Analytic Trigonometry

8.3 Trigonometric Identities

1. $\csc\theta\ \cos\theta = \frac{1}{\sin\theta}\ \cos\theta = \frac{\cos\theta}{\sin\theta} = \cot\theta$

2. $\sec\theta\ \sin\theta = \frac{1}{\cos\theta}\ \sin\theta = \frac{\sin\theta}{\cos\theta} = \tan\theta$

3. $1 + \tan^2(-\theta) = 1 + (-\tan\theta)^2 = 1 + \tan^2\theta = \sec^2\theta$

4. $1 + \cot^2(-\theta) = 1 + (-\cot\theta)^2 = 1 + \cot^2\theta = \csc^2\theta$

5. $\cos\theta(\tan\theta + \cot\theta) = \cos\theta\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right) = \cos\theta\left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right) = \frac{1}{\sin\theta} = \csc\theta$

6. $\sin\theta(\cot\theta + \tan\theta) = \sin\theta\left(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\right) = \sin\theta\left(\frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}\right) = \frac{1}{\cos\theta} = \sec\theta$

7. $\tan\theta\cot\theta - \cos^2\theta = \tan\theta\ \frac{1}{\tan\theta} - \cos^2\theta = 1 - \cos^2\theta = \sin^2\theta$

8. $\sin\theta\csc\theta - \cos^2\theta = \sin\theta\ \frac{1}{\sin\theta} - \cos^2\theta = 1 - \cos^2\theta = \sin^2\theta$

9. $(\sec\theta - 1)(\sec\theta + 1) = \sec^2\theta - 1 = \tan^2\theta$

10. $(\csc\theta - 1)(\csc\theta + 1) = \csc^2\theta - 1 = \cot^2\theta$

11. $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = \sec^2\theta - \tan^2\theta = 1$

12. $(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = \csc^2\theta - \cot^2\theta = 1$

13. $\cos^2\theta(1 + \tan^2\theta) = \cos^2\theta\ \sec^2\theta = \cos^2\theta\ \frac{1}{\cos^2\theta} = 1$

14. $(1 - \cos^2\theta)(1 + \cot^2\theta) = \sin^2\theta\ \csc^2\theta = \sin^2\theta\ \frac{1}{\sin^2\theta} = 1$

15. $(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2$
$=\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta+\sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta$
$=2\sin^2\theta+2\cos^2\theta=2(\sin^2\theta+\cos^2\theta)=2\ 1=2$

16. $\tan^2\theta\cos^2\theta+\cot^2\theta\sin^2\theta=\dfrac{\sin^2\theta}{\cos^2\theta}\ \cos^2\theta+\dfrac{\cos^2\theta}{\sin^2\theta}\ \sin^2\theta=\sin^2\theta+\cos^2\theta=1$

17. $\sec^4\theta-\sec^2\theta=\sec^2\theta(\sec^2\theta-1)=(\tan^2\theta+1)\tan^2\theta=\tan^4\theta+\tan^2\theta$

18. $\csc^4\theta-\csc^2\theta=\csc^2\theta(\csc^2\theta-1)=(\cot^2\theta+1)\cot^2\theta=\cot^4\theta+\cot^2\theta$

19. $\sec\theta-\tan\theta=\dfrac{1}{\cos\theta}-\dfrac{\sin\theta}{\cos\theta}=\left(\dfrac{1-\sin\theta}{\cos\theta}\right)\left(\dfrac{1+\sin\theta}{1+\sin\theta}\right)=\dfrac{1-\sin^2\theta}{\cos\theta(1+\sin\theta)}$
$=\dfrac{\cos^2\theta}{\cos\theta(1+\sin\theta)}=\dfrac{\cos\theta}{1+\sin\theta}$

20. $\csc\theta-\cot\theta=\dfrac{1}{\sin\theta}-\dfrac{\cos\theta}{\sin\theta}=\left(\dfrac{1-\cos\theta}{\sin\theta}\right)\left(\dfrac{1+\cos\theta}{1+\cos\theta}\right)=\dfrac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)}$
$=\dfrac{\sin^2\theta}{\sin\theta(1+\cos\theta)}=\dfrac{\sin\theta}{1+\cos\theta}$

21. $3\sin^2\theta+4\cos^2\theta=3\sin^2\theta+3\cos^2\theta+\cos^2\theta=3(\sin^2\theta+\cos^2\theta)+\cos^2\theta$
$=3\ 1+\cos^2\theta=3+\cos^2\theta$

22. $9\sec^2\theta-5\tan^2\theta=4\sec^2\theta+5\sec^2\theta-5\tan^2\theta=4\sec^2\theta+5(\sec^2\theta-\tan^2\theta)$
$=4\sec^2\theta+5\ 1=5+4\sec^2\theta$

