Chapter 8

Analytic Trigonometry

8.4 Sum and Difference Formulas

1. $\sin\left(\frac{5\pi}{12}\right)=\sin\left(\frac{3\pi}{12}+\frac{2\pi}{12}\right)=\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}$
$=\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$

2. $\sin\left(\frac{\pi}{12}\right)=\sin\left(\frac{3\pi}{12}-\frac{2\pi}{12}\right)=\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)-\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}$
$=\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$

3. $\cos\left(\frac{7\pi}{12}\right)=\cos\left(\frac{4\pi}{12}+\frac{3\pi}{12}\right)=\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)=\frac{1}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$
$=\frac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$

4. $\tan\left(\frac{7\pi}{12}\right)=\tan\left(\frac{3\pi}{12}+\frac{4\pi}{12}\right)=\frac{\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{3}\right)}{1-\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{3}\right)}=\frac{1+\sqrt{3}}{1-1\cdot\sqrt{3}}=\left(\frac{1+\sqrt{3}}{1-\sqrt{3}}\right)\left(\frac{1+\sqrt{3}}{1+\sqrt{3}}\right)$
$=\frac{1+2\sqrt{3}+3}{1-3}=\frac{4+2\sqrt{3}}{-2}=-2-\sqrt{3}$

5. $\cos\left(165^\circ\right)=\cos\left(120^\circ+45^\circ\right)=\cos\left(120^\circ\right)\cos\left(45^\circ\right)-\sin\left(120^\circ\right)\sin\left(45^\circ\right)$
$=-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=-\frac{1}{4}\left(\sqrt{2}+\sqrt{6}\right)$

6. $\sin\left(105^\circ\right)=\sin\left(60^\circ+45^\circ\right)=\sin\left(60^\circ\right)\cos\left(45^\circ\right)+\cos\left(60^\circ\right)\sin\left(45^\circ\right)$
$=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$

7. $\tan(15^\circ)=\tan(45^\circ-30^\circ)=\dfrac{\tan(45^\circ)-\tan(30^\circ)}{1+\tan(45^\circ)\tan(30^\circ)}=\dfrac{\left(1-\dfrac{\sqrt{3}}{3}\right)}{\left(1+1\dfrac{\sqrt{3}}{3}\right)}=\dfrac{\left(\dfrac{3-\sqrt{3}}{3}\right)}{\left(\dfrac{3+\sqrt{3}}{3}\right)}$

$=\left(\dfrac{3-\sqrt{3}}{3+\sqrt{3}}\right)\left(\dfrac{3-\sqrt{3}}{3-\sqrt{3}}\right)=\dfrac{9-6\sqrt{3}+3}{9-3}=\dfrac{12-6\sqrt{3}}{6}=\dfrac{6\left(2-\sqrt{3}\right)}{6}=2-\sqrt{3}$

8. $\tan(195^\circ)=\tan(135^\circ+60^\circ)=\dfrac{\tan(135^\circ)+\tan(60^\circ)}{1-\tan(135^\circ)\tan(60^\circ)}=\dfrac{-1+\sqrt{3}}{1-(-1)\sqrt{3}}=\left(\dfrac{-1+\sqrt{3}}{1+\sqrt{3}}\right)\left(\dfrac{1-\sqrt{3}}{1-\sqrt{3}}\right)$

$=\dfrac{-1+2\sqrt{3}-3}{1-3}=\dfrac{-4+2\sqrt{3}}{-2}=2-\sqrt{3}$

9. $\sin\left(\dfrac{17\pi}{12}\right)=\sin\left(\dfrac{15\pi}{12}+\dfrac{2\pi}{12}\right)=\sin\left(\dfrac{5\pi}{4}\right)\cos\left(\dfrac{\pi}{6}\right)+\cos\left(\dfrac{5\pi}{4}\right)\sin\left(\dfrac{\pi}{6}\right)=-\dfrac{\sqrt{2}}{2}\,\dfrac{\sqrt{3}}{2}+-\dfrac{\sqrt{2}}{2}\,\dfrac{1}{2}$

$=-\dfrac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$

10. $\tan\left(\dfrac{19\pi}{12}\right)=\tan\left(\dfrac{15\pi}{12}+\dfrac{4\pi}{12}\right)=\dfrac{\tan\left(\dfrac{5\pi}{4}\right)+\tan\left(\dfrac{\pi}{3}\right)}{1-\tan\left(\dfrac{5\pi}{4}\right)\tan\left(\dfrac{\pi}{3}\right)}=\dfrac{1+\sqrt{3}}{1-1\sqrt{3}}=\left(\dfrac{1+\sqrt{3}}{1-\sqrt{3}}\right)\left(\dfrac{1+\sqrt{3}}{1+\sqrt{3}}\right)$

$=\dfrac{1+2\sqrt{3}+3}{1-3}=\dfrac{4+2\sqrt{3}}{-2}=-2-\sqrt{3}$

11. $\sec\left(-\dfrac{\pi}{12}\right)=\dfrac{1}{\cos\left(-\dfrac{\pi}{12}\right)}=\dfrac{1}{\cos\left(\dfrac{3\pi}{12}-\dfrac{4\pi}{12}\right)}=\dfrac{1}{\cos\left(\dfrac{\pi}{4}\right)\cos\left(\dfrac{\pi}{3}\right)+\sin\left(\dfrac{\pi}{4}\right)\sin\left(\dfrac{\pi}{3}\right)}$

$=\dfrac{1}{\left(\dfrac{\sqrt{2}}{2}\,\dfrac{1}{2}+\dfrac{\sqrt{2}}{2}\,\dfrac{\sqrt{3}}{2}\right)}=\dfrac{1}{\left(\dfrac{\sqrt{2}+\sqrt{6}}{4}\right)}=\left(\dfrac{4}{\sqrt{2}+\sqrt{6}}\right)\left(\dfrac{\sqrt{2}-\sqrt{6}}{\sqrt{2}-\sqrt{6}}\right)$

$=\dfrac{4\left(\sqrt{2}-\sqrt{6}\right)}{2-6}=\dfrac{4\left(\sqrt{2}-\sqrt{6}\right)}{-4}=-\left(\sqrt{2}-\sqrt{6}\right)=\sqrt{6}-\sqrt{2}$

12. $\cot\left(-\dfrac{5\pi}{12}\right)=-\cot\dfrac{5\pi}{12}=\dfrac{-1}{\tan\left(\dfrac{5\pi}{12}\right)}=\dfrac{-1}{\tan\left(\dfrac{3\pi}{12}+\dfrac{2\pi}{12}\right)}=\dfrac{-1}{\left(\dfrac{\tan\left(\dfrac{\pi}{4}\right)+\tan\left(\dfrac{\pi}{6}\right)}{1-\tan\left(\dfrac{\pi}{4}\right)\tan\left(\dfrac{\pi}{6}\right)}\right)}$

$$= -\left(\frac{1-\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}{\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{6}\right)}\right) = -\left(\frac{1-1\left(\frac{1}{\sqrt{3}}\right)}{1+\frac{1}{\sqrt{3}}}\right) = \left(\frac{1-\sqrt{3}}{\sqrt{3}+1}\right)\left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right)$$

$$= \frac{1-2\sqrt{3}+3}{1-3} = \frac{4-2\sqrt{3}}{-2} = -2+\sqrt{3}$$

13. $\sin(20^\circ)\cos(10^\circ)+\cos(20^\circ)\sin(10^\circ) = \sin(20^\circ+10^\circ) = \sin(30^\circ) = \frac{1}{2}$

14. $\sin(20^\circ)\cos(80^\circ)-\cos(20^\circ)\sin(80^\circ) = \sin(20^\circ-80^\circ) = \sin(-60^\circ) = -\sin(60^\circ) = -\frac{\sqrt{3}}{2}$

15. $\cos(70^\circ)\cos(20^\circ)-\sin(70^\circ)\sin(20^\circ) = \cos(70^\circ+20^\circ) = \cos(90^\circ) = 0$

