Chapter 7

Trigonometric Functions

7.5 Trigonometric Functions: Unit Circle Approach

1. $P=\left(\frac{1}{4},\frac{\sqrt{15}}{4}\right)\quad a=\frac{1}{4}, b=\frac{\sqrt{15}}{4}$

$\sin t=\frac{\sqrt{15}}{4}$

$\csc t=\frac{1}{\left(\frac{\sqrt{15}}{4}\right)}=1\cdot\frac{4}{\sqrt{15}}\cdot\frac{\sqrt{15}}{\sqrt{15}}=\frac{4\sqrt{15}}{15}$

$\cos t=\frac{1}{4}$

$\sec t=\frac{1}{\left(\frac{1}{4}\right)}=1\cdot\frac{4}{1}=4$

$\tan t=\frac{\left(\frac{\sqrt{15}}{4}\right)}{\left(\frac{1}{4}\right)}=\frac{\sqrt{15}}{4}\cdot\frac{4}{1}=\sqrt{15}$

$\cot t=\frac{\left(\frac{1}{4}\right)}{\left(\frac{\sqrt{15}}{4}\right)}=\frac{1}{4}\cdot\frac{4}{\sqrt{15}}=\frac{1}{\sqrt{15}}\cdot\frac{\sqrt{15}}{\sqrt{15}}=\frac{\sqrt{15}}{15}$

2. $P=\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)\quad a=-\frac{\sqrt{3}}{2}, b=-\frac{1}{2}$

$\sin t=-\frac{1}{2}$

$\csc t=\frac{1}{\left(-\frac{1}{2}\right)}=1\cdot\left(-\frac{2}{1}\right)=-2$

$\cos t=-\frac{\sqrt{3}}{2}$

$\sec t=\frac{1}{\left(-\frac{\sqrt{3}}{2}\right)}=1\cdot\left(-\frac{2}{\sqrt{3}}\right)=-\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{2}}=-\frac{2\sqrt{3}}{3}$

$\tan t=\frac{\left(-\frac{1}{2}\right)}{\left(-\frac{\sqrt{3}}{2}\right)}=\left(-\frac{1}{2}\right)\left(-\frac{2}{\sqrt{3}}\right)$

$=\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}}{3}$

$\cot t=\frac{\left(-\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}\right)}=\left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{2}{1}\right)=\sqrt{3}$

3. $P=\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$ $\quad a=-\frac{\sqrt{2}}{2}, b=-\frac{\sqrt{2}}{2}$

$\sin t=-\frac{\sqrt{2}}{2}$ $\qquad \csc t=\frac{1}{\left(-\frac{\sqrt{2}}{2}\right)}=1\left(-\frac{2}{\sqrt{2}}\right)=-\frac{2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=-\sqrt{2}$

$\cos t=-\frac{\sqrt{2}}{2}$ $\qquad \sec t=\frac{1}{\left(-\frac{\sqrt{2}}{2}\right)}=1\left(-\frac{2}{\sqrt{2}}\right)=-\frac{2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=-\sqrt{2}$

$\tan t=\frac{\left(-\frac{\sqrt{2}}{2}\right)}{\left(-\frac{\sqrt{2}}{2}\right)}=1$ $\qquad \cot t=\frac{\left(-\frac{\sqrt{2}}{2}\right)}{\left(-\frac{\sqrt{2}}{2}\right)}=1$

4. $P=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$ $\quad a=\frac{\sqrt{2}}{2}, b=-\frac{\sqrt{2}}{2}$

$\sin t=-\frac{\sqrt{2}}{2}$ $\qquad \csc t=\frac{1}{\left(-\frac{\sqrt{2}}{2}\right)}=1\left(-\frac{2}{\sqrt{2}}\right)=-\frac{2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=-\sqrt{2}$

$\cos t=\frac{\sqrt{2}}{2}$ $\qquad \sec t=\frac{1}{\left(\frac{\sqrt{2}}{2}\right)}=1\left(\frac{2}{\sqrt{2}}\right)=\frac{2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\sqrt{2}$

$\tan t=\frac{\left(-\frac{\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{2}}{2}\right)}=-1$ $\qquad \cot t=\frac{\left(\frac{\sqrt{2}}{2}\right)}{\left(-\frac{\sqrt{2}}{2}\right)}=-1$