23. $1-\dfrac{\cos^2\theta}{1+\sin\theta}=1-\dfrac{1-\sin^2\theta}{1+\sin\theta}=1-\dfrac{(1-\sin\theta)(1+\sin\theta)}{1+\sin\theta}=1-1+\sin\theta=\sin\theta$

24. $1-\dfrac{\sin^2\theta}{1-\cos\theta}=1-\dfrac{1-\cos^2\theta}{1-\cos\theta}=1-\dfrac{(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta}=1-1-\cos\theta=-\cos\theta$

25. $\dfrac{1+\tan\theta}{1-\tan\theta}=\dfrac{\left(1+\dfrac{1}{\cot\theta}\right)}{\left(1-\dfrac{1}{\cot\theta}\right)}=\dfrac{\left(\dfrac{\cot\theta+1}{\cot\theta}\right)}{\left(\dfrac{\cot\theta-1}{\cot\theta}\right)}=\dfrac{\cot\theta+1}{\cot\theta}\ \dfrac{\cot\theta}{\cot\theta-1}=\dfrac{\cot\theta+1}{\cot\theta-1}$

26. $\dfrac{\csc\theta-1}{\csc\theta+1}=\dfrac{\left(\dfrac{1}{\sin\theta}-1\right)}{\left(\dfrac{1}{\sin\theta}+1\right)}=\dfrac{\left(\dfrac{1-\sin\theta}{\sin\theta}\right)}{\left(\dfrac{1+\sin\theta}{\sin\theta}\right)}=\dfrac{1-\sin\theta}{\sin\theta}\ \dfrac{\sin\theta}{1+\sin\theta}=\dfrac{1-\sin\theta}{1+\sin\theta}$

27. $\dfrac{\sec\theta}{\csc\theta}+\dfrac{\sin\theta}{\cos\theta}=\dfrac{\left(\frac{1}{\cos\theta}\right)}{\left(\frac{1}{\sin\theta}\right)}+\dfrac{\sin\theta}{\cos\theta}=\dfrac{\sin\theta}{\cos\theta}+\dfrac{\sin\theta}{\cos\theta}=\tan\theta+\tan\theta=2\tan\theta$

28. $\dfrac{\csc\theta-1}{\cot\theta}=\dfrac{\csc\theta-1}{\cot\theta}\ \dfrac{\csc\theta+1}{\csc\theta+1}=\dfrac{\csc^2\theta-1}{\cot\theta(\csc\theta+1)}=\dfrac{\cot^2\theta}{\cot\theta(\csc\theta+1)}=\dfrac{\cot\theta}{\csc\theta+1}$

29. $\dfrac{1+\sin\theta}{1-\sin\theta}=\dfrac{\left(1+\frac{1}{\csc\theta}\right)}{\left(1-\frac{1}{\csc\theta}\right)}=\dfrac{\left(\frac{\csc\theta+1}{\csc\theta}\right)}{\left(\frac{\csc\theta-1}{\csc\theta}\right)}=\dfrac{\csc\theta+1}{\csc\theta}\ \dfrac{\csc\theta}{\csc\theta-1}=\dfrac{\csc\theta+1}{\csc\theta-1}$

30. $\dfrac{\cos\theta+1}{\cos\theta-1}=\dfrac{(\cos\theta+1)\left(\frac{1}{\cos\theta}\right)}{(\cos\theta-1)\left(\frac{1}{\cos\theta}\right)}=\dfrac{\left(1+\frac{1}{\cos\theta}\right)}{\left(1-\frac{1}{\cos\theta}\right)}=\dfrac{1+\sec\theta}{1-\sec\theta}$

31. $\dfrac{1-\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{1-\sin\theta}=\dfrac{(1-\sin\theta)^2+\cos^2\theta}{\cos\theta(1-\sin\theta)}=\dfrac{1-2\sin\theta+\sin^2\theta+\cos^2\theta}{\cos\theta(1-\sin\theta)}$
$=\dfrac{1-2\sin\theta+1}{\cos\theta(1-\sin\theta)}=\dfrac{2-2\sin\theta}{\cos\theta(1-\sin\theta)}=\dfrac{2(1-\sin\theta)}{\cos\theta(1-\sin\theta)}=\dfrac{2}{\cos\theta}=2\sec\theta$

32. $\dfrac{\cos\theta}{1+\sin\theta}+\dfrac{1+\sin\theta}{\cos\theta}=\dfrac{\cos^2\theta+(1+\sin\theta)^2}{\cos\theta(1+\sin\theta)}=\dfrac{\cos^2\theta+1+2\sin\theta+\sin^2\theta}{\cos\theta(1+\sin\theta)}$
$=\dfrac{1+2\sin\theta+1}{\cos\theta(1+\sin\theta)}=\dfrac{2+2\sin\theta}{\cos\theta(1+\sin\theta)}=\dfrac{2(1+\sin\theta)}{\cos\theta(1+\sin\theta)}=\dfrac{2}{\cos\theta}=2\sec\theta$