16. $\cos(40^\circ)\cos(10^\circ)+\sin(40^\circ)\sin(10^\circ) = \cos(40^\circ-10^\circ) = \cos(30^\circ) = \frac{\sqrt{3}}{2}$

17. $\dfrac{\tan(20^\circ)+\tan(25^\circ)}{1-\tan(20^\circ)\tan(25^\circ)} = \tan(20^\circ+25^\circ) = \tan(45^\circ) = 1$

18. $\dfrac{\tan(40^\circ)-\tan(10^\circ)}{1+\tan(40^\circ)\tan(10^\circ)} = \tan(40^\circ-10^\circ) = \tan(30^\circ) = \frac{\sqrt{3}}{3}$

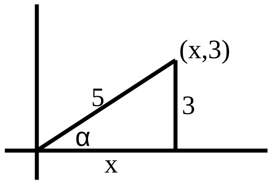
19. $\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{7\pi}{12}\right)-\cos\left(\frac{\pi}{12}\right)\sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{\pi}{12}-\frac{7\pi}{12}\right) = \sin\left(-\frac{\pi}{2}\right) = -1$

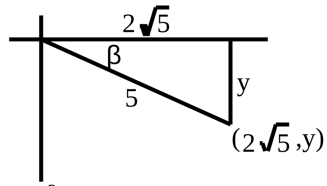
20. $\cos\left(\frac{5\pi}{12}\right)\cos\left(\frac{7\pi}{12}\right)-\sin\left(\frac{5\pi}{12}\right)\sin\left(\frac{7\pi}{12}\right) = \cos\left(\frac{5\pi}{12}+\frac{7\pi}{12}\right) = \cos\left(\frac{12\pi}{12}\right) = \cos(\pi) = -1$

21. $\cos\left(\frac{\pi}{12}\right)\cos\left(\frac{5\pi}{12}\right)+\sin\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{12}-\frac{5\pi}{12}\right) = \cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

22. $\sin\left(\frac{\pi}{18}\right)\cos\left(\frac{5\pi}{18}\right)+\cos\left(\frac{\pi}{18}\right)\sin\left(\frac{5\pi}{18}\right) = \sin\left(\frac{\pi}{18}+\frac{5\pi}{18}\right) = \sin\left(\frac{6\pi}{18}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

23. $\sin\alpha = \frac{3}{5},\ 0 < \alpha < \frac{\pi}{2};\qquad \cos\beta = \frac{2\sqrt{5}}{5},\ -\frac{\pi}{2} < \beta < 0$





$$x^2 + 3^2 = 5^2,\quad x > 0$$
$$x^2 = 25 - 9 = 16,\quad x > 0$$
$$x = 4$$
$$\cos\alpha = \frac{4}{5},\quad \tan\alpha = \frac{3}{4}$$

$$\left(2\sqrt{5}\right)^2 + y^2 = 5^2,\quad y < 0$$
$$y^2 = 25 - 20 = 5.\quad y < 0$$
$$y = -\sqrt{5}$$
$$\sin\beta = -\frac{\sqrt{5}}{5},\quad \tan\beta = \frac{-\sqrt{5}}{2\sqrt{5}} = -\frac{1}{2}$$

(a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{3}{5}\cdot\frac{2\sqrt{5}}{5} + \frac{4}{5}\cdot\left(-\frac{\sqrt{5}}{5}\right) = \frac{6\sqrt{5} - 4\sqrt{5}}{25} = \frac{2\sqrt{5}}{25}$

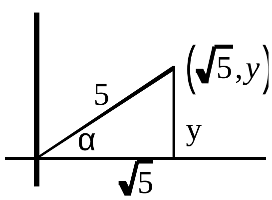
(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{4}{5}\cdot\frac{2\sqrt{5}}{5} - \frac{3}{5}\cdot\left(-\frac{\sqrt{5}}{5}\right) = \frac{8\sqrt{5} + 3\sqrt{5}}{25} = \frac{11\sqrt{5}}{25}$

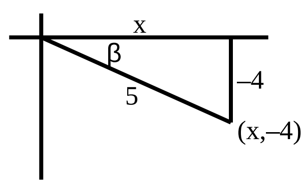
(c) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{3}{5}\cdot\frac{2\sqrt{5}}{5} - \frac{4}{5}\cdot\left(-\frac{\sqrt{5}}{5}\right) = \frac{6\sqrt{5} + 4\sqrt{5}}{25}$

$$= \frac{10\sqrt{5}}{25} = \frac{2\sqrt{5}}{5}$$

(d) $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \dfrac{\left(\frac{3}{4}\right) - \left(-\frac{1}{2}\right)}{1 + \left(\frac{3}{4}\right)\left(-\frac{1}{2}\right)} = \dfrac{\left(\frac{5}{4}\right)}{\left(\frac{5}{8}\right)} = 2$

24. $\cos\alpha = \frac{\sqrt{5}}{5},\ 0 < \alpha < \frac{\pi}{2};\qquad \sin\beta = -\frac{4}{5},\ -\frac{\pi}{2} < \beta < 0$





$$\sqrt{5}^2 + y^2 = 5^2,\quad y > 0$$
$$y^2 = 25 - 5 = 20,\quad y > 0$$
$$y = \sqrt{20} = 2\sqrt{5}$$
$$\sin\alpha = \frac{2\sqrt{5}}{5},\quad \tan\alpha = \frac{2\sqrt{5}}{\sqrt{5}} = 2$$

$$x^2 + (-4)^2 = 5^2,\quad x > 0$$
$$x^2 = 25 - 16 = 9.\quad x > 0$$
$$x = 3$$
$$\cos\beta = \frac{3}{5},\quad \tan\beta = \frac{-4}{3} = -\frac{4}{3}$$

(a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \left(\frac{2\sqrt{5}}{5}\right)\left(\frac{3}{5}\right) + \left(\frac{\sqrt{5}}{5}\right)\left(-\frac{4}{5}\right) = \frac{6\sqrt{5} - 4\sqrt{5}}{25} = \frac{2\sqrt{5}}{25}$

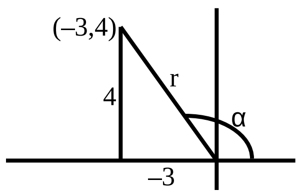
(b) $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta=\left(\dfrac{\sqrt{5}}{5}\right)\left(\dfrac{3}{5}\right)-\left(\dfrac{2\sqrt{5}}{5}\right)\left(-\dfrac{4}{5}\right)=\dfrac{3\sqrt{5}+8\sqrt{5}}{25}=\dfrac{11\sqrt{5}}{25}$

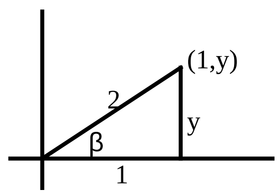
(c) $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta=\left(\dfrac{2\sqrt{5}}{5}\right)\left(\dfrac{3}{5}\right)-\left(\dfrac{\sqrt{5}}{5}\right)\left(-\dfrac{4}{5}\right)=\dfrac{6\sqrt{5}+4\sqrt{5}}{25}$

$=\dfrac{10\sqrt{5}}{25}=\dfrac{2\sqrt{5}}{5}$

(d) $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\dfrac{\left(2-\left(-\dfrac{4}{3}\right)\right)}{\left(1+2\left(-\dfrac{4}{3}\right)\right)}=\dfrac{\left(\dfrac{10}{3}\right)}{\left(-\dfrac{5}{3}\right)}=-2$

25. $\tan\alpha=-\dfrac{4}{3},\ \dfrac{\pi}{2}<\alpha<\pi;\qquad \cos\beta=\dfrac{1}{2},\ 0<\beta<\dfrac{\pi}{2}$





