# Analytic Trigonometry

## 8.2 The Inverse Trigonometric Functions (Continued)

1. $\cos\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

   Find the angle $\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $\frac{\sqrt{2}}{2}$.

   $$\sin\theta = \frac{\sqrt{2}}{2} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

   $$\theta = \frac{\pi}{4} \rightarrow \cos\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

2. $\sin\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$

   Find the angle $\theta$, $0 \le \theta \le \pi$, whose cosine equals $\frac{1}{2}$.

   $$\cos\theta = \frac{1}{2} \qquad 0 \le \theta \le \pi$$

   $$\theta = \frac{\pi}{3} \rightarrow \sin\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

3. $\tan\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

   Find the angle $\theta$, $0 \le \theta \le \pi$, whose cosine equals $-\frac{\sqrt{3}}{2}$.

   $$\cos\theta = -\frac{\sqrt{3}}{2} \qquad 0 \le \theta \le \pi$$

   $$\theta = \frac{5\pi}{6} \rightarrow \tan\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \tan\frac{5\pi}{6} = -\frac{\sqrt{3}}{3}$$

4. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

Find the angle $\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

$$\sin\theta = -\frac{1}{2} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{6} \rightarrow \tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$$

5. $\sec\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$

Find the angle $\theta,\ 0 \le \theta \le \pi$, whose cosine equals $\frac{1}{2}$.

$$\cos\theta = \frac{1}{2} \qquad 0 \le \theta \le \pi$$

$$\theta = \frac{\pi}{3} \rightarrow \sec\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = \sec\left(\frac{\pi}{3}\right) = 2$$

6. $\cot\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

Find the angle $\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

$$\sin\theta = -\frac{1}{2} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{6} \rightarrow \cot\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \cot\left(-\frac{\pi}{6}\right) = -\sqrt{3}$$

7. $\csc\left(\tan^{-1}(1)\right)$

Find the angle $\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals 1.

$$\tan\theta = 1 \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\theta = \frac{\pi}{4} \rightarrow \csc\left(\tan^{-1}(1)\right) = \csc\left(\frac{\pi}{4}\right) = \sqrt{2}$$

8. $\sec\left(\tan^{-1}\left(\sqrt{3}\right)\right)$

Find the angle $\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $\sqrt{3}$.

$$\tan\theta = \sqrt{3} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\theta = \frac{\pi}{3} \rightarrow \sec\left(\tan^{-1}\left(\sqrt{3}\right)\right) = \sec\left(\frac{\pi}{3}\right) = 2$$

9. $\sin\left(\tan^{-1}(-1)\right)$

Find the angle $\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals –1.

$$\tan\theta = -1 \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{4} \rightarrow \sin\left(\tan^{-1}(-1)\right) = \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

10. $\cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

Find the angle $\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{\sqrt{3}}{2}$.

$$\sin\theta = -\frac{\sqrt{3}}{2} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{3} \rightarrow \cos\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$$

11. $\sec\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

Find the angle $\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

$$\sin\theta = -\frac{1}{2} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{6} \rightarrow \sec\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \sec\left(-\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}$$

12. $\csc\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

Find the angle $\theta$, $0 \le \theta \le \pi$, whose cosine equals $-\frac{\sqrt{3}}{2}$.

$$\cos\theta = -\frac{\sqrt{3}}{2} \qquad 0 \le \theta \le \pi$$

$$\theta = \frac{5\pi}{6} \rightarrow \csc\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \csc\left(\frac{5\pi}{6}\right) = 2$$

13. $\cos^{-1}\left(\cos\frac{5\pi}{4}\right) = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Find the angle $\theta$, $0 \le \theta \le \pi$, whose cosine equals $-\frac{\sqrt{2}}{2}$.

$$\cos\theta = -\frac{\sqrt{2}}{2} \qquad 0 \le \theta \le \pi$$

$$\theta = \frac{3\pi}{4}$$

14. $\tan^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\right) = \tan^{-1}\left(-\sqrt{3}\right)$

Find the angle $\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $-\sqrt{3}$.

$$\tan\theta = -\sqrt{3} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{3}$$