33. $\dfrac{\sin\theta}{\sin\theta-\cos\theta}=\dfrac{\sin\theta}{\sin\theta-\cos\theta}\ \dfrac{\left(\frac{1}{\sin\theta}\right)}{\left(\frac{1}{\sin\theta}\right)}=\dfrac{1}{\left(1-\frac{\cos\theta}{\sin\theta}\right)}=\dfrac{1}{1-\cot\theta}$

34. $1-\dfrac{\sin^2\theta}{1+\cos\theta}=\dfrac{1+\cos\theta-\sin^2\theta}{1+\cos\theta}=\dfrac{1+\cos\theta-(1-\cos^2\theta)}{1+\cos\theta}=\dfrac{\cos\theta+\cos^2\theta}{1+\cos\theta}$
$=\dfrac{\cos\theta(1+\cos\theta)}{1+\cos\theta}=\cos\theta$

35. $(\sec\theta-\tan\theta)^2=\sec^2\theta-2\sec\theta\tan\theta+\tan^2\theta=\dfrac{1}{\cos^2\theta}-2\ \dfrac{1}{\cos\theta}\ \dfrac{\sin\theta}{\cos\theta}+\dfrac{\sin^2\theta}{\cos^2\theta}$
$=\dfrac{1-2\sin\theta+\sin^2\theta}{\cos^2\theta}=\dfrac{(1-\sin\theta)(1-\sin\theta)}{1-\sin^2\theta}=\dfrac{(1-\sin\theta)(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}=\dfrac{1-\sin\theta}{1+\sin\theta}$

36. $(\csc\theta-\cot\theta)^2=\csc^2\theta-2\csc\theta\cot\theta+\cot^2\theta=\dfrac{1}{\sin^2\theta}-2\ \dfrac{1}{\sin\theta}\ \dfrac{\cos\theta}{\sin\theta}+\dfrac{\cos^2\theta}{\sin^2\theta}$

$$= \frac{1-2\cos\theta+\cos^2\theta}{\sin^2\theta} = \frac{(1-\cos\theta)(1-\cos\theta)}{1-\cos^2\theta} = \frac{(1-\cos\theta)(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)} = \frac{1-\cos\theta}{1+\cos\theta}$$

37. $$\frac{\cos\theta}{1-\tan\theta}+\frac{\sin\theta}{1-\cot\theta} = \frac{\cos\theta}{\left(1-\frac{\sin\theta}{\cos\theta}\right)}+\frac{\sin\theta}{\left(1-\frac{\cos\theta}{\sin\theta}\right)} = \frac{\cos\theta}{\left(\frac{\cos\theta-\sin\theta}{\cos\theta}\right)}+\frac{\sin\theta}{\left(\frac{\sin\theta-\cos\theta}{\sin\theta}\right)}$$
$$= \frac{\cos^2\theta}{\cos\theta-\sin\theta}+\frac{\sin^2\theta}{\sin\theta-\cos\theta} = \frac{\cos^2\theta-\sin^2\theta}{\cos\theta-\sin\theta}$$
$$= \frac{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}{\cos\theta-\sin\theta} = \cos\theta+\sin\theta = \sin\theta+\cos\theta$$

38. $$\frac{\cot\theta}{1-\tan\theta}+\frac{\tan\theta}{1-\cot\theta} = \frac{\left(\frac{\cos\theta}{\sin\theta}\right)}{\left(1-\frac{\sin\theta}{\cos\theta}\right)}+\frac{\left(\frac{\sin\theta}{\cos\theta}\right)}{\left(1-\frac{\cos\theta}{\sin\theta}\right)} = \frac{\left(\frac{\cos\theta}{\sin\theta}\right)}{\left(\frac{\cos\theta-\sin\theta}{\cos\theta}\right)}+\frac{\left(\frac{\sin\theta}{\cos\theta}\right)}{\left(\frac{\sin\theta-\cos\theta}{\sin\theta}\right)}$$
$$= \frac{\cos^2\theta}{\sin\theta(\cos\theta-\sin\theta)}+\frac{\sin^2\theta}{\cos\theta(\sin\theta-\cos\theta)}$$
$$= \frac{-\cos^2\theta\ \cos\theta+\sin^2\theta\ \sin\theta}{\sin\theta\cos\theta(\sin\theta-\cos\theta)} = \frac{\sin^3\theta-\cos^3\theta}{\sin\theta\cos\theta(\sin\theta-\cos\theta)}$$
$$= \frac{(\sin\theta-\cos\theta)(\sin^2\theta+\sin\theta\cos\theta+\cos^2\theta)}{\sin\theta\cos\theta(\sin\theta-\cos\theta)}$$
$$= \frac{\sin^2\theta+\sin\theta\cos\theta+\cos^2\theta}{\sin\theta\cos\theta} = \frac{\sin^2\theta}{\sin\theta\cos\theta}+\frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}+\frac{\cos^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{\sin\theta}{\cos\theta}+1+\frac{\cos\theta}{\sin\theta} = 1+\tan\theta+\cot\theta$$