$r^2=(-3)^2+4^2=25$

$r=5$

$\sin\alpha=\dfrac{4}{5},\ \cos\alpha=\dfrac{-3}{5}=-\dfrac{3}{5}$

$1^2+y^2=2^2,\quad y>0$

$y^2=4-1=3.\quad y>0$

$y=\sqrt{3}$

$\sin\beta=\dfrac{\sqrt{3}}{2},\quad \tan\beta=\dfrac{\sqrt{3}}{1}=\sqrt{3}$

(a) $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta=\left(\dfrac{4}{5}\right)\left(\dfrac{1}{2}\right)+\left(-\dfrac{3}{5}\right)\left(\dfrac{\sqrt{3}}{2}\right)=\dfrac{4-3\sqrt{3}}{10}$

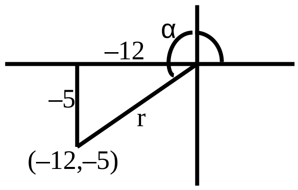
(b) $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta=\left(-\dfrac{3}{5}\right)\left(\dfrac{1}{2}\right)-\left(\dfrac{4}{5}\right)\left(\dfrac{\sqrt{3}}{2}\right)=\dfrac{-3-4\sqrt{3}}{10}$

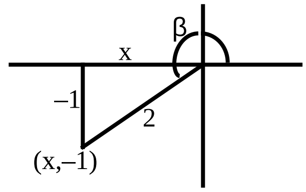
(c) $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta=\left(\dfrac{4}{5}\right)\left(\dfrac{1}{2}\right)-\left(-\dfrac{3}{5}\right)\left(\dfrac{\sqrt{3}}{2}\right)=\dfrac{4+3\sqrt{3}}{10}$

(d) $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\dfrac{\left(-\dfrac{4}{3}-\sqrt{3}\right)}{\left(1+\left(-\dfrac{4}{3}\right)\sqrt{3}\right)}=\dfrac{\left(\dfrac{-4-3\sqrt{3}}{3}\right)}{\left(\dfrac{3-4\sqrt{3}}{3}\right)}=\left(\dfrac{-4-3\sqrt{3}}{3-4\sqrt{3}}\right)\left(\dfrac{3+4\sqrt{3}}{3+4\sqrt{3}}\right)$

$=\dfrac{-48-25\sqrt{3}}{-39}=\dfrac{48+25\sqrt{3}}{39}$

26. $\tan\alpha = \frac{5}{12},\ \pi < \alpha < \frac{3\pi}{2};\qquad \sin\beta = -\frac{1}{2},\ \pi < \beta < \frac{3\pi}{2}$





$r^2 = (-12)^2 + (-5)^2 = 169$

$r = 13$

$\sin\alpha = \frac{-5}{13} = -\frac{5}{13},\ \cos\alpha = \frac{-12}{13} = -\frac{12}{13}$

$x^2 + (-1)^2 = 2^2,\ \ x < 0$

$x^2 = 4 - 1 = 3.\ \ x < 0$

$x = -\sqrt{3}$

$\cos\beta = -\frac{\sqrt{3}}{2},\quad \tan\beta = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$

(a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \left(-\frac{5}{13}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(-\frac{12}{13}\right)\left(-\frac{1}{2}\right) = \frac{5\sqrt{3}+12}{26}$

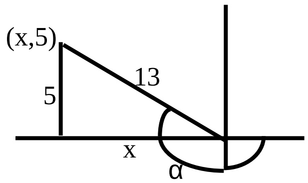
(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(-\frac{12}{13}\right)\left(-\frac{\sqrt{3}}{2}\right) - \left(-\frac{5}{13}\right)\left(-\frac{1}{2}\right) = \frac{12\sqrt{3}-5}{26}$

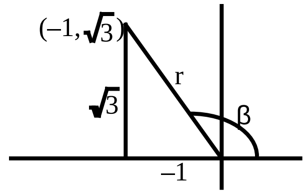
(c) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \left(-\frac{5}{13}\right)\left(-\frac{\sqrt{3}}{2}\right) - \left(-\frac{12}{13}\right)\left(-\frac{1}{2}\right) = \frac{5\sqrt{3}-12}{26}$

(d) $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{\left(\frac{5}{12} - \frac{\sqrt{3}}{3}\right)}{\left(1+\frac{5}{12}\cdot\frac{\sqrt{3}}{3}\right)} = \frac{\left(\frac{5-4\sqrt{3}}{12}\right)}{\left(\frac{36+5\sqrt{3}}{36}\right)} = \left(\frac{15-12\sqrt{3}}{36+5\sqrt{3}}\right)\left(\frac{36-5\sqrt{3}}{36-5\sqrt{3}}\right)$

$= \frac{540-507\sqrt{3}+180}{1296-75} = \frac{720-507\sqrt{3}}{1221} = \frac{240-169\sqrt{3}}{407}$

27. $\sin\alpha = \dfrac{5}{13},\ -\dfrac{3\pi}{2} < \alpha < -\pi; \qquad \tan\beta = -\sqrt{3},\ \dfrac{\pi}{2} < \beta < \pi$





$x^2 + 5^2 = 13^2,\ x < 0$

$x^2 = 169 - 25 = 144,\ x < 0$

$x = -12$

$\cos\alpha = \dfrac{-12}{13} = -\dfrac{12}{13},\ \tan\alpha = -\dfrac{5}{12}$

$r^2 = (-1)^2 + \sqrt{3}^{\,2} = 4$

$r = 2$

$\sin\beta = \dfrac{\sqrt{3}}{2}, \quad \cos\beta = \dfrac{-1}{2} = -\dfrac{1}{2}$

(a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \left(\dfrac{5}{13}\right)\left(-\dfrac{1}{2}\right) + \left(-\dfrac{12}{13}\right)\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{-5-12\sqrt{3}}{26}$

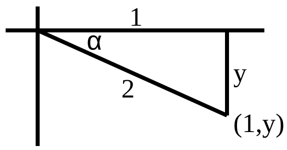
(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(-\dfrac{12}{13}\right)\left(-\dfrac{1}{2}\right) - \left(\dfrac{5}{13}\right)\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{12-5\sqrt{3}}{26}$

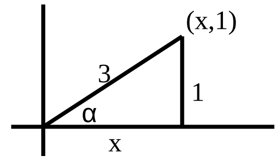
(c) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \left(\dfrac{5}{13}\right)\left(-\dfrac{1}{2}\right) - \left(-\dfrac{12}{13}\right)\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{-5+12\sqrt{3}}{26}$

(d) $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \dfrac{\left(-\dfrac{5}{12} - \left(-\sqrt{3}\right)\right)}{\left(1 + \left(-\dfrac{5}{12}\right)\left(-\sqrt{3}\right)\right)} = \dfrac{\left(\dfrac{-5+12\sqrt{3}}{12}\right)}{\left(\dfrac{12+5\sqrt{3}}{12}\right)}$

$$= \left(\frac{-5+12\sqrt{3}}{12+5\sqrt{3}}\right)\left(\frac{12-5\sqrt{3}}{12-5\sqrt{3}}\right) = \frac{-240+169\sqrt{3}}{69}$$

28. $\cos\alpha = \frac{1}{2},\quad -\frac{\pi}{2} < \alpha < 0; \qquad \sin\beta = \frac{1}{3},\quad 0 < \beta < \frac{\pi}{2}$





$1^2 + y^2 = 2^2,\ y < 0 \qquad\qquad x^2 + 1^2 = 3^2,\ x > 0$

$y^2 = 4 - 1 = 3,\ y < 0 \qquad\qquad x^2 = 9 - 1 = 8.\ x > 0$

$y = -\sqrt{3} \qquad\qquad x = \sqrt{8} = 2\sqrt{2}$

$\sin\alpha = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2},\ \tan\alpha = \frac{-\sqrt{3}}{1} = -\sqrt{3} \qquad \cos\beta = \frac{2\sqrt{2}}{3},\quad \tan\beta = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

(a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{2\sqrt{2}}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{-2\sqrt{6}+1}{6}$

(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(\frac{1}{2}\right)\left(\frac{2\sqrt{2}}{3}\right) - \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{3}\right) = \frac{2\sqrt{2}+\sqrt{3}}{6}$