15. $\sin^{-1}\left(\sin\left(-\frac{7\pi}{6}\right)\right) = \sin^{-1}\left(\frac{1}{2}\right)$

Find the angle $\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $\frac{1}{2}$.

$$\sin\theta = \frac{1}{2} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\theta = \frac{\pi}{6}$$

16. $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \cos^{-1}\left(\frac{1}{2}\right)$

Find the angle $\theta$, $0 \le \theta \le \pi$, whose cosine equals $\frac{1}{2}$.

$$\cos\theta = \frac{1}{2} \qquad 0 \le \theta \le \pi$$

$$\theta = \frac{\pi}{3}$$

17. $\tan\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$

Since $\sin\theta = \frac{1}{3}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, let $y = 1$ and $r = 3$. Solve for x:

$$x^2 + 1 = 9 \rightarrow x^2 = 8 \rightarrow x = \pm\sqrt{8} = \pm 2\sqrt{2}$$

Since $\theta$ is in quadrant I, $x = 2\sqrt{2}$.

$$\tan\left(\sin^{-1}\left(\frac{1}{3}\right)\right) = \tan\theta = \frac{y}{x} = \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4}$$

18. $\tan\left(\cos^{-1}\left(\frac{1}{3}\right)\right)$

Since $\cos\theta = \frac{1}{3}$, $0 \le \theta \le \pi$, let $x = 1$ and $r = 3$. Solve for y:

$$1 + y^2 = 9 \rightarrow y^2 = 8 \rightarrow y = \pm\sqrt{8} = \pm 2\sqrt{2}$$

Since $\theta$ is in quadrant I, $y = 2\sqrt{2}$.

$$\tan\left(\cos^{-1}\left(\frac{1}{3}\right)\right) = \tan\theta = \frac{y}{x} = \frac{2\sqrt{2}}{1} = 2\sqrt{2}$$

19. $\sec\left(\tan^{-1}\left(\frac{1}{2}\right)\right)$

Since $\tan\theta = \frac{1}{2}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, let $x = 2$ and $y = 1$. Solve for r:

$$2^2 + 1 = r^2 \rightarrow r^2 = 5 \rightarrow r = \sqrt{5}$$

$\theta$ is in quadrant I.

$$\sec\left(\tan^{-1}\left(\frac{1}{2}\right)\right) = \sec\theta = \frac{r}{x} = \frac{\sqrt{5}}{2}$$

20. $\cos\left(\sin^{-1}\left(\frac{\sqrt{2}}{3}\right)\right)$

Since $\sin\theta = \frac{\sqrt{2}}{3}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, let $y = \sqrt{2}$ and $r = 3$. Solve for x:

$$x^2 + 2 = 9 \rightarrow x^2 = 7 \rightarrow x = \pm\sqrt{7}$$

Since $\theta$ is in quadrant I, $x = \sqrt{7}$.

$$\cos\left(\sin^{-1}\left(\frac{\sqrt{2}}{3}\right)\right) = \cos\theta = \frac{x}{r} = \frac{\sqrt{7}}{3}$$

21. $\cot\left(\sin^{-1}\left(-\frac{\sqrt{2}}{3}\right)\right)$

Since $\sin\theta = -\frac{\sqrt{2}}{3}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, let $y = -\sqrt{2}$ and $r = 3$. Solve for x:

$$x^2 + 2 = 9 \rightarrow x^2 = 7 \rightarrow x = \pm\sqrt{7}$$

Since $\theta$ is in quadrant IV, $x = \sqrt{7}$.

$$\cot\left(\sin^{-1}\left(-\frac{\sqrt{2}}{3}\right)\right) = \cot\theta = \frac{x}{y} = \frac{\sqrt{7}}{-\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{14}}{2}$$

22. $\csc\left(\tan^{-1}(-2)\right)$

Since $\tan\theta = -2$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, let $x = 1$ and $y = -2$. Solve for r:

$$1 + 4 = r^2 \quad \rightarrow \quad r^2 = 5 \quad \rightarrow \quad r = \sqrt{5}$$

$\theta$ is in quadrant IV.