39. $$\tan\theta+\frac{\cos\theta}{1+\sin\theta} = \frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{1+\sin\theta} = \frac{\sin\theta(1+\sin\theta)+\cos^2\theta}{\cos\theta(1+\sin\theta)}$$
$$= \frac{\sin\theta+\sin^2\theta+\cos^2\theta}{\cos\theta(1+\sin\theta)} = \frac{\sin\theta+1}{\cos\theta(1+\sin\theta)} = \frac{1}{\cos\theta} = \sec\theta$$

40. $$\frac{\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta} = \frac{(\sin\theta\cos\theta)\left(\frac{1}{\cos^2\theta}\right)}{(\cos^2\theta-\sin^2\theta)\left(\frac{1}{\cos^2\theta}\right)} = \frac{\left(\frac{\sin\theta}{\cos\theta}\right)}{\left(1-\frac{\sin^2\theta}{\cos^2\theta}\right)} = \frac{\tan\theta}{1-\tan^2\theta}$$

41. $$\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \frac{\tan\theta+(\sec\theta-1)}{\tan\theta-(\sec\theta-1)}\ \frac{\tan\theta+(\sec\theta-1)}{\tan\theta+(\sec\theta-1)}$$

$$= \frac{\tan^2\theta + 2\tan\theta(\sec\theta - 1) + \sec^2\theta - 2\sec\theta + 1}{\tan^2\theta - (\sec^2\theta - 2\sec\theta + 1)}$$
$$= \frac{\sec^2\theta - 1 + 2\tan\theta(\sec\theta - 1) + \sec^2\theta - 2\sec\theta + 1}{\sec^2\theta - 1 - \sec^2\theta + 2\sec\theta - 1}$$
$$= \frac{2\sec^2\theta - 2\sec\theta + 2\tan\theta(\sec\theta - 1)}{2\sec\theta - 2}$$
$$= \frac{2\sec\theta(\sec\theta - 1) + 2\tan\theta(\sec\theta - 1)}{2\sec\theta - 2}$$
$$= \frac{2(\sec\theta - 1)(\sec\theta + \tan\theta)}{2(\sec\theta - 1)} = \sec\theta + \tan\theta = \tan\theta + \sec\theta$$

42. $$\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \frac{(\sin\theta - \cos\theta) + 1}{(\sin\theta + \cos\theta) - 1} \cdot \frac{(\sin\theta + \cos\theta) + 1}{(\sin\theta + \cos\theta) + 1}$$
$$= \frac{\sin^2\theta - \cos^2\theta + \sin\theta + \cos\theta + \sin\theta - \cos\theta + 1}{(\sin\theta + \cos\theta)^2 - 1}$$
$$= \frac{\sin^2\theta - \cos^2\theta + 2\sin\theta + 1}{\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta - 1}$$
$$= \frac{\sin^2\theta - (1 - \sin^2\theta) + 2\sin\theta + 1}{2\sin\theta\cos\theta + 1 - 1}$$
$$= \frac{2\sin^2\theta + 2\sin\theta}{2\sin\theta\cos\theta} = \frac{2\sin\theta(\sin\theta + 1)}{2\sin\theta\cos\theta} = \frac{\sin\theta + 1}{\cos\theta}$$

43. $$\frac{\tan\theta - \cot\theta}{\tan\theta + \cot\theta} = \frac{\left(\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}\right)}{\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)} = \frac{\left(\frac{\sin^2\theta - \cos^2\theta}{\cos\theta\sin\theta}\right)}{\left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right)} = \frac{\sin^2\theta - \cos^2\theta}{1} = \sin^2\theta - \cos^2\theta$$

44. $$\frac{\sec\theta - \cos\theta}{\sec\theta + \cos\theta} = \frac{\left(\frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta}\right)}{\left(\frac{1}{\cos\theta} + \frac{\cos^2\theta}{\cos\theta}\right)} = \frac{\left(\frac{1 - \cos^2\theta}{\cos\theta}\right)}{\left(\frac{1 + \cos^2\theta}{\cos\theta}\right)} = \frac{\sin^2\theta}{1 + \cos^2\theta}$$

45. $$\frac{\tan\theta - \cot\theta}{\tan\theta + \cot\theta} + 1 = \frac{\left(\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}\right)}{\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)} + 1 = \frac{\left(\frac{\sin^2\theta - \cos^2\theta}{\cos\theta\sin\theta}\right)}{\left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right)} + 1 = \frac{\sin^2\theta - \cos^2\theta}{1} + 1$$
$$= \sin^2\theta - \cos^2\theta + 1 = \sin^2\theta + (1 - \cos^2\theta) = \sin^2\theta + \sin^2\theta = 2\sin^2\theta$$