5. $P=\left(\frac{\sqrt{5}}{3},\frac{2}{3}\right)\quad a=\frac{\sqrt{5}}{3}, b=\frac{2}{3}$

$\sin t=\frac{2}{3}$ $\qquad$ $\csc t=\frac{1}{\left(\frac{2}{3}\right)}=1\cdot\left(\frac{3}{2}\right)=\frac{3}{2}$

$\cos t=\frac{\sqrt{5}}{3}$ $\qquad$ $\sec t=\frac{1}{\left(\frac{\sqrt{5}}{3}\right)}=1\cdot\left(\frac{3}{\sqrt{5}}\right)=\frac{3}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\frac{3\sqrt{5}}{5}$

$\tan t=\frac{\left(\frac{2}{3}\right)}{\left(\frac{\sqrt{5}}{3}\right)}=\left(\frac{2}{3}\right)\cdot\left(\frac{3}{\sqrt{5}}\right)=\frac{2}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$ $\qquad$ $\cot t=\frac{\left(\frac{\sqrt{5}}{3}\right)}{\left(\frac{2}{3}\right)}=\left(\frac{\sqrt{5}}{3}\right)\cdot\left(\frac{3}{2}\right)=\frac{\sqrt{5}}{2}$

6. $P=\left(-\frac{\sqrt{5}}{5},\frac{2\sqrt{5}}{5}\right)\quad a=-\frac{\sqrt{5}}{5}, b=\frac{2\sqrt{5}}{5}$

$\sin t=\frac{2\sqrt{5}}{5}$ $\qquad$ $\csc t=\frac{1}{\left(\frac{2\sqrt{5}}{5}\right)}=1\cdot\left(\frac{5}{2\sqrt{5}}\right)\cdot\frac{\sqrt{5}}{\sqrt{5}}=\frac{\sqrt{5}}{2}$

$\cos t=-\frac{\sqrt{5}}{5}$ $\qquad$ $\sec t=\frac{1}{\left(-\frac{\sqrt{5}}{5}\right)}=1\cdot\left(-\frac{5}{\sqrt{5}}\right)\cdot\frac{\sqrt{5}}{\sqrt{5}}=-\sqrt{5}$

$\tan t=\frac{\left(\frac{2\sqrt{5}}{5}\right)}{\left(-\frac{\sqrt{5}}{5}\right)}=\left(\frac{2\sqrt{5}}{5}\right)\cdot\left(-\frac{5}{\sqrt{5}}\right)=-2$ $\qquad$ $\cot t=\frac{\left(-\frac{\sqrt{5}}{5}\right)}{\left(\frac{2\sqrt{5}}{5}\right)}=\left(-\frac{\sqrt{5}}{5}\right)\cdot\left(\frac{5}{2\sqrt{5}}\right)=-\frac{1}{2}$

7. For the point (3, -4), $x=3,\ y=-4,\ r=\sqrt{x^2+y^2}=\sqrt{9+16}=\sqrt{25}=5$

$\sin\theta=-\frac{4}{5}\qquad\cos\theta=\frac{3}{5}\qquad\tan\theta=-\frac{4}{3}$

$\csc\theta=-\frac{5}{4}\qquad\sec\theta=\frac{5}{3}\qquad\cot\theta=-\frac{3}{4}$

8. For the point (4, –3), $x = 4,\ y = -3,\ r = \sqrt{x^2 + y^2} = \sqrt{16+9} = \sqrt{25} = 5$

$$\sin\theta = -\frac{3}{5} \qquad \cos\theta = \frac{4}{5} \qquad \tan\theta = -\frac{3}{4}$$
$$\csc\theta = \frac{5}{3} \qquad \sec\theta = \frac{5}{4} \qquad \cot\theta = -\frac{4}{3}$$