(c) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{-2\sqrt{6}-1}{6}$

(d) $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{\left(-\sqrt{3} - \frac{\sqrt{2}}{4}\right)}{\left(1 + \left(-\sqrt{3}\right)\frac{\sqrt{2}}{4}\right)} = \frac{\left(\frac{-4\sqrt{3}-\sqrt{2}}{4}\right)}{\left(\frac{4-\sqrt{6}}{4}\right)}$

$= \left(\frac{-4\sqrt{3}-\sqrt{2}}{4-\sqrt{6}}\right)\left(\frac{4+\sqrt{6}}{4+\sqrt{6}}\right) = \frac{-16\sqrt{3} - 4\sqrt{2} - 4\sqrt{18} - \sqrt{12}}{16-6}$

$= \frac{-18\sqrt{3} - 16\sqrt{2}}{10} = \frac{-9\sqrt{3} - 8\sqrt{2}}{5}$

29. $\sin\theta = \frac{1}{3},\quad \theta$ in quadrant II

(a) $\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-\left(\frac{1}{3}\right)^2} = -\sqrt{1-\frac{1}{9}} = -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3}$

(b) $\sin\left(\theta + \frac{\pi}{6}\right) = \sin\theta\cos\left(\frac{\pi}{6}\right) + \cos\theta\sin\left(\frac{\pi}{6}\right) = \left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(-\frac{2\sqrt{2}}{3}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}-2\sqrt{2}}{6}$

(c) $\cos\left(\theta - \frac{\pi}{3}\right) = \cos\theta\cos\left(\frac{\pi}{3}\right) + \sin\theta\sin\left(\frac{\pi}{3}\right) = \left(-\frac{2\sqrt{2}}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{-2\sqrt{2}+\sqrt{3}}{6}$

(d) $\tan\left(\theta + \frac{\pi}{4}\right) = \dfrac{\tan\theta + \tan\left(\frac{\pi}{4}\right)}{1 - \tan\theta\tan\left(\frac{\pi}{4}\right)} = \dfrac{\left(-\frac{1}{2\sqrt{2}} + 1\right)}{\left(1 - \left(-\frac{1}{2\sqrt{2}}\right)1\right)} = \dfrac{\left(\frac{-1+2\sqrt{2}}{2\sqrt{2}}\right)}{\left(\frac{2\sqrt{2}+1}{2\sqrt{2}}\right)}$

$= \left(\frac{2\sqrt{2}-1}{2\sqrt{2}+1}\right)\left(\frac{2\sqrt{2}-1}{2\sqrt{2}-1}\right) = \frac{9-4\sqrt{2}}{7}$

30. $\cos\theta = \frac{1}{4}$, θ in quadrant IV

(a) $\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{1-\left(\frac{1}{4}\right)^2} = -\sqrt{1-\frac{1}{16}} = -\sqrt{\frac{15}{16}} = -\frac{\sqrt{15}}{4}$

(b) $\sin\left(\theta - \frac{\pi}{6}\right) = \sin\theta\cos\left(\frac{\pi}{6}\right) - \cos\theta\sin\left(\frac{\pi}{6}\right) = \left(-\frac{\sqrt{15}}{4}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{-3\sqrt{5}-1}{8}$

(c) $\cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta\cos\left(\frac{\pi}{3}\right) - \sin\theta\sin\left(\frac{\pi}{3}\right) = \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) - \left(-\frac{\sqrt{15}}{4}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{1+3\sqrt{5}}{8}$

(d) $\tan\left(\theta - \frac{\pi}{4}\right) = \dfrac{\tan\theta - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\theta\tan\left(\frac{\pi}{4}\right)} = \dfrac{-\sqrt{15}-1}{1+\left(-\sqrt{15}\right)1} = \left(\frac{-1-\sqrt{15}}{1-\sqrt{15}}\right)\left(\frac{1+\sqrt{15}}{1+\sqrt{15}}\right)$

$= \frac{-1-2\sqrt{15}-15}{1-15} = \frac{-16-2\sqrt{15}}{-14} = \frac{-8-\sqrt{15}}{-7}$

31. $\sin\left(\frac{\pi}{2}+\theta\right) = \sin\left(\frac{\pi}{2}\right)\cos\theta + \cos\left(\frac{\pi}{2}\right)\sin\theta = 1\ \cos\theta + 0\ \sin\theta = \cos\theta$

32. $\cos\left(\frac{\pi}{2}+\theta\right) = \cos\left(\frac{\pi}{2}\right)\cos\theta - \sin\left(\frac{\pi}{2}\right)\sin\theta = 0\ \cos\theta - 1\ \sin\theta = -\sin\theta$

33. $\sin(\pi-\theta) = \sin(\pi)\cos\theta - \cos(\pi)\sin\theta = 0\ \cos\theta - (-1)\sin\theta = \sin\theta$

34. $\cos(\pi-\theta) = \cos(\pi)\cos\theta + \sin(\pi)\sin\theta = -1\ \cos\theta + (0)\sin\theta = -\cos\theta$

35. $\sin(\pi+\theta) = \sin(\pi)\cos\theta + \cos(\pi)\sin\theta = 0\ \cos\theta + (-1)\sin\theta = -\sin\theta$

36. $\cos(\pi+\theta)=\cos(\pi)\cos\theta-\sin(\pi)\sin\theta=-1\ \cos\theta-(0)\sin\theta=-\cos\theta$

37. $\tan(\pi-\theta)=\dfrac{\tan(\pi)-\tan\theta}{1+\tan(\pi)\tan\theta}=\dfrac{0-\tan\theta}{1+0\ \tan\theta}=\dfrac{-\tan\theta}{1}=-\tan\theta$

38. $\tan(2\pi-\theta)=\dfrac{\tan(2\pi)-\tan\theta}{1+\tan(2\pi)\tan\theta}=\dfrac{0-\tan\theta}{1+0\ \tan\theta}=\dfrac{-\tan\theta}{1}=-\tan\theta$

39. $\sin\left(\dfrac{3\pi}{2}+\theta\right)=\sin\left(\dfrac{3\pi}{2}\right)\cos\theta+\cos\left(\dfrac{3\pi}{2}\right)\sin\theta=-1\ \cos\theta+0\ \sin\theta=-\cos\theta$

40. $\cos\left(\dfrac{3\pi}{2}+\theta\right)=\cos\left(\dfrac{3\pi}{2}\right)\cos\theta-\sin\left(\dfrac{3\pi}{2}\right)\sin\theta=0\ \cos\theta-(-1)\ \sin\theta=\sin\theta$

41. $$\begin{aligned}\sin(\alpha+\beta)+\sin(\alpha-\beta)&=\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta\\&=2\sin\alpha\cos\beta\end{aligned}$$

42. $$\begin{aligned}\cos(\alpha+\beta)+\cos(\alpha-\beta)&=\cos\alpha\cos\beta-\sin\alpha\sin\beta+\cos\alpha\cos\beta+\sin\alpha\sin\beta\\&=2\cos\alpha\cos\beta\end{aligned}$$

43. $\dfrac{\sin(\alpha+\beta)}{\sin\alpha\cos\beta}=\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta}=\dfrac{\sin\alpha\cos\beta}{\sin\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\sin\alpha\cos\beta}=1+\cot\alpha\tan\beta$

44. $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}=\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}=\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}=\tan\alpha+\tan\beta$

45. $\dfrac{\cos(\alpha+\beta)}{\cos\alpha\cos\beta}=\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}=\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}=1-\tan\alpha\tan\beta$

46. $\dfrac{\cos(\alpha-\beta)}{\sin\alpha\cos\beta}=\dfrac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta}=\dfrac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta}+\dfrac{\sin\alpha\sin\beta}{\sin\alpha\cos\beta}=\cot\alpha+\tan\beta$

47. $$\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}=\frac{\left(\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}\right)}{\left(\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}\right)}=\frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}$$