46. $$\frac{\tan\theta - \cot\theta}{\tan\theta + \cot\theta} + 2\cos^2\theta = \frac{\left(\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}\right)}{\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)} + 2\cos^2\theta = \frac{\left(\frac{\sin^2\theta - \cos^2\theta}{\cos\theta\sin\theta}\right)}{\left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right)} + 2\cos^2\theta$$

$$= \frac{\sin^2\theta - \cos^2\theta}{1} + 2\cos^2\theta = \sin^2\theta + \cos^2\theta = 1$$

47. $$\frac{\sec\theta + \tan\theta}{\cot\theta + \cos\theta} = \frac{\left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)}{\left(\frac{\cos\theta}{\sin\theta} + \cos\theta\right)} = \frac{\left(\frac{1+\sin\theta}{\cos\theta}\right)}{\left(\frac{\cos\theta + \cos\theta\sin\theta}{\sin\theta}\right)} = \frac{1+\sin\theta}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta(1+\sin\theta)}$$
$$= \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\cos\theta} = \tan\theta\sec\theta$$

48. $$\frac{\sec\theta}{1+\sec\theta} = \frac{\left(\frac{1}{\cos\theta}\right)}{\left(1 + \frac{1}{\cos\theta}\right)} = \frac{\left(\frac{1}{\cos\theta}\right)}{\left(\frac{\cos\theta + 1}{\cos\theta}\right)} = \left(\frac{1}{1+\cos\theta}\right)\left(\frac{1-\cos\theta}{1-\cos\theta}\right) = \frac{1-\cos\theta}{1-\cos^2\theta} = \frac{1-\cos\theta}{\sin^2\theta}$$

49. $$\frac{1-\tan^2\theta}{1+\tan^2\theta} + 1 = \frac{1-\tan^2\theta + 1 + \tan^2\theta}{1+\tan^2\theta} = \frac{2}{\sec^2\theta} = 2 \cdot \frac{1}{\sec^2\theta} = 2\cos^2\theta$$

50. $$\frac{1-\cot^2\theta}{1+\cot^2\theta} + 2\cos^2\theta = \frac{1-\cot^2\theta}{\csc^2\theta} + 2\cos^2\theta = \frac{1}{\csc^2\theta} - \frac{\cot^2\theta}{\csc^2\theta} + 2\cos^2\theta$$
$$= \sin^2\theta - \frac{\left(\frac{\cos^2\theta}{\sin^2\theta}\right)}{\left(\frac{1}{\sin^2\theta}\right)} + 2\cos^2\theta = \sin^2\theta - \cos^2\theta + 2\cos^2\theta = \sin^2\theta + \cos^2\theta = 1$$

51. $$\frac{\sec\theta - \csc\theta}{\sec\theta\csc\theta} = \frac{\left(\frac{1}{\cos\theta} - \frac{1}{\sin\theta}\right)}{\left(\frac{1}{\cos\theta} \cdot \frac{1}{\sin\theta}\right)} = \frac{\left(\frac{\sin\theta - \cos\theta}{\cos\theta\sin\theta}\right)}{\left(\frac{1}{\cos\theta\sin\theta}\right)} = \sin\theta - \cos\theta$$

52. $$\frac{\sin^2\theta - \tan\theta}{\cos^2\theta - \cot\theta} = \frac{\left(\sin^2\theta - \frac{\sin\theta}{\cos\theta}\right)}{\left(\cos^2\theta - \frac{\cos\theta}{\sin\theta}\right)} = \frac{\left(\frac{\sin^2\theta\cos\theta - \sin\theta}{\cos\theta}\right)}{\left(\frac{\cos^2\theta\sin\theta - \cos\theta}{\sin\theta}\right)} = \frac{\left(\frac{\sin\theta(\sin\theta\cos\theta - 1)}{\cos\theta}\right)}{\left(\frac{\cos\theta(\cos\theta\sin\theta - 1)}{\sin\theta}\right)}$$
$$= \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$$

53. $$\sec\theta - \cos\theta - \sin\theta\tan\theta = \frac{1}{\cos\theta} - \cos\theta - \sin\theta \cdot \frac{\sin\theta}{\cos\theta} = \frac{1-\cos^2\theta - \sin^2\theta}{\cos\theta}$$
$$= \frac{\sin^2\theta - \sin^2\theta}{\cos\theta} = 0$$

54. $$\tan\theta + \cot\theta - \sec\theta\csc\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} - \frac{1}{\cos\theta} \cdot \frac{1}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta - 1}{\cos\theta\sin\theta}$$

$$= \frac{1-1}{\cos\theta\sin\theta} = 0$$

55. $$\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta} = \frac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)} = \frac{2}{1-\sin^2\theta} = \frac{2}{\cos^2\theta} = 2\sec^2\theta$$