$$\csc\left(\tan^{-1}(-2)\right) = \csc\theta = \frac{r}{y} = \frac{\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2}$$

23. $\sin\left(\tan^{-1}(-3)\right)$

Since $\tan\theta = -3$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, let $x = 1$ and $y = -3$. Solve for r:

$$1 + 9 = r^2 \quad \rightarrow \quad r^2 = 10 \quad \rightarrow \quad r = \sqrt{10}$$

$\theta$ is in quadrant IV.

$$\sin\left(\tan^{-1}(-3)\right) = \sin\theta = \frac{y}{r} = \frac{-3}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$$

24. $\cot\left(\cos^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right)$

Since $\cos\theta = -\frac{\sqrt{3}}{3}$, $0 \le \theta \le \pi$, let $x = -\sqrt{3}$ and $r = 3$. Solve for y:

$$3 + y^2 = 9 \rightarrow y^2 = 6 \rightarrow y = \pm\sqrt{6}$$

Since $\theta$ is in quadrant II, $y = \sqrt{6}$.

$$\cot\left(\cos^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right) = \cot\theta = \frac{x}{y} = \frac{-\sqrt{3}}{\sqrt{6}} = \frac{-1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

25. $\sec\left(\sin^{-1}\left(\frac{2\sqrt{5}}{5}\right)\right)$

Since $\sin\theta = \frac{2\sqrt{5}}{5}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, let $y = 2\sqrt{5}$ and $r = 5$. Solve for x:

$$x^2 + 20 = 25 \rightarrow x^2 = 5 \rightarrow x = \pm\sqrt{5}$$

Since $\theta$ is in quadrant I, $x = \sqrt{5}$.

$$\sec\left(\sin^{-1}\left(\frac{2\sqrt{5}}{5}\right)\right) = \sec\theta = \frac{r}{x} = \frac{5}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \sqrt{5}$$

26. $\csc\left(\tan^{-1}\left(\frac{1}{2}\right)\right)$

Since $\tan\theta = \frac{1}{2}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, let $x = 2$ and $y = 1$. Solve for r:

$$2^2 + 1 = r^2 \quad \rightarrow \quad r^2 = 5 \quad \rightarrow \quad r = \sqrt{5}$$

$\theta$ is in quadrant I.

$$\csc\left(\tan^{-1}\left(\frac{1}{2}\right)\right) = \csc\theta = \frac{r}{y} = \frac{\sqrt{5}}{1} = \sqrt{5}$$

27. $\sin^{-1}\left(\cos\frac{3\pi}{4}\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

28. $\cos^{-1}\left(\sin\frac{7\pi}{6}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

29. $\cot^{-1}\left(\sqrt{3}\right)$

We are finding the angle $\theta$, $0 < \theta < \pi$, whose cotangent equals $\sqrt{3}$.

$\cot\theta = \sqrt{3} \quad 0 < \theta < \pi$

$\theta = \frac{\pi}{6} \rightarrow \cot^{-1}\left(\sqrt{3}\right) = \frac{\pi}{6}$

30. $\cot^{-1}(1)$

We are finding the angle $\theta$, $0 < \theta < \pi$, whose cotangent equals 1.

$\cot\theta = 1 \quad 0 < \theta < \pi$

$\theta = \frac{\pi}{4} \rightarrow \cot^{-1}(1) = \frac{\pi}{4}$

31. $\csc^{-1}(-1)$

We are finding the angle $\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta \ne 0$, whose cosecant equals –1.

$\csc\theta = -1 \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ \theta \ne 0$

$\theta = -\frac{\pi}{2} \rightarrow \csc^{-1}(-1) = -\frac{\pi}{2}$

32. $\csc^{-1}\left(\sqrt{2}\right)$

We are finding the angle $\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta \ne 0$, whose cosecant equals $\sqrt{2}$.

$\csc\theta = \sqrt{2} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ \theta \ne 0$

$\theta = \frac{\pi}{4} \rightarrow \csc^{-1}\left(\sqrt{2}\right) = \frac{\pi}{4}$

33. $\sec^{-1}\left(\frac{2\sqrt{3}}{3}\right)$

We are finding the angle $\theta$, $0 \le \theta \le \pi$, $\theta \ne \frac{\pi}{2}$, whose secant equals $\frac{2\sqrt{3}}{3}$.