48. $$\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}=\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta+\sin\alpha\sin\beta}=\frac{\left(\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\right)}{\left(\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\right)}=\frac{1-\tan\alpha\tan\beta}{1+\tan\alpha\tan\beta}$$

49. $\cot(\alpha+\beta)=\dfrac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}=\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}$

$$=\frac{\left(\dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}-\dfrac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}\right)}{\left(\dfrac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta}+\dfrac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}\right)}=\frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}$$

50. $\cot(\alpha-\beta)=\dfrac{\cos(\alpha-\beta)}{\sin(\alpha-\beta)}=\dfrac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$

$$=\frac{\left(\dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}+\dfrac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}\right)}{\left(\dfrac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta}-\dfrac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}\right)}=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$$

51. $\sec(\alpha+\beta)=\dfrac{1}{\cos(\alpha+\beta)}=\dfrac{1}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$

$$=\frac{\left(\dfrac{1}{\sin\alpha\sin\beta}\right)}{\left(\dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}-\dfrac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}\right)}=\frac{\csc\alpha\csc\beta}{\cot\alpha\cot\beta-1}$$

52. $\sec(\alpha-\beta)=\dfrac{1}{\cos(\alpha-\beta)}=\dfrac{1}{\cos\alpha\cos\beta+\sin\alpha\sin\beta}$

$$=\frac{\left(\dfrac{1}{\cos\alpha\cos\beta}\right)}{\left(\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\right)}=\frac{\sec\alpha\sec\beta}{1+\tan\alpha\tan\beta}$$

53. $\sin(\alpha-\beta)\sin(\alpha+\beta)=\left(\sin\alpha\cos\beta-\cos\alpha\sin\beta\right)\left(\sin\alpha\cos\beta+\cos\alpha\sin\beta\right)$
$=\sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta=\sin^2\alpha(1-\sin^2\beta)-(1-\sin^2\alpha)\sin^2\beta$
$\sin^2\alpha-\sin^2\alpha\sin^2\beta-\sin^2\beta+\sin^2\alpha\sin^2\beta=\sin^2\alpha-\sin^2\beta$

54. $\cos(\alpha-\beta)\cos(\alpha+\beta)=\left(\cos\alpha\cos\beta+\sin\alpha\sin\beta\right)\left(\cos\alpha\cos\beta-\sin\alpha\sin\beta\right)$
$=\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta=\cos^2\alpha(1-\sin^2\beta)-(1-\cos^2\alpha)\sin^2\beta$
$=\cos^2\alpha-\cos^2\alpha\sin^2\beta-\sin^2\beta+\cos^2\alpha\sin^2\beta=\cos^2\alpha-\sin^2\beta$

55. $\sin(\theta+k\pi)=\sin\theta\cos(k\pi)+\cos\theta\sin(k\pi)=\sin\theta(-1)^k+\cos\theta\;\;0$
$=(-1)^k\sin\theta$, k any integer

56. $\cos(\theta + k\pi) = \cos\theta\cos(k\pi) - \sin\theta\sin(k\pi) = \cos\theta(-1)^k - \sin\theta \cdot 0$
$= (-1)^k \cos\theta$, k any integer

57. $\sin\left(\sin^{-1}\left(\frac{1}{2}\right) + \cos^{-1}(0)\right) = \sin\left(\frac{\pi}{6} + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{2}\right) = \frac{1}{2}\cdot 0 + \frac{\sqrt{3}}{2}\cdot 1 = \frac{\sqrt{3}}{2}$

58. $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \cos^{-1}(1)\right) = \sin\left(\frac{\pi}{3} + 0\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

59. $\sin\left[\sin^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(-\frac{4}{5}\right)\right]$

Let $\alpha = \sin^{-1}\left(\frac{3}{5}\right)$ and $\beta = \cos^{-1}\left(-\frac{4}{5}\right)$. α is in quadrant I; β is in quadrant II.

Then $\sin\alpha = \frac{3}{5}$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, and $\cos\beta = -\frac{4}{5}$, $0 \le \beta \le \pi$.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(-\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\sin\left[\sin^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(-\frac{4}{5}\right)\right] = \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) - \left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = -\frac{12}{25} - \frac{12}{25} = -\frac{24}{25}$$

60. $\sin\left(\sin^{-1}\left(-\frac{4}{5}\right) - \tan^{-1}\left(\frac{3}{4}\right)\right)$

Let $\alpha = \sin^{-1}\left(-\frac{4}{5}\right)$ and $\beta = \tan^{-1}\left(\frac{3}{4}\right)$. α is in quadrant IV; β is in quadrant I.

Then $\sin\alpha = -\frac{4}{5}$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, and $\tan\beta = \frac{3}{4}$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(-\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\sec\beta = \sqrt{1+\tan^2\beta} = \sqrt{1+\left(\frac{3}{4}\right)^2} = \sqrt{1+\frac{9}{16}} = \sqrt{\frac{25}{15}} = \frac{5}{4}; \quad \cos\beta = \frac{4}{5}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\sin\left(\sin^{-1}\left(-\frac{4}{5}\right)-\tan^{-1}\left(\frac{3}{4}\right)\right)=\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$=\left(-\frac{4}{5}\right)\left(\frac{4}{5}\right)-\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)=-\frac{16}{25}-\frac{9}{25}=-\frac{25}{25}=-1$$

61. $\cos\left(\tan^{-1}\left(\frac{4}{3}\right)+\cos^{-1}\left(\frac{5}{13}\right)\right)$

Let $\alpha=\tan^{-1}\left(\frac{4}{3}\right)$ and $\beta=\cos^{-1}\left(\frac{5}{13}\right)$. α is in quadrant I; β is in quadrant I.

Then $\tan\alpha=\frac{4}{3},\ -\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, and $\cos\beta=\frac{5}{13},\ 0\le\beta\le\pi$.

$$\sec\alpha=\sqrt{1+\tan^2\alpha}=\sqrt{1+\left(\frac{4}{3}\right)^2}=\sqrt{1+\frac{16}{9}}=\sqrt{\frac{25}{9}}=\frac{5}{3};\quad \cos\alpha=\frac{3}{5}$$

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

$$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-\left(\frac{5}{13}\right)^2}=\sqrt{1-\frac{25}{169}}=\sqrt{\frac{144}{169}}=\frac{12}{13}$$

$$\cos\left(\tan^{-1}\left(\frac{4}{3}\right)+\cos^{-1}\left(\frac{5}{13}\right)\right)=\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$=\left(\frac{3}{5}\right)\left(\frac{5}{13}\right)-\left(\frac{4}{5}\right)\left(\frac{12}{13}\right)=\frac{15}{65}-\frac{48}{65}=-\frac{33}{65}$$

62. $\sin\left(\tan^{-1}\left(\frac{5}{12}\right)-\sin^{-1}\left(-\frac{3}{5}\right)\right)$

Let $\alpha=\tan^{-1}\left(\frac{5}{12}\right)$ and $\beta=\sin^{-1}\left(-\frac{3}{5}\right)$. α is in quadrant I; β is in quadrant IV.

Then $\tan\alpha=\frac{5}{12},\ -\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, and $\sin\beta=-\frac{3}{5},-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$.

$$\sec\alpha=\sqrt{1+\tan^2\alpha}=\sqrt{1+\left(\frac{5}{12}\right)^2}=\sqrt{1+\frac{25}{144}}=\sqrt{\frac{169}{144}}=\frac{13}{12};\quad \cos\alpha=\frac{12}{13}$$

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\left(\frac{12}{13}\right)^2}=\sqrt{1-\frac{144}{169}}=\sqrt{\frac{25}{169}}=\frac{5}{13}$$

$$\cos\beta=\sqrt{1-\sin^2\beta}=\sqrt{1-\left(-\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{15}}=\frac{4}{5}$$

$$\sin\left(\tan^{-1}\left(\frac{5}{12}\right)-\sin^{-1}\left(-\frac{3}{5}\right)\right)=\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$=\left(\frac{5}{13}\right)\left(\frac{4}{5}\right)-\left(\frac{12}{13}\right)\left(-\frac{3}{5}\right)=\frac{20}{65}+\frac{36}{65}=\frac{56}{65}$$

63. $\sec\left(\sin^{-1}\left(\frac{5}{13}\right)-\tan^{-1}\left(\frac{3}{4}\right)\right)$

Let $\alpha=\sin^{-1}\left(\frac{5}{13}\right)$ and $\beta=\tan^{-1}\left(\frac{3}{4}\right)$. α is in quadrant I; β is in quadrant I.