56. $$\frac{1+\sin\theta}{1-\sin\theta} - \frac{1-\sin\theta}{1+\sin\theta} = \frac{(1+\sin\theta)^2 - (1-\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)}$$
$$= \frac{1+2\sin\theta+\sin^2\theta - (1-2\sin\theta+\sin^2\theta)}{1-\sin^2\theta}$$
$$= \frac{4\sin\theta}{\cos^2\theta} = 4\ \frac{\sin\theta}{\cos\theta}\ \frac{1}{\cos\theta} = 4\tan\theta\sec\theta$$

57. $$\frac{\sec\theta}{1-\sin\theta} = \left(\frac{\sec\theta}{1-\sin\theta}\right)\left(\frac{1+\sin\theta}{1+\sin\theta}\right) = \frac{\sec\theta(1+\sin\theta)}{1-\sin^2\theta} = \frac{\sec\theta(1+\sin\theta)}{\cos^2\theta}$$
$$= \frac{1}{\cos\theta}\ \frac{1+\sin\theta}{\cos^2\theta} = \frac{1+\sin\theta}{\cos^3\theta}$$

58. $$(\sec\theta-\tan\theta)^2 = \sec^2\theta - 2\sec\theta\tan\theta + \tan^2\theta = \frac{1}{\cos^2\theta} - 2\ \frac{1}{\cos\theta}\ \frac{\sin\theta}{\cos\theta} + \frac{\sin^2\theta}{\cos^2\theta}$$
$$= \frac{1-2\sin\theta+\sin^2\theta}{\cos^2\theta} = \frac{(1-\sin\theta)(1-\sin\theta)}{1-\sin^2\theta} = \frac{(1-\sin\theta)(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)} = \frac{1-\sin\theta}{1+\sin\theta}$$

59. $$\frac{(\sec\theta-\tan\theta)^2+1}{\csc\theta(\sec\theta-\tan\theta)} = \frac{\sec^2\theta - 2\sec\theta\tan\theta + \tan^2\theta + 1}{\csc\theta(\sec\theta-\tan\theta)} = \frac{2\sec^2\theta - 2\sec\theta\tan\theta}{\csc\theta(\sec\theta-\tan\theta)}$$
$$= \frac{2\sec\theta(\sec\theta-\tan\theta)}{\csc\theta(\sec\theta-\tan\theta)} = \frac{2\sec\theta}{\csc\theta} = \frac{\left(\frac{1}{\cos\theta}\right)^2}{\frac{1}{\sin\theta}} = 2\ \frac{1}{\cos\theta}\ \frac{\sin\theta}{1} = 2\tan\theta$$

60. $$\frac{\sec^2\theta - \tan^2\theta + \tan\theta}{\sec\theta} = \frac{1+\tan\theta}{\sec\theta} = \frac{1+\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} = \left(1+\frac{\sin\theta}{\cos\theta}\right)\cos\theta = \cos\theta + \sin\theta$$

61. $$\frac{\sin\theta+\cos\theta}{\cos\theta} - \frac{\sin\theta-\cos\theta}{\sin\theta} = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\cos\theta} - \frac{\sin\theta}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin\theta}{\cos\theta} + 1 - 1 + \frac{\cos\theta}{\sin\theta}$$
$$= \frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta} = \frac{1}{\cos\theta\sin\theta} = \sec\theta\csc\theta$$

62. $$\frac{\sin\theta+\cos\theta}{\sin\theta} - \frac{\cos\theta-\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sin\theta} + \frac{\cos\theta}{\sin\theta} - \frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = 1 + \frac{\cos\theta}{\sin\theta} - 1 + \frac{\sin\theta}{\cos\theta}$$
$$= \frac{\cos^2\theta+\sin^2\theta}{\cos\theta\sin\theta} = \frac{1}{\cos\theta\sin\theta} = \sec\theta\csc\theta$$

63. $$\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta} = \frac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{\sin\theta+\cos\theta} = 1-\sin\theta\cos\theta$$

64. $$\frac{\sin^3\theta+\cos^3\theta}{1-2\cos^2\theta}=\frac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{1-\cos^2\theta-\cos^2\theta}$$
$$=\frac{(\sin\theta+\cos\theta)(1-\sin\theta\cos\theta)}{\sin^2\theta-\cos^2\theta}=\frac{(\sin\theta+\cos\theta)(1-\sin\theta\cos\theta)}{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}$$
$$=\frac{1-\sin\theta\cos\theta}{(\sin\theta-\cos\theta)}\cdot\frac{\left(\frac{1}{\cos\theta}\right)}{\left(\frac{1}{\cos\theta}\right)}=\frac{\left(\frac{1}{\cos\theta}-\sin\theta\right)}{\left(\frac{\sin\theta}{\cos\theta}-1\right)}=\frac{\sec\theta-\sin\theta}{\tan\theta-1}$$

65. $$\frac{\cos^2\theta-\sin^2\theta}{1-\tan^2\theta}=\frac{\cos^2\theta-\sin^2\theta}{\left(1-\frac{\sin^2\theta}{\cos^2\theta}\right)}=\frac{\cos^2\theta-\sin^2\theta}{\left(\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}\right)}=\cos^2\theta$$