9. For the point (-2, 3), $x = -2,\ y = 3,\ r = \sqrt{x^2 + y^2} = \sqrt{4+9} = \sqrt{13}$

$$\sin\theta = \frac{3}{\sqrt{13}}\ \frac{\sqrt{13}}{\sqrt{13}} = \frac{3\sqrt{13}}{13} \qquad \cos\theta = -\frac{2}{\sqrt{13}}\ \frac{\sqrt{13}}{\sqrt{13}} = -\frac{2\sqrt{13}}{13} \qquad \tan\theta = -\frac{3}{2}$$
$$\csc\theta = \frac{\sqrt{13}}{3} \qquad \sec\theta = -\frac{\sqrt{13}}{2} \qquad \cot\theta = -\frac{2}{3}$$

10. For the point (2, -4), $x = 2,\ y = -4,\ r = \sqrt{x^2 + y^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

$$\sin\theta = \frac{-4}{2\sqrt{5}}\ \frac{\sqrt{5}}{\sqrt{5}} = -\frac{2\sqrt{5}}{5} \qquad \cos\theta = \frac{2}{\sqrt{5}}\ \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \qquad \tan\theta = \frac{-4}{2} = -2$$
$$\csc\theta = \frac{2\sqrt{5}}{-4} = -\frac{\sqrt{5}}{2} \qquad \sec\theta = \frac{\sqrt{5}}{2} \qquad \cot\theta = -\frac{1}{2}$$

11. For the point (–1, –1), $x = -1,\ y = -1,\ r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2} = \sqrt{2}$

$$\sin\theta = \frac{-1}{\sqrt{2}}\ \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \qquad \cos\theta = \frac{-1}{\sqrt{2}}\ \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \qquad \tan\theta = \frac{-1}{-1} = 1$$
$$\csc\theta = \frac{\sqrt{2}}{-1} = -\sqrt{2} \qquad \sec\theta = \frac{\sqrt{2}}{-1} = -\sqrt{2} \qquad \cot\theta = \frac{-1}{-1} = 1$$

12. For the point (–3, 1), $x = -3,\ y = 1,\ r = \sqrt{x^2 + y^2} = \sqrt{9+1} = \sqrt{10}$

$$\sin\theta = \frac{1}{\sqrt{10}}\ \frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{10}}{10} \qquad \cos\theta = \frac{-3}{\sqrt{10}}\ \frac{\sqrt{10}}{\sqrt{10}} = \frac{-3\sqrt{10}}{13} \qquad \tan\theta = \frac{1}{-3} = -\frac{1}{3}$$
$$\csc\theta = \frac{\sqrt{10}}{1} = \sqrt{10} \qquad \sec\theta = \frac{\sqrt{10}}{-3} = -\frac{\sqrt{10}}{3} \qquad \cot\theta = \frac{-3}{1} = -3$$

13. $\sin(405^\circ) = \sin(360^\circ + 45^\circ) = \sin(45^\circ) = \frac{\sqrt{2}}{2}$

14. $\cos(420^\circ) = \cos(360^\circ + 60^\circ) = \cos(60^\circ) = \frac{1}{2}$

15. $\tan(405^\circ) = \tan(180^\circ + 180^\circ + 45^\circ) = \tan(45^\circ) = 1$

16. $\sin(390^\circ) = \sin(360^\circ + 30^\circ) = \sin(30^\circ) = \frac{1}{2}$

17. $\csc(450^\circ) = \csc(360^\circ + 90^\circ) = \csc(90^\circ) = 1$

18. $\sec(540^\circ) = \sec(360^\circ + 180^\circ) = \sec(180^\circ) = -1$

19. $\cot(390^\circ) = \cot(180^\circ + 180^\circ + 30^\circ) = \cot(30^\circ) = \sqrt{3}$

20. $\sec(420^\circ) = \sec(360^\circ + 60^\circ) = \sec(60^\circ) = 2$

21. $\cos\left(\frac{33\pi}{4}\right) = \cos\left(\frac{\pi}{4} + \frac{32\pi}{4}\right) = \cos\left(\frac{\pi}{4} + 8\pi\right) = \cos\left(\frac{\pi}{4} + 4 \cdot 2\pi\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

22. $\sin\left(\frac{9\pi}{4}\right) = \sin\left(\frac{\pi}{4} + \frac{8\pi}{4}\right) = \sin\left(\frac{\pi}{4} + 2\pi\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