Then $\sin\alpha=\frac{5}{13},\ -\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$, and $\tan\beta=\frac{3}{4},-\frac{\pi}{2}<\beta<\frac{\pi}{2}$.

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(\frac{5}{13}\right)^2}=\sqrt{1-\frac{25}{169}}=\sqrt{\frac{144}{169}}=\frac{12}{13}$$

$$\sec\beta=\sqrt{1+\tan^2\beta}=\sqrt{1+\left(\frac{3}{4}\right)^2}=\sqrt{1+\frac{9}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4};\quad \cos\beta=\frac{4}{5}$$

$$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

$$\sec\left[\sin^{-1}\left(\frac{5}{13}\right)-\tan^{-1}\left(\frac{3}{4}\right)\right]=\frac{1}{\cos\left[\sin^{-1}\left(\frac{5}{13}\right)-\tan^{-1}\left(\frac{3}{4}\right)\right]}=\frac{1}{\cos(\alpha-\beta)}$$

$$=\frac{1}{\cos\alpha\cos\beta+\sin\alpha\sin\beta}=\frac{1}{\left(\frac{12}{13}\cdot\frac{4}{5}+\frac{5}{13}\cdot\frac{3}{5}\right)}$$

$$=\frac{1}{\left(\frac{48}{65}+\frac{15}{65}\right)}=\frac{1}{\left(\frac{63}{65}\right)}=\frac{65}{63}$$

64. $\cos\left(\tan^{-1}\left(\frac{4}{3}\right)+\cos^{-1}\left(\frac{12}{13}\right)\right)$

Let $\alpha=\tan^{-1}\left(\frac{4}{3}\right)$ and $\beta=\cos^{-1}\left(\frac{12}{13}\right)$. α is in quadrant I; β is in quadrant I.

Then $\tan\alpha=\frac{4}{3},\ -\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, and $\cos\beta=\frac{12}{13},\ 0\le\beta\le\pi$.

$$\sec\alpha=\sqrt{1+\tan^2\alpha}=\sqrt{1+\left(\frac{4}{3}\right)^2}=\sqrt{1+\frac{16}{9}}=\sqrt{\frac{25}{9}}=\frac{5}{3};\quad \cos\alpha=\frac{3}{5}$$

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

$$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-\left(\frac{12}{13}\right)^2}=\sqrt{1-\frac{144}{169}}=\sqrt{\frac{25}{169}}=\frac{5}{13}$$

$$\cos\left(\tan^{-1}\left(\frac{4}{3}\right)+\cos^{-1}\left(\frac{12}{13}\right)\right)=\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$=\left(\frac{3}{5}\right)\left(\frac{12}{13}\right)-\left(\frac{4}{5}\right)\left(\frac{5}{13}\right)=\frac{36}{65}-\frac{20}{65}=\frac{16}{65}$$

65. $\tan\left(\sin^{-1}\left(\frac{5}{3}\right)+\frac{\pi}{6}\right)$

$\sin^{-1}\left(\frac{5}{3}\right)$ is undefined since $\frac{5}{3}$ is not in the domain of $f(x)=\sin^{-1}(x)$

therefore $\tan\left(\sin^{-1}\left(\frac{5}{3}\right)+\frac{\pi}{6}\right)$ is undefined

66. $\tan\left(\frac{\pi}{4}-\cos^{-1}\left(\frac{3}{5}\right)\right)$

Let $\alpha=\cos^{-1}\left(\frac{3}{5}\right)$. α is in quadrant I.

Then $\cos\alpha=\frac{3}{5},\ 0\le\alpha\le\pi$.

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

$$\tan\alpha=\frac{\sin\alpha}{\cos\alpha}=\frac{\left(\frac{4}{5}\right)}{\left(\frac{3}{5}\right)}=\frac{4}{5}\cdot\frac{5}{3}=\frac{4}{3}$$

$$\tan\left(\frac{\pi}{4}-\cos^{-1}\left(\frac{3}{5}\right)\right)=\frac{\tan\left(\frac{\pi}{4}\right)-\tan\left(\cos^{-1}\left(\frac{3}{5}\right)\right)}{1+\tan\left(\frac{\pi}{4}\right)\tan\left(\cos^{-1}\left(\frac{3}{5}\right)\right)}=\frac{\left(1-\frac{4}{3}\right)}{\left(1+1\cdot\frac{4}{3}\right)}=\frac{\left(-\frac{1}{3}\right)}{\left(\frac{7}{3}\right)}=-\frac{1}{3}\cdot\frac{3}{7}=-\frac{1}{7}$$

67. $\tan\left(\sin^{-1}\left(\frac{4}{5}\right)+\cos^{-1}(1)\right)$

Let $\alpha=\sin^{-1}\left(\frac{4}{5}\right)$ and $\beta=\cos^{-1}(1)$; α is in quadrant I.

Then $\sin\alpha=\frac{4}{5},\ -\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$ and $\cos\beta=1,\ 0\le\beta\le\pi$.

$\cos\beta=1,\ 0\le\beta\le\pi \quad \beta=0 \quad \cos^{-1}1=0$

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\left(\frac{4}{5}\right)}{\left(\frac{3}{5}\right)} = \frac{4}{5}\cdot\frac{5}{3} = \frac{4}{3}$$

$$\tan\left(\sin^{-1}\left(\frac{4}{5}\right)+\cos^{-1}(1)\right) = \frac{\tan\left(\sin^{-1}\left(\frac{4}{5}\right)\right)+\tan\left(\cos^{-1}(1)\right)}{1-\tan\left(\sin^{-1}\left(\frac{4}{5}\right)\right)\tan\left(\cos^{-1}(1)\right)} = \frac{\left(\frac{4}{3}+0\right)}{\left(1-\frac{4}{3}\cdot 0\right)} = \frac{\left(\frac{4}{3}\right)}{(1)} = \frac{4}{3}$$

68. $\tan\left(\cos^{-1}\left(\frac{4}{5}\right)+\sin^{-1}(1)\right)$

Let $\alpha = \cos^{-1}\left(\frac{4}{5}\right)$ and $\beta = \sin^{-1}(1)$; α is in quadrant I.

Then $\cos\alpha = \frac{4}{5},\ 0 \le \alpha \le \pi$ and $\sin\beta = 1,\ -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$.

$$\sin\beta = 1,\ -\frac{\pi}{2} \le \beta \le \frac{\pi}{2} \qquad \beta = \frac{\pi}{2} \qquad \sin^{-1}(1) = \frac{\pi}{2}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\left(\frac{3}{5}\right)}{\left(\frac{4}{5}\right)} = \frac{3}{5}\cdot\frac{5}{4} = \frac{3}{4}; \quad \text{but } \tan\left(\frac{\pi}{2}\right) \text{ is undefined.}$$

therefore $\tan\left(\cos^{-1}\left(\frac{4}{5}\right)+\sin^{-1}(1)\right)$ is undefined.

69. $\cos\left(\cos^{-1}u+\sin^{-1}v\right)$

Let $\alpha = \cos^{-1}u$ and $\beta = \sin^{-1}v$.

Then $\cos\alpha = u,\ 0 \le \alpha \le \pi$, and $\sin\beta = v,\ -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-u^2}$$

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-v^2}$$

$$\cos\left(\cos^{-1}u+\sin^{-1}v\right) = \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = u\sqrt{1-v^2} - v\sqrt{1-u^2}$$

70. $\sin\left(\sin^{-1}u - \cos^{-1}v\right)$

Let $\alpha = \sin^{-1}u$ and $\beta = \cos^{-1}v$.