66. $$\frac{\cos\theta+\sin\theta-\sin^3\theta}{\sin\theta}=\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\sin\theta}-\frac{\sin^3\theta}{\sin\theta}=\cot\theta+1-\sin^2\theta=\cot\theta+\cos^2\theta$$

67. $$\frac{(2\cos^2\theta-1)^2}{\cos^4\theta-\sin^4\theta}=\frac{\left[2\cos^2\theta-(\sin^2\theta+\cos^2\theta)\right]^2}{(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)}$$
$$=\frac{(\cos^2\theta-\sin^2\theta)^2}{(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)}=\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}$$
$$=\cos^2\theta-\sin^2\theta=1-\sin^2\theta-\sin^2\theta=1-2\sin^2\theta$$

68. $$\frac{1-2\cos^2\theta}{\sin\theta\cos\theta}=\frac{1-\cos^2\theta-\cos^2\theta}{\sin\theta\cos\theta}=\frac{\sin^2\theta-\cos^2\theta}{\sin\theta\cos\theta}=\frac{\sin^2\theta}{\sin\theta\cos\theta}-\frac{\cos^2\theta}{\sin\theta\cos\theta}$$
$$=\frac{\sin\theta}{\cos\theta}-\frac{\cos\theta}{\sin\theta}=\tan\theta-\cot\theta$$

69. $$\frac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}=\frac{(1+\sin\theta)+\cos\theta}{(1+\sin\theta)-\cos\theta}\cdot\frac{(1+\sin\theta)+\cos\theta}{(1+\sin\theta)+\cos\theta}$$
$$=\frac{1+2\sin\theta+\sin^2\theta+2\cos\theta(1+\sin\theta)+\cos^2\theta}{1+2\sin\theta+\sin^2\theta-\cos^2\theta}$$
$$=\frac{1+2\sin\theta+\sin^2\theta+2\cos\theta(1+\sin\theta)+(1-\sin^2\theta)}{1+2\sin\theta+\sin^2\theta-(1-\sin^2\theta)}$$
$$=\frac{2+2\sin\theta+2\cos\theta(1+\sin\theta)}{2\sin\theta+2\sin^2\theta}=\frac{2(1+\sin\theta)+2\cos\theta(1+\sin\theta)}{2\sin\theta(1+\sin\theta)}$$
$$=\frac{2(1+\sin\theta)(1+\cos\theta)}{2\sin\theta(1+\sin\theta)}=\frac{1+\cos\theta}{\sin\theta}$$

70. $$\frac{1+\cos\theta+\sin\theta}{1+\cos\theta-\sin\theta}=\frac{(1+\cos\theta)+\sin\theta}{(1+\cos\theta)-\sin\theta}\cdot\frac{(1+\cos\theta)+\sin\theta}{(1+\cos\theta)+\sin\theta}$$

$$= \frac{1+2\cos\theta+\cos^2\theta+2\sin\theta(1+\cos\theta)+\sin^2\theta}{1+2\cos\theta+\cos^2\theta-\sin^2\theta}$$
$$= \frac{1+2\cos\theta+\cos^2\theta+2\sin\theta(1+\cos\theta)+\sin^2\theta}{1+2\cos\theta+\cos^2\theta-(1-\cos^2\theta)}$$
$$= \frac{2+2\cos\theta+2\sin\theta(1+\cos\theta)}{2\cos\theta+2\cos^2\theta} = \frac{2(1+\cos\theta)+2\sin\theta(1+\cos\theta)}{2\cos\theta(1+\cos\theta)}$$
$$= \frac{2(1+\cos\theta)(1+\sin\theta)}{2\cos\theta(1+\cos\theta)} = \frac{1+\sin\theta}{\cos\theta} = \frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta} = \sec\theta+\tan\theta$$

71. $(a\sin\theta+b\cos\theta)^2+(a\cos\theta-b\sin\theta)^2$
$= a^2\sin^2\theta+2ab\sin\theta\cos\theta+b^2\cos^2\theta+a^2\cos^2\theta-2ab\sin\theta\cos\theta+b^2\sin^2\theta$
$= a^2(\sin^2\theta+\cos^2\theta)+b^2(\sin^2\theta+\cos^2\theta) = a^2+b^2$

72. $(2a\sin\theta\cos\theta)^2+a^2(\cos^2\theta-\sin^2\theta)^2$
$= 4a^2\sin^2\theta\cos^2\theta+a^2\left(\cos^4\theta-2\cos^2\theta\sin^2\theta+\sin^4\theta\right)$
$= a^2\left(4\sin^2\theta\cos^2\theta+\cos^4\theta-2\cos^2\theta\sin^2\theta+\sin^4\theta\right)$
$= a^2\left(\cos^4\theta+2\cos^2\theta\sin^2\theta+\sin^4\theta\right)$
$= a^2\left(\cos^2\theta+\sin^2\theta\right)^2 = a^2(1)^2 = a^2$

