^2\alpha}{1-\sin^2\alpha}\right) = (1)\left(1-\sin^2\alpha\right) \rightarrow \sin^2\alpha - \sin^4\alpha + \sin^2\alpha = 1 - \sin^2\alpha$$

$$\sin^4\alpha - 3\sin^2\alpha + 1 = 0 \rightarrow \sin^2\alpha = \frac{3\pm\sqrt{5}}{2} \rightarrow \sin\alpha = \sqrt{\frac{3\pm\sqrt{5}}{2}}$$

66. Consider the right triangle:



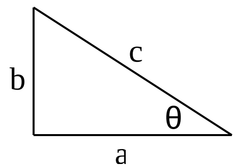
If θ is an acute angle in this triangle, then $a > 0$, $b > 0$ and $c > 0$.

So $\cos\theta = \dfrac{a}{c} > 0$. Also, since $a^2 + b^2 = c^2$, we know that $0 < a^2 < c^2 \rightarrow 0 < a < c$.

Now $0 < a < c \rightarrow 0 < \dfrac{a}{c} < 1$. So we now know that $0 < \cos\theta < 1$, which implies that

$$\frac{1}{\cos\theta} > \frac{1}{1} \rightarrow \frac{1}{\cos\theta} > 1 \rightarrow \sec\theta > 1.$$

67. Consider the right triangle:



If θ is an acute angle in this triangle, then $a > 0$, $b > 0$ and $c > 0$.

So $\sin\theta = \dfrac{b}{c} > 0$. Also, since $a^2 + b^2 = c^2$, we know that $0 < b^2 < c^2 \rightarrow 0 < b < c$.

Now $0 < b < c \rightarrow 0 < \dfrac{b}{c} < 1$. Therefore, $0 < \sin\theta < 1$.