Then $\sin\alpha = u,\ -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, and $\cos\beta = v,\ 0 \le \beta \le \pi$

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-u^2}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-v^2}$$

$$\sin\left(\sin^{-1}u - \cos^{-1}v\right) = \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= u \cdot v - \sqrt{1-u^2}\cdot\sqrt{1-v^2}$$

71. $\sin\left(\tan^{-1}u - \sin^{-1}v\right)$

Let $\alpha = \tan^{-1}u$ and $\beta = \sin^{-1}v$.

Then $\tan\alpha = u,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, and $\sin\beta = v,\ -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$

$$\sec\alpha = \sqrt{\tan^2\alpha+1} = \sqrt{u^2+1}; \qquad \cos\alpha = \frac{1}{\sqrt{u^2+1}}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{1}{u^2+1}} = \sqrt{\frac{u^2+1-1}{u^2+1}} = \sqrt{\frac{u^2}{u^2+1}} = \frac{u}{\sqrt{u^2+1}}$$

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-v^2}$$

$$\sin\left(\tan^{-1}u - \sin^{-1}v\right) = \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \frac{u}{\sqrt{u^2+1}}\cdot\sqrt{1-v^2} - \frac{1}{\sqrt{u^2+1}}\cdot v = \frac{u\sqrt{1-v^2}-v}{\sqrt{u^2+1}}$$

72. $\cos\left(\tan^{-1}u + \tan^{-1}v\right)$

Let $\alpha = \tan^{-1}u$ and $\beta = \tan^{-1}v$.

Then $\tan\alpha = u,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, and $\tan\beta = v,\ -\frac{\pi}{2} < \beta < \frac{\pi}{2}$

$$\sec\alpha = \sqrt{\tan^2\alpha+1} = \sqrt{u^2+1}; \qquad \cos\alpha = \frac{1}{\sqrt{u^2+1}}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{1}{u^2+1}} = \sqrt{\frac{u^2+1-1}{u^2+1}} = \sqrt{\frac{u^2}{u^2+1}} = \frac{u}{\sqrt{u^2+1}}$$

$$\sec\beta = \sqrt{\tan^2\beta+1} = \sqrt{v^2+1}; \qquad \cos\beta = \frac{1}{\sqrt{v^2+1}}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\frac{1}{v^2+1}} = \sqrt{\frac{v^2+1-1}{v^2+1}} = \sqrt{\frac{v^2}{v^2+1}} = \frac{v}{\sqrt{v^2+1}}$$

$$\cos\left(\tan^{-1}u + \tan^{-1}v\right) = \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \frac{1}{\sqrt{u^2+1}}\cdot\frac{1}{\sqrt{v^2+1}} - \frac{u}{\sqrt{u^2+1}}\cdot\frac{v}{\sqrt{v^2+1}} = \frac{1-uv}{\sqrt{u^2+1}\sqrt{v^2+1}}$$

73. $\tan\left(\sin^{-1}u - \cos^{-1}v\right)$

Let $\alpha = \sin^{-1}u$ and $\beta = \cos^{-1}v$.

Then $\sin\alpha = u,\ -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, and $\cos\beta = v,\ 0 \le \beta \le \pi$

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-u^2}; \qquad \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{u}{\sqrt{1-u^2}}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-v^2}; \qquad \tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{\sqrt{1-v^2}}{v}$$

$$\tan\left(\sin^{-1}u - \cos^{-1}v\right) = \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{\left(\frac{u}{\sqrt{1-u^2}} - \frac{\sqrt{1-v^2}}{v}\right)}{\left(1+\frac{u}{\sqrt{1-u^2}}\ \frac{\sqrt{1-v^2}}{v}\right)}$$

$$= \frac{\left(\frac{uv-\sqrt{1-u^2}\sqrt{1-v^2}}{v\sqrt{1-u^2}}\right)}{\left(\frac{v\sqrt{1-u^2}+u\sqrt{1-v^2}}{v\sqrt{1-u^2}}\right)} = \frac{uv-\sqrt{1-u^2}\sqrt{1-v^2}}{v\sqrt{1-u^2}+u\sqrt{1-v^2}}$$

74. $\sec\left(\tan^{-1}u + \cos^{-1}v\right)$

Let $\alpha = \tan^{-1}u$ and $\beta = \cos^{-1}v$.

Then $\tan\alpha = u,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, and $\cos\beta = v,\ 0 \le \beta \le \pi$

$$\sec\alpha = \sqrt{\tan^2\alpha+1} = \sqrt{u^2+1}; \qquad \cos\alpha = \frac{1}{\sqrt{u^2+1}}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{1}{u^2+1}} = \sqrt{\frac{u^2+1-1}{u^2+1}} = \sqrt{\frac{u^2}{u^2+1}} = \frac{u}{\sqrt{u^2+1}}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-v^2}$$

$$\sec\left(\tan^{-1}u+\cos^{-1}v\right) = \sec(\alpha+\beta) = \frac{1}{\cos(\alpha+\beta)} = \frac{1}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

$$= \frac{1}{\left(\frac{1}{\sqrt{u^2+1}}\ v - \frac{u}{\sqrt{u^2+1}}\ \sqrt{1-v^2}\right)} = \frac{1}{\left(\frac{v}{\sqrt{u^2+1}} - \frac{u\sqrt{1-v^2}}{\sqrt{u^2+1}}\right)}$$

$$= \frac{1}{\left(\frac{v-u\sqrt{1-v^2}}{\sqrt{u^2+1}}\right)} = \frac{\sqrt{u^2+1}}{v-u\sqrt{1-v^2}}$$

75. Show that $\sin^{-1}v + \cos^{-1}v = \frac{\pi}{2}$.

Let $\alpha = \sin^{-1}v$ and $\beta = \cos^{-1}v$.

Then $\sin\alpha = v = \cos\beta$, and since $\sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right)$, $\cos\left(\frac{\pi}{2} - \alpha\right) = \cos\beta$. If $v \ge 0$, then $0 \le \alpha \le \frac{\pi}{2}$, so that $\left(\frac{\pi}{2} - \alpha\right)$ and β both lie in the interval $\left[0, \frac{\pi}{2}\right]$. If $v < 0$, then $-\frac{\pi}{2} \le \alpha < 0$, so that $\left(\frac{\pi}{2} - \alpha\right)$ and β both lie in the interval $\left[\frac{\pi}{2}, \pi\right]$. Either way, $\cos\left(\frac{\pi}{2} - \alpha\right) = \cos\beta$ implies $\frac{\pi}{2} - \alpha = \beta$, or $\alpha + \beta = \frac{\pi}{2}$. Thus, $\sin^{-1} v + \cos^{-1} v = \frac{\pi}{2}$.

76. Show that $\tan^{-1} v + \cot^{-1} v = \frac{\pi}{2}$.

Let $\alpha = \tan^{-1}\text{v}$ and $\beta = \cot^{-1} v$.

Then $\tan\alpha = v = \cot\beta$, and since $\tan\alpha = \cot\left(\frac{\pi}{2} - \alpha\right)$, $\cot\left(\frac{\pi}{2} - \alpha\right) = \cot\beta$. If $v \ge 0$, then $0 \le \alpha \le \frac{\pi}{2}$, so that $\left(\frac{\pi}{2} - \alpha\right)$ and β both lie in the interval $\left[0, \frac{\pi}{2}\right]$. If $v < 0$, then $-\frac{\pi}{2} \le \alpha < 0$, so that $\left(\frac{\pi}{2} - \alpha\right)$ and β both lie in the interval $\left[\frac{\pi}{2}, \pi\right]$. Either way, $\cot\left(\frac{\pi}{2} - \alpha\right) = \cot\beta$ implies $\frac{\pi}{2} - \alpha = \beta$, or $\alpha + \beta = \frac{\pi}{2}$. Thus, $\tan^{-1} v + \cot^{-1} v = \frac{\pi}{2}$.