23. $\tan(21\pi) = \tan(0 + 21\pi) = \tan(0) = 0$

24. $\csc\left(\frac{9\pi}{2}\right) = \csc\left(\frac{\pi}{2} + \frac{8\pi}{2}\right) = \csc\left(\frac{\pi}{2} + 4\pi\right) = \csc\left(\frac{\pi}{2} + 2 \cdot 2\pi\right) = \csc\left(\frac{\pi}{2}\right) = 1$

25. $\sec\left(\frac{17\pi}{4}\right) = \sec\left(\frac{\pi}{4} + \frac{16\pi}{4}\right) = \sec\left(\frac{\pi}{4} + 4\pi\right) = \sec\left(\frac{\pi}{4} + 2 \cdot 2\pi\right) = \sec\left(\frac{\pi}{4}\right) = \sqrt{2}$

26. $\cot\left(\frac{17\pi}{4}\right) = \cot\left(\frac{\pi}{4} + \frac{16\pi}{4}\right) = \cot\left(\frac{\pi}{4} + 4\pi\right) = \cot\left(\frac{\pi}{4} + 2 \cdot 2\pi\right) = \cot\left(\frac{\pi}{4}\right) = 1$

27. $\tan\left(\frac{19\pi}{6}\right) = \tan\left(\frac{\pi}{6} + \frac{18\pi}{6}\right) = \tan\left(\frac{\pi}{6} + 3\pi\right) = \tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

28. $\sec\left(\frac{25\pi}{6}\right) = \sec\left(\frac{\pi}{6} + \frac{24\pi}{6}\right) = \sec\left(\frac{\pi}{6} + 4\pi\right) = \sec\left(\frac{\pi}{6} + 2 \cdot 2\pi\right) = \sec\left(\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}$

29. $\sin(-60^\circ) = -\sin(60^\circ) = -\frac{\sqrt{3}}{2}$

30. $\cos(-30^\circ) = \cos(30^\circ) = \frac{\sqrt{3}}{2}$

31. $\tan(-30^\circ) = -\tan(30^\circ) = -\frac{\sqrt{3}}{3}$

32. $\sin(-135^\circ) = -\sin(135^\circ) = -\frac{\sqrt{2}}{2}$

33. $\sec(-60^\circ) = \sec(60^\circ) = 2$

34. $\csc(-30^\circ) = -\csc(30^\circ) = -2$

35. $\sin(-90^\circ) = -\sin(90^\circ) = -1$

36. $\cos(-270^\circ) = \cos(270^\circ) = 0$

37. $\tan\left(-\frac{\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -1$

38. $\sin(-\pi) = -\sin(\pi) = 0$

39. $\cos\left(-\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$

40. $\sin\left(-\frac{\pi}{3}\right)=-\sin\left(\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$

41. $\tan(-\pi)=-\tan(\pi)=0$

42. $\sin\left(-\frac{3\pi}{2}\right)=-\sin\left(\frac{3\pi}{2}\right)=-(-1)=1$

43. $\csc\left(-\frac{\pi}{4}\right)=-\csc\left(\frac{\pi}{4}\right)=-\sqrt{2}$

44. $\sec(-\pi)=\sec(\pi)=-1$

45. $\sec\left(-\frac{\pi}{6}\right)=\sec\left(\frac{\pi}{6}\right)=\frac{2\sqrt{3}}{3}$

46. $\csc\left(-\frac{\pi}{3}\right)=-\csc\left(\frac{\pi}{3}\right)=-\frac{2\sqrt{3}}{3}$

47. $\sin(-\pi)+\cos(5\pi)=-\sin(\pi)+\cos(\pi+4\pi)=0+\cos(\pi)=-1$

48. $\tan\left(-\frac{5\pi}{6}\right)-\cot\left(\frac{7\pi}{2}\right)=-\tan\left(\frac{5\pi}{6}\right)-\cot\left(\frac{3\pi}{2}+2\pi\right)=-\tan\left(\frac{5\pi}{6}\right)-\cot\left(\frac{3\pi}{2}\right)$

$$=-\frac{\sqrt{3}}{3}-0=\frac{\sqrt{3}}{3}$$

49. $\sec(-\pi)+\csc\left(-\frac{\pi}{2}\right)=\sec(\pi)-\csc\left(\frac{\pi}{2}\right)=-1-1=-2$