$\sec\theta = \frac{2\sqrt{3}}{3} \quad 0 \le \theta \le \pi,\ \theta \ne \frac{\pi}{2}$

$\theta = \frac{\pi}{6} \rightarrow \sec^{-1}\left(\frac{2\sqrt{3}}{3}\right) = \frac{\pi}{6}$

34. $\sec^{-1}(-2)$

We are finding the angle $\theta$, $0 \le \theta \le \pi$, $\theta \ne \frac{\pi}{2}$, whose secant equals –2.

$$\sec\theta = -2 \qquad 0 \le \theta \le \pi,\ \theta \ne \frac{\pi}{2}$$

$$\theta = \frac{2\pi}{3} \rightarrow \sec^{-1}(-2) = \frac{2\pi}{3}$$

35. $\cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

We are finding the angle $\theta$, $0 < \theta < \pi$, whose cotangent equals $-\frac{\sqrt{3}}{3}$.

$$\cot\theta = -\frac{\sqrt{3}}{3} \qquad 0 < \theta < \pi$$

$$\theta = \frac{2\pi}{3} \rightarrow \cot^{-1}\left(-\frac{\sqrt{3}}{3}\right) = \frac{2\pi}{3}$$

36. $\csc^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$

We are finding the angle $\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta \ne 0$, whose cosecant equals $-\frac{2\sqrt{3}}{3}$.

$$\csc\theta = -\frac{2\sqrt{3}}{3} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ \theta \ne 0$$

$$\theta = -\frac{\pi}{3} \rightarrow \csc^{-1}\left(-\frac{2\sqrt{3}}{3}\right) = -\frac{\pi}{3}$$

37. $\sec^{-1}(4) = \cos^{-1}\left(\frac{1}{4}\right) \approx 1.32$

38. $\csc^{-1}(5) = \sin^{-1}\left(\frac{1}{5}\right) \approx 0.20$

39. $\cot^{-1}(2) = \tan^{-1}\left(\frac{1}{2}\right) \approx 0.46$

40. $\sec^{-1}(-3) = \cos^{-1}\left(-\frac{1}{3}\right) \approx 1.91$

41. $\csc^{-1}(-3) = \sin^{-1}\left(-\frac{1}{3}\right) \approx -0.34$

42. $\cot^{-1}\left(-\frac{1}{2}\right) = \tan^{-1}(-2) \approx -1.11$

43. $\cot^{-1}\left(-\sqrt{5}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{5}}\right) \approx -0.42$

44. $\cot^{-1}(-8.1) = \tan^{-1}\left(-\frac{1}{8.1}\right) \approx -0.12$

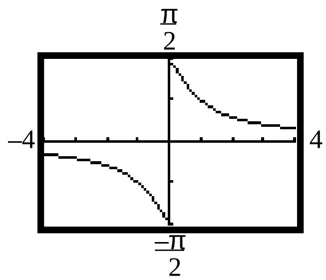
45. $\csc^{-1}\left(-\frac{3}{2}\right) = \sin^{-1}\left(-\frac{2}{3}\right) \approx -0.73$

46. $\sec^{-1}\left(-\frac{4}{3}\right) = \cos^{-1}\left(-\frac{3}{4}\right) \approx 2.42$

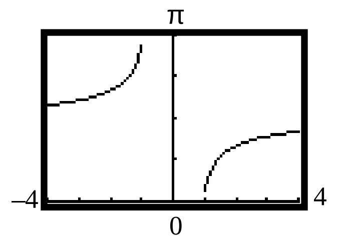
47. $\cot^{-1}\left(-\frac{3}{2}\right) = \tan^{-1}\left(-\frac{2}{3}\right) \approx -0.59$

48. $\cot^{-1}\left(-\sqrt{10}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{10}}\right) \approx -0.31$

49. $y = \cot^{-1} x$




50. $y = \sec^{-1} x$




51. $y = \csc^{-1} x$