Chapter 7

Trigonometric Functions

7.3 Computing the Values of Trigonometric Functions of Given Angles

1.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$

2.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30^o	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
60^o	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

3. $\sin\left(60^o\right) = \frac{\sqrt{3}}{2}$

4. $\cos\left(60^o\right) = \frac{1}{2}$

5. $\sin\left(30^o\right) = \frac{1}{2}$

6. $\cos\left(30^o\right) = \frac{\sqrt{3}}{2}$

7. $\left(\sin\left(60^o\right)\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$

8. $\left(\cos\left(60^o\right)\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

9. $2\sin\left(60^o\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

10. $2\cos\left(60^o\right) = 2\left(\frac{1}{2}\right) = 1$

11. $\frac{\sin\left(60^o\right)}{2} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{2} = \frac{\sqrt{3}}{4}$

12. $\frac{\cos\left(60^o\right)}{2} = \frac{\left(\frac{1}{2}\right)}{2} = \frac{1}{4}$

13. $4\cos\left(45^o\right) - 2\sin\left(45^o\right) = 4\left(\frac{\sqrt{2}}{2}\right) - 2\left(\frac{\sqrt{2}}{2}\right) = 2\sqrt{2} - \sqrt{2} = \sqrt{2}$

14. $2\sin\left(45^\circ\right)+\cos\left(30^\circ\right)=2\left(\frac{\sqrt{2}}{2}\right)+\frac{\sqrt{3}}{2}=\sqrt{2}+ +\frac{\sqrt{3}}{2}=\frac{2\sqrt{2}+\sqrt{3}}{2}$

15. $6\tan\left(45^\circ\right)-8\cos\left(60^\circ\right)=6(1)-8\left(\frac{1}{2}\right)=6-4=2$

16. $\sin\left(30^\circ\right)\tan\left(60^\circ\right)=\frac{1}{2}\left(\sqrt{3}\right)=\frac{\sqrt{3}}{2}$

17. $\sec\left(\frac{\pi}{4}\right)+2\csc\left(\frac{\pi}{3}\right)=\sqrt{2}+2\left(\frac{\sqrt{3}}{3}\right)=\frac{3\sqrt{2}+2\sqrt{3}}{3}$

18. $\tan\left(\frac{\pi}{4}\right)+\cot\left(\frac{\pi}{4}\right)=1+1=2$

19. $\sec^2\left(\frac{\pi}{6}\right)-4=\left(\frac{2\sqrt{3}}{3}\right)^2-4=\frac{12}{9}-4=\frac{4}{3}-4=\frac{4-12}{3}=-\frac{8}{3}$

20. $4+\tan^2\left(\frac{\pi}{3}\right)=4+\left(\sqrt{3}\right)^2=4+3=7$

21. $\sin^2\left(30^\circ\right)+\cos^2\left(60^\circ\right)=\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2=\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$

22. $\sec^2\left(60^\circ\right)-\tan^2\left(45^\circ\right)=(2)^2-(1)^2=4-1=3$

23. $1-\cos^2\left(30^\circ\right)-\cos^2\left(60^\circ\right)=1-\left(\frac{\sqrt{3}}{2}\right)^2-\left(\frac{1}{2}\right)^2=1-\frac{3}{4}-\frac{1}{4}=1-1=0$

24. $1+\tan^2\left(30^\circ\right)-\csc^2\left(45^\circ\right)=1+\left(\frac{\sqrt{3}}{3}\right)^2-\left(\sqrt{2}\right)^2=1+\frac{3}{9}-2=-\frac{2}{3}$