50. $\tan(-6\pi)+\cos\left(\frac{9\pi}{4}\right)=-\tan(0+6\pi)+\cos\left(\frac{\pi}{4}+2\pi\right)=-\tan(0)+\cos\left(\frac{\pi}{4}\right)=0+\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}$

51. $\sin\left(-\frac{9\pi}{4}\right)-\tan\left(-\frac{9\pi}{4}\right)=-\sin\left(\frac{9\pi}{4}\right)+\tan\left(\frac{9\pi}{4}\right)=-\sin\left(\frac{\pi}{4}+\frac{8\pi}{4}\right)+\tan\left(\frac{\pi}{4}+\frac{8\pi}{4}\right)$

$$=-\sin\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}+1$$

52. $\cos\left(-\frac{17\pi}{4}\right)-\sin\left(-\frac{3\pi}{2}\right)=\cos\left(\frac{17\pi}{4}\right)+\sin\left(\frac{3\pi}{2}\right)=\cos\left(\frac{\pi}{4}+2\cdot 2\pi\right)+\sin\left(\frac{3\pi}{2}\right)$

$$=\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{3\pi}{2}\right)=\frac{\sqrt{2}}{2}+(-1)=\frac{\sqrt{2}}{2}-1$$

53. The domain of the sine function is the set of all real numbers.

54. The domain of the cosine function is the set of all real numbers.

55. $f(\theta)=\tan\theta$ is not defined for numbers that are odd multiples of $\frac{\pi}{2}$.

56. $f(\theta)=\cot\theta$ is not defined for numbers that are multiples of π.

57. $f(\theta) = \sec\theta$ is not defined for numbers that are odd multiples of $\frac{\pi}{2}$.

58. $f(\theta) = \csc\theta$ is not defined for numbers that are multiples of π.

59. The range of the sine function is the set of all real numbers between –1 and 1, inclusive.

60. The range of the cosine function is the set of all real numbers between –1 and 1, inclusive.

61. The range of the tangent function is the set of all real numbers.

62. The range of the cotangent function is the set of all real numbers.

63. The range of the secant function is the set of all real number greater than or equal to 1 and all real numbers less than or equal to –1.

64. The range of the cosecant function is the set of all real number greater than or equal to 1 and all real numbers less than or equal to –1.

65. The sine function is odd because $\sin(-\theta) = -\sin\theta$. Its graph is symmetric to the origin.

66. The cosine function is even because $\cos(-\theta) = \cos\theta$. Its graph is symmetric to the y-axis.

67. The tangent function is odd because $\tan(-\theta) = -\tan\theta$. Its graph is symmetric to the origin.

68. The cotangent function is odd because $\cot(-\theta) = -\cot\theta$. Its graph is symmetric to the origin.

69. The secant function is even because $\sec(-\theta) = \sec\theta$. Its graph is symmetric to the y-axis.

70. The cosecant function is odd because $\csc(-\theta) = -\csc\theta$. Its graph is symmetric to the origin.

71. If $\sin\theta = 0.3$, then $\sin(\theta + \pi) = -0.3$

72. If $\cos\theta = 0.2$, then $\cos(\theta + \pi) = -0.2$

73. If $\tan\theta = 3$, then $\tan(\theta + \pi) = 3$

74. If $\cot\theta = -2$, then $\cot(\theta + \pi) = -2$

75. (a) $f(-a) = -f(a) = -\frac{1}{3}$

(b) $f(a) + f(a+2\pi) + f(a+4\pi) = f(a) + f(a) + f(a) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

76. (a) $f(-a) = f(a) = \frac{1}{4}$

(b) $f(a) + f(a+2\pi) + f(a-2\pi) = f(a) + f(a) + f(a) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$

77. (a) $f(-a) = -f(a) = -2$

(b) $f(a) + f(a+\pi) + f(a+2\pi) = f(a) + f(a) + f(a) = 2+2+2 = 6$

78. (a) $f(-a) = -f(a) = -(-3) = 3$

(b) $f(a) + f(a+\pi) + f(a+4\pi) = f(a) + f(a) + f(a) = -3 + (-3) + (-3) = -9$

79. (a) $f(-a) = f(a) = -4$

(b) $f(a) + f(a+2\pi) + f(a+4\pi) = f(a) + f(a) + f(a) = (-4) + (-4) + (-4) = -12$