77. Show that $\tan^{-1}\left(\frac{1}{v}\right) = \frac{\pi}{2} - \tan^{-1} v$, if $v > 0$.

Let $\alpha = \tan^{-1}\left(\frac{1}{\text{v}}\right)$ and $\beta = \tan^{-1} v$. Because $\frac{1}{v}$ must be defined, $v \ne 0$ and so $\alpha, \beta \ne 0$.

Then $\tan\alpha = \frac{1}{v} = \frac{1}{\tan\beta} = \cot\beta$, and since $\tan\alpha = \cot\left(\frac{\pi}{2} - \alpha\right)$, $\cot\left(\frac{\pi}{2} - \alpha\right) = \cot\beta$.

Because $v > 0$, $0 < \alpha < \frac{\pi}{2}$ and so $\frac{\pi}{2} - \alpha$ and β both lie in the interval $\left(0, \frac{\pi}{2}\right)$. Then, $\cot\left(\frac{\pi}{2} - \alpha\right) = \cot\beta$ implies $\frac{\pi}{2} - \alpha = \beta$ or $\alpha = \frac{\pi}{2} - \beta$. Thus,

$\tan^{-1}\left(\frac{1}{v}\right) = \frac{\pi}{2} - \tan^{-1} v$, if $v > 0$.

78. Show that $\cot^{-1}\left(e^{v}\right) = \tan^{-1}\left(e^{-v}\right)$.

Let $\theta = \tan^{-1}\left(e^{-v}\right)$. Then $\tan\theta = e^{-v}$, so $\cot\theta = \dfrac{1}{e^{-v}} = e^{v}$. Since $0 \le \theta \le \dfrac{\pi}{2}$ (because $e^{-v} > 0$), $\cot^{-1}\left(e^{v}\right) = \cot^{-1}(\cot\theta) = \theta = \tan^{-1}\left(e^{-v}\right)$.

79. $$\sin\left(\sin^{-1} v + \cos^{-1} v\right) = \sin\left(\sin^{-1} v\right)\cos\left(\cos^{-1} v\right) + \cos\left(\sin^{-1} v\right)\sin\left(\cos^{-1} v\right)$$
$$= v\,v + \sqrt{1-v^2}\sqrt{1-v^2} = v^2 + 1 - v^2 = 1$$

80. $$\cos\left(\sin^{-1} v + \cos^{-1} v\right) = \cos\left(\sin^{-1} v\right)\cos\left(\cos^{-1} v\right) - \sin\left(\sin^{-1} v\right)\sin\left(\cos^{-1} v\right)$$
$$= \sqrt{1-v^2}\,v - v\sqrt{1-v^2} = 0$$

81. $$\frac{\sin(x+h) - \sin x}{h} = \frac{\sin x\cos h + \cos x\sin h - \sin x}{h} = \frac{\cos x\sin h - \sin x + \sin x\cos h}{h}$$
$$= \cos x\left(\frac{\sin h}{h}\right) \sin x\left(\frac{1-\cos h}{h}\right)$$

82. $$\frac{\cos(x+h) - \cos x}{h} = \frac{\cos x\cos h - \sin x\sin h - \cos x}{h} = \frac{-\sin x\sin h + \cos x\cos h - \cos x}{h}$$
$$= -\sin x\left(\frac{\sin h}{h}\right) - \cos x\left(\frac{1-\cos h}{h}\right)$$

83. $$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{\tan\left(\frac{\pi}{2}\right) - \tan\theta}{1 + \tan\left(\frac{\pi}{2}\right)\tan\theta}$$ This is impossible because $\tan\left(\frac{\pi}{2}\right)$ is undefined.

$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

84. If $\tan\alpha = x + 1$ and $\tan\beta = x - 1$, then
$$2\cot(\alpha - \beta) = 2\,\frac{1}{\tan(\alpha - \beta)} = \frac{2}{\left(\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\right)} = \frac{2\left(1 + \tan\alpha\tan\beta\right)}{\tan\alpha - \tan\beta}$$
$$= \frac{2\left(1 + (x+1)(x-1)\right)}{x + 1 - (x-1)} = \frac{2\left(1 + (x^2 - 1)\right)}{x + 1 - x + 1} = \frac{2\left(1 + x^2 - 1\right)}{2} = x^2$$

85. $$\tan\theta = \tan\left(\theta_2 - \theta_1\right) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2\tan\theta_1} = \frac{m_2 - m_1}{1 + m_2 m_1}$$

86. $\sin(\alpha-\theta)\sin(\beta-\theta)\sin(\gamma-\theta)$

$$=\left(\sin\alpha\cos\theta-\cos\alpha\sin\theta\right)\left(\sin\beta\cos\theta-\cos\beta\sin\theta\right)\left(\sin\gamma\cos\theta-\cos\gamma\sin\theta\right)$$

$$=\sin\theta\left(\sin\alpha\left(\frac{\cos\theta}{\sin\theta}\right)-\cos\alpha\right)\sin\theta\left(\sin\beta\left(\frac{\cos\theta}{\sin\theta}\right)-\cos\beta\right)\sin\theta\left(\sin\gamma\left(\frac{\cos\theta}{\sin\theta}\right)-\cos\gamma\right)$$

$$=\sin^3\theta\left(\sin\alpha\left(\frac{\cos\theta}{\sin\theta}-\frac{\cos\alpha}{\sin\alpha}\right)\right)\left(\sin\beta\left(\frac{\cos\theta}{\sin\theta}-\frac{\cos\beta}{\sin\beta}\right)\right)\left(\sin\gamma\left(\frac{\cos\theta}{\sin\theta}-\frac{\cos\gamma}{\sin\gamma}\right)\right)$$

$$=\sin^3\theta\left(\sin\alpha\left(\cot\theta-\cot\alpha\right)\right)\left(\sin\beta\left(\cot\theta-\cot\beta\right)\right)\left(\sin\gamma\left(\cot\theta-\cot\gamma\right)\right)$$

$$=\sin^3\theta\sin\alpha\sin\beta\sin\gamma\left(\cot\beta+\cot\gamma\right)\left(\cot\alpha+\cot\gamma\right)\left(\cot\alpha+\cot\beta\right)$$

$$=\sin^3\theta\sin\alpha\sin\beta\sin\gamma\left(\frac{\cos\beta}{\sin\beta}+\frac{\cos\gamma}{\sin\gamma}\right)\left(\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\gamma}{\sin\gamma}\right)\left(\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\beta}{\sin\beta}\right)$$

$$=\sin^3\theta\sin\alpha\sin\beta\sin\gamma\left(\frac{\sin(\gamma+\beta)}{\sin\beta\sin\gamma}\right)\left(\frac{\sin(\gamma+\alpha)}{\sin\alpha\sin\gamma}\right)\left(\frac{\sin(\beta+\alpha)}{\sin\alpha\sin\beta}\right)$$

$$=\sin^3\theta\sin\alpha\sin\beta\sin\gamma\left(\frac{\sin(180^\circ-\alpha)}{\sin\beta\sin\gamma}\right)\left(\frac{\sin(180^\circ-\beta)}{\sin\alpha\sin\gamma}\right)\left(\frac{\sin(180^\circ-\gamma)}{\sin\alpha\sin\beta}\right)$$

$$=\sin^3\theta\sin\alpha\sin\beta\sin\gamma\left(\frac{\sin\alpha}{\sin\beta\sin\gamma}\right)\left(\frac{\sin\beta}{\sin\alpha\sin\gamma}\right)\left(\frac{\sin\gamma}{\sin\alpha\sin\beta}\right)$$

$$=\sin^3\theta$$

87. The first step in the derivation

$$\tan\left(\theta+\frac{\pi}{2}\right)=\frac{\tan\theta+\tan\left(\frac{\pi}{2}\right)}{1-\tan\theta\tan\left(\frac{\pi}{2}\right)}$$

is impossible because $\tan\left(\frac{\pi}{2}\right)$ is undefined.