25. Set the calculator to degree mode: $\sin\left(28^\circ\right)\approx 0.47$.

26. Set the calculator to degree mode: $\cos\left(14^\circ\right)\approx 0.97$.

27. Set the calculator to degree mode: $\tan\left(21^\circ\right)\approx 0.38$.

28. Set the calculator to degree mode: $\cot\left(70^\circ\right)=\frac{1}{\tan\left(70^\circ\right)}\approx 0.36$.

29. Set the calculator to degree mode: $\sec(41^\circ) = \dfrac{1}{\cos(41^\circ)} \approx 1.33$.

30. Set the calculator to degree mode: $\csc(55^\circ) = \dfrac{1}{\sin(55^\circ)} \approx 1.22$.

31. Set the calculator to radian mode: $\sin\left(\dfrac{\pi}{10}\right) \approx 0.31$.

32. Set the calculator to radian mode: $\cos\left(\dfrac{\pi}{8}\right) \approx 0.92$.

33. Set the calculator to radian mode: $\tan\left(\dfrac{5\pi}{12}\right) \approx 3.73$.

34. Set the calculator to radian mode: $\cot\left(\dfrac{\pi}{18}\right) = \dfrac{1}{\tan\left(\dfrac{\pi}{18}\right)} \approx 5.67$.

35. Set the calculator to radian mode: $\sec\left(\dfrac{\pi}{12}\right) = \dfrac{1}{\cos\left(\dfrac{\pi}{12}\right)} \approx 1.04$.

36. Set the calculator to radian mode: $\csc\left(\dfrac{5\pi}{13}\right) = \dfrac{1}{\sin\left(\dfrac{5\pi}{13}\right)} \approx 1.07$.

37. Set the calculator to radian mode: $\sin(1) \approx 0.84$.

38. Set the calculator to radian mode: $\tan(1) \approx 1.56$.

39. Set the calculator to degree mode: $\sin(1^\circ) \approx 0.02$.

40. Set the calculator to degree mode: $\tan(1^\circ) \approx 0.02$.

41. Set the calculator to radian mode: $\tan(0.3) \approx 0.31$.

42. Set the calculator to radian mode: $\tan(0.1) \approx 0.1$.

43. Use the formula $R = \dfrac{v_0^2 \sin(2\theta)}{g}$ with $g = 32.2\text{ft} / \sec^2$; $\theta = 45^\circ$; $v_0 = 100$ ft / sec:

$$R = \frac{100^2 \sin(2(45^\circ))}{32.2} = \frac{10000\sin(90^\circ)}{32.2} = \frac{10000}{32.2} \approx 310.56 \text{ feet}$$

Use the formula $H = \dfrac{v_0^{\,2}\sin^2\theta}{2g}$ with $g = 32.2\text{ft / sec}^2$; $\theta = 45^{\circ}$; $v_0 = 100$ ft / sec:

$$H = \frac{100^2\sin^2\left(45^{\circ}\right)}{2(32.2)} \approx \frac{10000(0.7071)^2}{64.4} \approx 77.64 \text{ feet}$$

44. Use the formula $R = \dfrac{v_0^{\,2}\sin(2\theta)}{g}$ with $g = 9.8$ m / sec^2; $\theta = 30^{\circ}$; $v_0 = 150$ m / sec:

$$R = \frac{150^2\sin(2(30^{\circ}))}{9.8} = \frac{22500\sin\left(60^{\circ}\right)}{9.8} \approx 1988.32 \text{ meters}$$

Use the formula $H = \dfrac{v_0^{\,2}\sin^2\theta}{2g}$ with $g = 9.8$ m / sec^2; $\theta = 30^{\circ}$; $v_0 = 150$ m / sec:

$$H = \frac{150^2\sin^2\left(30^{\circ}\right)}{2(9.8)} = \frac{22500(0.5)^2}{19.6} \approx 286.99 \text{ meters}$$

45. Use the formula $R = \dfrac{v_0^{\,2}\sin(2\theta)}{g}$ with $g = 9.8$ m / sec^2; $\theta = 25^{\circ}$; $v_0 = 500$ m / sec:

$$R = \frac{500^2\sin(2(25^{\circ}))}{9.8} = \frac{250{,}000\sin\left(50^{\circ}\right)}{9.8} \approx 19{,}542 \text{ meters}$$