73. $\dfrac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta} = \dfrac{\tan\alpha+\tan\beta}{\left(\frac{1}{\tan\alpha}+\frac{1}{\tan\beta}\right)} = \dfrac{\tan\alpha+\tan\beta}{\left(\frac{\tan\beta+\tan\alpha}{\tan\alpha\tan\beta}\right)}$
$= (\tan\alpha+\tan\beta)\left(\frac{\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}\right) = \tan\alpha\tan\beta$

74. $(\tan\alpha+\tan\beta)(1-\cot\alpha\cot\beta)+(\cot\alpha+\cot\beta)(1-\tan\alpha\tan\beta)$
$= \tan\alpha+\tan\beta-\tan\alpha\cot\alpha\cot\beta-\tan\beta\cot\alpha\cot\beta$
$\quad+\cot\alpha+\cot\beta-\cot\alpha\tan\alpha\tan\beta-\cot\beta\tan\alpha\tan\beta$
$= \tan\alpha+\tan\beta-\cot\beta-\cot\alpha+\cot\alpha+\cot\beta-\tan\beta-\tan\alpha = 0$

75. $(\sin\alpha+\cos\beta)^2+(\cos\beta+\sin\alpha)(\cos\beta-\sin\alpha)$
$= \sin^2\alpha+2\sin\alpha\cos\beta+\cos^2\beta+\cos^2\beta-\sin^2\alpha$
$= 2\sin\alpha\cos\beta+2\cos^2\beta = 2\cos\beta(\sin\alpha+\cos\beta)$

76. $(\sin\alpha-\cos\beta)^2+(\cos\beta+\sin\alpha)(\cos\beta-\sin\alpha)$
$= \sin^2\alpha-2\sin\alpha\cos\beta+\cos^2\beta+\cos^2\beta-\sin^2\alpha$
$= -2\sin\alpha\cos\beta+2\cos^2\beta = -2\cos\beta(\sin\alpha-\cos\beta)$

77. $\ln|\sec\theta| = \ln\left|\dfrac{1}{\cos\theta}\right| = \ln|\cos\theta|^{-1} = -\ln|\cos\theta|$

78. $\ln|\tan\theta| = \ln\left|\dfrac{\sin\theta}{\cos\theta}\right| = \ln|\sin\theta|-\ln|\cos\theta|$

79. $\ln|1+\cos\theta| + \ln|1-\cos\theta| = \ln\left(|1+\cos\theta|\,|1-\cos\theta|\right) = \ln|1-\cos^2\theta|$
$= \ln|\sin^2\theta| = 2\ln|\sin\theta|$

80. $\ln|\sec\theta+\tan\theta| + \ln|\sec\theta-\tan\theta| = \ln\left(|\sec\theta+\tan\theta|\,|\sec\theta-\tan\theta|\right)$
$= \ln|\sec^2\theta - \tan^2\theta| = \ln|1| = 0$

81. Show that $\sec\left(\tan^{-1}v\right) = \sqrt{1+v^2}$.
Let $\alpha = \tan^{-1}v$. Then $\tan\alpha = v$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.
$\sec\left(\tan^{-1}v\right) = \sec\alpha = \sqrt{1+\tan^2\alpha} = \sqrt{1+v^2}$

82. Show that $\tan\left(\sin^{-1}v\right) = \frac{v}{\sqrt{1-v^2}}$.
Let $\alpha = \sin^{-1}v$. Then $\sin\alpha = v$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$.
$\tan\left(\sin^{-1}v\right) = \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}} = \frac{v}{\sqrt{1-v^2}}$

83. Show that $\tan\left(\cos^{-1}v\right) = \frac{\sqrt{1-v^2}}{v}$.
Let $\alpha = \cos^{-1}v$. Then $\cos\alpha = v$, $0 \le \alpha \le \pi$.
$\tan\left(\cos^{-1}v\right) = \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha} = \frac{\sqrt{1-v^2}}{v}$

84. Show that $\sin\left(\cos^{-1}v\right) = \sqrt{1-v^2}$.
Let $\alpha = \cos^{-1}v$. Then $\cos\alpha = v$, $0 \le \alpha \le \pi$.
$\sin\left(\cos^{-1}v\right) = \sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-v^2}$

85. Show that $\cos\left(\sin^{-1}v\right) = \sqrt{1-v^2}$.
Let $\alpha = \sin^{-1}v$. Then $\sin\alpha = v$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$.
$\cos\left(\sin^{-1}v\right) = \cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-v^2}$

86. From Problem 81 we know that $\sec\left(\tan^{-1}v\right) = \sqrt{1+v^2}$.
Since $\sec\alpha = \frac{1}{\cos\alpha}$, then $\cos\left(\tan^{-1}v\right) = \frac{1}{\sqrt{1+v^2}}$.