80. (a) $f(-a) = -f(a) = -2$

(b) $f(a) + f(a+2\pi) + f(a+4\pi) = f(a) + f(a) + f(a) = 2+2+2 = 6$

81. Let $P = (x, y)$ be the point on the unit circle that corresponds to an angle θ. Consider the equation $\tan\theta = \frac{y}{x} = a$. Then $y = ax$. Now $x^2 + y^2 = 1$, so $x^2 + a^2x^2 = 1$. Thus, $x = \pm\frac{1}{\sqrt{1+a^2}}$ and $y = \pm\frac{a}{\sqrt{1+a^2}}$; that is, for any real number a, there is a point $P = (x, y)$ on the unit circle for which $\tan\theta = a$. In other words, $-\infty < \tan\theta < +\infty$, and the range of the tangent function is the set of all real numbers.

82. Let $P = (x, y)$ be the point on the unit circle that corresponds to an angle θ. Consider the equation $\cot\theta = \frac{x}{y} = a$. Then $x = ay$. Now $x^2 + y^2 = 1$, so $a^2y^2 + y^2 = 1$. Thus, $y = \pm\frac{1}{\sqrt{1+a^2}}$ and $x = \pm\frac{a}{\sqrt{1+a^2}}$; that is, for any real number a, there is a point $P = (x, y)$ on the unit circle for which $\cot\theta = a$. In other words, $-\infty < \cot\theta < +\infty$, and the range of the tangent function is the set of all real numbers.

83. Suppose there is a number p, $0 < p < 2\pi$, for which $\sin(\theta + p) = \sin\theta$ for all θ. If $\theta = 0$, then $\sin(0 + p) = \sin p = \sin 0 = 0$; so that $p = \pi$. If $\theta = \frac{\pi}{2}$, then $\sin\left(\frac{\pi}{2} + p\right) = \sin\left(\frac{\pi}{2}\right)$. But $p = \pi$. Thus, $\sin\left(\frac{3\pi}{2}\right) = -1 = \sin\left(\frac{\pi}{2}\right) = 1$, or $-1 = 1$. This is impossible. The smallest positive number p for which $\sin(\theta + p) = \sin\theta$ for all θ is therefore $p = 2\pi$.

84. Suppose there is a number p, $0 < p < 2\pi$, for which $\cos(\theta + p) = \cos\theta$ for all θ. If $\theta = \frac{\pi}{2}$, then $\cos\left(\frac{\pi}{2} + p\right) = \cos\left(\frac{\pi}{2}\right) = 0$; so that $p = \pi$. If $\theta = 0$, then $\cos(0 + p) = \cos 0$. But $p = \pi$. Thus, $\cos(\pi) = -1 = \cos(0) = 1$, or $-1 = 1$. This is impossible. The smallest positive number p for which $\cos(\theta + p) = \cos\theta$ for all θ is therefore $p = 2\pi$.

85. $\sec\theta = \dfrac{1}{\cos\theta}$: since $\cos\theta$ has period 2π, so does $\sec\theta$.

86. $\csc\theta = \dfrac{1}{\sin\theta}$: since $\sin\theta$ has period 2π, so does $\csc\theta$.

87. If $P = (a,b)$ is the point on the unit circle corresponding to θ, then $Q = (-a,-b)$ is the point on the unit circle corresponding to $\theta + \pi$.
Thus, $\tan(\theta + \pi) = \dfrac{-b}{-a} = \dfrac{b}{a} = \tan\theta$; that is, the period of the tangent function is π.

88. If $P = (a,b)$ is the point on the unit circle corresponding to θ, then $Q = (-a,-b)$ is the point on the unit circle corresponding to $\theta + \pi$.
Thus, $\cot(\theta + \pi) = \dfrac{-a}{-b} = \dfrac{a}{b} = \cot\theta$; that is, the period of the cotangent function is π.

89. Slope of $L^* = \dfrac{\sin\theta - 0}{\cos\theta - 0} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$
Since L is parallel to L^*, then the slope of $L = \tan\theta$.