Use the formula $H = \dfrac{v_0^{\,2}\sin^2\theta}{2g}$ with $g = 9.8$ m / sec^2; $\theta = 25^{\circ}$; $v_0 = 500$ m / sec:

$$H = \frac{500^2\sin^2\left(25^{\circ}\right)}{2(9.8)} \approx \frac{250{,}000(0.4226)^2}{19.6} \approx 2278 \text{ meters}$$

46. Use the formula $R = \dfrac{v_0^{\,2}\sin(2\theta)}{g}$ with $g = 32.2\text{ft / sec}^2$; $\theta = 50^{\circ}$; $v_0 = 200$ ft / sec:

$$R = \frac{200^2\sin(2(50^{\circ}))}{32.2} = \frac{40000\sin\left(100^{\circ}\right)}{32.2} \approx 1223.36 \text{ feet}$$

Use the formula $H = \dfrac{v_0^{\,2}\sin^2\theta}{2g}$ with $g = 32.2\text{ft / sec}^2$; $\theta = 50^{\circ}$; $v_0 = 200$ ft / sec:

$$H = \frac{200^2\sin^2 50^{\circ}}{2(32.2)} = \frac{40000(0.7660)^2}{64.4} \approx 364.49 \text{ feet}$$

47. Use the formula $t = \sqrt{\dfrac{2a}{g\sin\theta\cos\theta}}$ with $g = 32$ ft / sec^2 and $a = 10$ feet:

(a) $$t = \sqrt{\frac{2(10)}{32\sin\left(30^{\circ}\right)\cos\left(30^{\circ}\right)}} = \sqrt{\frac{20}{\left(32 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}\right)}} = \sqrt{\frac{20}{8\sqrt{3}}} = \sqrt{\frac{5}{2\sqrt{3}}} \approx 1.20 \text{ seconds}$$

(b) $$t = \sqrt{\frac{2(10)}{32\sin\left(45^{\circ}\right)\cos\left(45^{\circ}\right)}} = \sqrt{\frac{20}{\left(32 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}\right)}} = \sqrt{\frac{20}{16}} = \sqrt{\frac{5}{4}} \approx 1.12 \text{ seconds}$$

(c) $t=\sqrt{\dfrac{2(10)}{32\sin(60^\circ)\cos(60^\circ)}}=\sqrt{\dfrac{20}{\left(32\cdot\dfrac{\sqrt{3}}{2}\cdot\dfrac{1}{2}\right)}}=\sqrt{\dfrac{20}{8\sqrt{3}}}=\sqrt{\dfrac{5}{2\sqrt{3}}}\approx 1.20$ seconds

48. Use the formula $x=\cos\theta+\sqrt{16+0.5\cos(2\theta)}$.
When $\theta=30^\circ$:

$$x=\cos(30^\circ)+\sqrt{16+0.5\cos(2\cdot 30^\circ)}=\cos(30^\circ)+\sqrt{16+0.5\cos(60^\circ)}\approx 4.897 \text{ cm}$$

When $\theta=45^\circ$:

$$x=\cos(45^\circ)+\sqrt{16+0.5\cos(2\cdot 45^\circ)}=\cos(45^\circ)+\sqrt{16+0.5\cos(90^\circ)}\approx 4.707 \text{ cm}$$

49. (a) $T(30^\circ)=1+\dfrac{2}{3\sin(30^\circ)}-\dfrac{1}{4\tan(30^\circ)}=1+\dfrac{2}{\left(3\cdot\dfrac{1}{2}\right)}-\dfrac{1}{\left(4\cdot\dfrac{1}{\sqrt{3}}\right)}=1+\dfrac{4}{3}-\dfrac{\sqrt{3}}{4}\approx 1.9$ hrs

$\dfrac{1}{x}=\tan\theta \;\rightarrow\; x=\dfrac{1}{\tan\theta}$

Distance traveled on road is: $8-2x=8-\dfrac{2}{\tan\theta}$

Time on road $=\dfrac{\text{distance on road}}{\text{rate on road}}=\dfrac{\left(8-\dfrac{2}{\tan\theta}\right)}{8}$

Time on road $=\dfrac{\left(8-\dfrac{2}{\tan(30^\circ)}\right)}{8}\approx 0.57$ hours

(b) $T(45^\circ)=1+\dfrac{2}{3\sin(45^\circ)}-\dfrac{1}{4\tan(45^\circ)}=1+\dfrac{2}{\left(3\cdot\dfrac{1}{\sqrt{2}}\right)}-\dfrac{1}{4\cdot 1}=1+\dfrac{2\sqrt{2}}{3}-\dfrac{1}{4}\approx 1.69$ hrs

Time on road $=\dfrac{\left(8-\dfrac{2}{\tan(45^\circ)}\right)}{8}\approx 0.75$ hours

(c) $T(60^\circ)=1+\dfrac{2}{3\sin(60^\circ)}-\dfrac{1}{4\tan(60^\circ)}=1+\dfrac{2}{\left(3\cdot\dfrac{\sqrt{3}}{2}\right)}-\dfrac{1}{4\cdot\sqrt{3}}$

$=1+\dfrac{4}{3\sqrt{3}}-\dfrac{1}{4\sqrt{3}}\approx 1.63$ hrs Time on road $=\dfrac{\left(8-\dfrac{2}{\tan(60^\circ)}\right)}{8}\approx 0.86$ hours

(d) $T(90^\circ)=1+\dfrac{2}{3\sin(90^\circ)}-\dfrac{1}{4\tan(90^\circ)}=$ (tan90° is undefinec

The distance would be 2 miles in the sand and 8 miles on the road. The total time would be: $\dfrac{2}{3}+1=\dfrac{5}{3}\approx 1.67$ hours.

50. Use the formula: $V(\theta) = \frac{1}{3}\pi R^3 \frac{(1+\sec\theta)^3}{\tan^2\theta}$

When $\theta = 30^\circ$:

$$V(30^\circ) = \left(\frac{1}{3}\right)\left(\pi 2^3\right)\frac{\left(1+\sec\left(30^\circ\right)\right)^3}{\tan^2\left(30^\circ\right)} \approx \left(\frac{8}{3}\right)(\pi)\frac{(1+1.1547)^3}{(0.5774)^2} \approx 251.4 \text{ cm}^3$$

When $\theta = 45^\circ$:

$$V(45^\circ) = \left(\frac{1}{3}\right)\left(\pi 2^3\right)\frac{\left(1+\sec\left(45^\circ\right)\right)^3}{\tan^2\left(45^\circ\right)} \approx \left(\frac{8}{3}\right)(\pi)\frac{(1+1.4142)^3}{(1)^2} \approx 117.9 \text{ cm}^3$$

When $\theta = 60^\circ$:

$$V(60^\circ) = \left(\frac{1}{3}\right)\left(\pi 2^3\right)\frac{\left(1+\sec\left(60^\circ\right)\right)^3}{\tan^2\left(60^\circ\right)} = \left(\frac{8}{3}\right)(\pi)\frac{(1+2)^3}{(1.7321)^2} \approx 75.4 \text{ cm}^3$$

51. Complete the table:

θ	0.5	0.4	0.2	0.1	0.01	0.001	0.0001	0.00001
$\sin\theta$	0.4794	0.3894	0.1987	0.0998	0.0100	0.0010	0.0001	0.00001
$\frac{\sin\theta}{\theta}$	0.9589	0.9735	0.9933	0.9983	1.0000	1.0000	1.0000	1.0000

The ratio $\frac{\sin\theta}{\theta}$ as θ approaches 0 is 1.

52. Complete the table:

θ	0.5	0.4	0.2	0.1	0.01	0.001	0.0001	0.00001
$\cos\theta-1$	–0.1224	–0.0789	–0.0199	–0.0050	–0.0000	–0.0000	–0.0000	–0.0000
$\frac{\cos\theta-1}{\theta}$	–0.2448	–0.1973	–0.0997	–0.0500	–0.0050	–0.0005	–0.0000	–0.0000

The ratio $\frac{\cos\theta-1}{\theta}$ as θ approaches 0 is (

53. $\tan 1^\circ \ \tan 2^\circ \ \tan 3^\circ \ \ldots \ \tan 89^\circ = 1$

We can rearrange the order of the terms in this product as follows:

$$\left(\tan\left(1^\circ\right)\ \tan\left(89^\circ\right)\right)\ \left(\tan\left(2^\circ\right)\ \tan\left(88^\circ\right)\right)\ \left(\tan\left(3^\circ\right)\ \tan\left(87^\circ\right)\right)\ \ldots\ \left(\tan\left(44^\circ\right)\ \tan\left(46^\circ\right)\right)\ \tan\left(45^\circ\right)$$

Now each set of parentheses contains a pair of complementary angles.

For example, $\left(\tan\left(1^\circ\right)\ \tan\left(89^\circ\right)\right)$. Using cofunction properties we have:

$$\left(\tan\left(1^\circ\right)\ \tan\left(89^\circ\right)\right) = \left(\tan\left(1^\circ\right)\ \tan\left(90^\circ-1^\circ\right)\right) = \left(\tan\left(1^\circ\right)\ \cot\left(1^\circ\right)\right) = 1$$

This result holds for each pair in our product. And since we know that $\tan\left(45^\circ\right) = 1$,

our product can be rewritten as $1\ 1\ 1\ \ldots\ 1 = 1$. Therefore, $\tan 1^\circ \ \tan 2^\circ \ \tan 3^\circ \ \ldots \ \tan 89^\circ = 1$.

54. $\cot\left(1^\circ\right)\ \cot\left(2^\circ\right)\ \cot\left(3^\circ\right)\ \ldots\ \cot\left(89^\circ\right) = 1$

We can rearrange the order of the terms in this product as follows:

$$\left(\cot\left(1^\circ\right)\ \cot\left(89^\circ\right)\right)\ \left(\cot\left(2^\circ\right)\ \cot\left(88^\circ\right)\right)\ \left(\cot\left(3^\circ\right)\ \cot\left(87^\circ\right)\right)\ \ldots\ \left(\cot\left(44^\circ\right)\ \cot\left(46^\circ\right)\right)\ \cot\left(45^\circ\right)$$

Now follow the strategy used in Problem 53.

55. $\cos(1^\circ)\ \cot(2^\circ)\ \ldots\ \cos(45^\circ)\ \csc(46^\circ)\ \ldots\ \csc(89^\circ)$

56. $\sin(1^\circ)\ \sin(2^\circ)\ \ldots\ \sin(45^\circ)\ \sec(46^\circ)\ \ldots\ \sec(89^\circ) = \dfrac{\sqrt{2}}{2}$

We can rearrange the order of the terms in this product as follows:

$\left(\sin(1^\circ)\ \sec(89^\circ)\right)\ \left(\sin(2^\circ)\ \sec(88^\circ)\right)\ \left(\sin(3^\circ)\ \sec(87^\circ)\right)\ \ldots\ \left(\sin(44^\circ)\ \sec(46^\circ)\right)\ \sin(45^\circ)$

Now each set of parentheses contains a pair of complementary angles.

For example, $\left(\sin(1^\circ)\ \sec(89^\circ)\right)$. Using cofunction properties we have:

$$\left(\sin(1^\circ)\ \sec(89^\circ)\right) = \left(\sin(1^\circ)\ \sec(90^\circ - 1^\circ)\right) = \left(\sin(1^\circ)\ \csc(1^\circ)\right) = 1$$

This result holds for each pair in our product. And since we know that $\sin(45^\circ) = \dfrac{\sqrt{2}}{2}$,

Our product can be rewritten as $1\ 1\ 1\ \ldots\ 1\ \dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{2}}{2}$.