Chapter 8

Analytic Trigonometry

8.5 Double-Angle and Half-Angle Formulas

1. $\sin\theta = \frac{3}{5}, \quad 0 < \theta < \frac{\pi}{2};$ thus, $0 < \frac{\theta}{2} < \frac{\pi}{4}$ $\quad \frac{\theta}{2}$ is in quadrant I.

   $y = 3, \quad r = 5$

   $x^2 + 3^2 = 5^2, \quad x > 0 \quad \rightarrow \quad x^2 = 25 - 9 = 16, \quad x > 0 \quad \rightarrow \quad x = 4$

   $\cos\theta = \frac{4}{5}$

   (a) $\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$

   (b) $\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$

   (c) $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{\left(1-\frac{4}{5}\right)}{2}} = \sqrt{\frac{\left(\frac{1}{5}\right)}{2}} = \sqrt{\frac{1}{10}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$

   (d) $\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{\left(1+\frac{4}{5}\right)}{2}} = \sqrt{\frac{\left(\frac{9}{5}\right)}{2}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$

2. $\cos\theta = \frac{3}{5}, \quad 0 < \theta < \frac{\pi}{2};$ thus, $0 < \frac{\theta}{2} < \frac{\pi}{4}$ $\quad \frac{\theta}{2}$ is in quadrant I.

   $x = 3, \quad r = 5$

   $3^2 + y^2 = 5^2, \quad y > 0 \quad \rightarrow \quad y^2 = 25 - 9 = 16, \quad y > 0 \quad \rightarrow \quad y = 4$

   $\sin\theta = \frac{4}{5}$

   (a) $\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}$

   (b) $\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$

   (c) $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{\left(1-\frac{3}{5}\right)}{2}} = \sqrt{\frac{\left(\frac{2}{5}\right)}{2}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

   (d) $\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{\left(1+\frac{3}{5}\right)}{2}} = \sqrt{\frac{\left(\frac{8}{5}\right)}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

3. $\tan\theta = \frac{4}{3},\quad \pi < \theta < \frac{3\pi}{2};\qquad$ thus, $\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}\qquad \frac{\theta}{2}$ is in quadrant II.

$$x = -3,\quad y = -4$$

$$r^2 = (-3)^2 + (-4)^2 = 9 + 16 = 25 \quad\to\quad r = 5$$

$$\sin\theta = -\frac{4}{5},\quad \cos\theta = -\frac{3}{5}$$

(a) $$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(-\frac{4}{5}\right)\left(-\frac{3}{5}\right) = \frac{24}{25}$$

(b) $$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(-\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

(c) $$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{\left(\frac{8}{5}\right)}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

(d) $$\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}} = -\sqrt{\frac{\left(\frac{2}{5}\right)}{2}} = -\sqrt{\frac{1}{5}} = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

4. $\tan\theta = \frac{1}{2},\quad \pi < \theta < \frac{3\pi}{2};\qquad$ thus, $\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}\qquad \frac{\theta}{2}$ is in quadrant II.

$$x = -2,\quad y = -1$$

$$r^2 = (-2)^2 + (-1)^2 = 4 + 1 = 5 \quad\to\quad r = \sqrt{5}$$

$$\sin\theta = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5},\quad \cos\theta = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

(a) $$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(-\frac{\sqrt{5}}{5}\right)\left(-\frac{2\sqrt{5}}{5}\right) = \frac{4}{5}$$

(b) $$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(-\frac{2\sqrt{5}}{5}\right)^2 - \left(-\frac{\sqrt{5}}{5}\right)^2 = \frac{20}{25} - \frac{5}{25} = \frac{15}{25} = \frac{3}{5}$$

(c) $$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\left(-\frac{2\sqrt{5}}{5}\right)}{2}} = \sqrt{\frac{\left(\frac{5+2\sqrt{5}}{5}\right)}{2}} = \sqrt{\frac{5+2\sqrt{5}}{10}}$$

(d) $$\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\left(-\frac{2\sqrt{5}}{5}\right)}{2}} = -\sqrt{\frac{\left(\frac{5-2\sqrt{5}}{5}\right)}{2}} = -\sqrt{\frac{5-2\sqrt{5}}{10}}$$

5. $\cos\theta = -\dfrac{\sqrt{6}}{3},\ \dfrac{\pi}{2} < \theta < \pi;$ thus, $\dfrac{\pi}{4} < \dfrac{\theta}{2} < \dfrac{\pi}{2}$ $\dfrac{\theta}{2}$ is in quadrant I.

$x = -\sqrt{6},\ r = 3$

$(-\sqrt{6})^2 + y^2 = 3^2 \rightarrow y^2 = 9 - 6 = 3 \rightarrow y = \sqrt{3}$

$\sin\theta = \dfrac{\sqrt{3}}{3}$

(a) $\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\dfrac{\sqrt{3}}{3}\right)\left(-\dfrac{\sqrt{6}}{3}\right) = -\dfrac{2\sqrt{18}}{9} = -\dfrac{6\sqrt{2}}{9} = -\dfrac{2\sqrt{2}}{3}$

(b) $\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(-\dfrac{\sqrt{6}}{3}\right)^2 - \left(\dfrac{\sqrt{3}}{3}\right)^2 = \dfrac{6}{9} - \dfrac{3}{9} = \dfrac{3}{9} = \dfrac{1}{3}$

(c) $\sin\left(\dfrac{\theta}{2}\right) = \sqrt{\dfrac{1-\cos\theta}{2}} = \sqrt{\dfrac{1-\left(-\dfrac{\sqrt{6}}{3}\right)}{2}} = \sqrt{\dfrac{\left(\dfrac{3+\sqrt{6}}{3}\right)}{2}} = \sqrt{\dfrac{3+\sqrt{6}}{6}}$

(d) $\cos\left(\dfrac{\theta}{2}\right) = \sqrt{\dfrac{1+\cos\theta}{2}} = \sqrt{\dfrac{1+\left(-\dfrac{\sqrt{6}}{3}\right)}{2}} = \sqrt{\dfrac{\left(\dfrac{3-\sqrt{6}}{3}\right)}{2}} = \sqrt{\dfrac{3-\sqrt{6}}{6}}$

6. $\sin\theta = -\dfrac{\sqrt{3}}{3},\ \dfrac{3\pi}{2} < \theta < 2\pi;$ thus, $\dfrac{3\pi}{4} < \dfrac{\theta}{2} < \pi$ $\dfrac{\theta}{2}$ is in quadrant II.

$y = -\sqrt{3},\ r = 3$

$x^2 + \left(-\sqrt{3}\right)^2 = 3^2 \rightarrow x^2 = 9 - 3 = 6 \rightarrow x = \sqrt{6}$

$\cos\theta = \dfrac{\sqrt{6}}{3}$

(a) $\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(-\dfrac{\sqrt{3}}{3}\right)\left(\dfrac{\sqrt{6}}{3}\right) = -\dfrac{2\sqrt{18}}{9} = -\dfrac{6\sqrt{2}}{9} = -\dfrac{2\sqrt{2}}{3}$

(b) $\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\dfrac{\sqrt{6}}{3}\right)^2 - \left(-\dfrac{\sqrt{3}}{3}\right)^2 = \dfrac{6}{9} - \dfrac{3}{9} = \dfrac{3}{9} = \dfrac{1}{3}$

(c) $\sin\left(\dfrac{\theta}{2}\right) = \sqrt{\dfrac{1-\cos\theta}{2}} = \sqrt{\dfrac{\left(1-\dfrac{\sqrt{6}}{3}\right)}{2}} = \sqrt{\dfrac{\left(\dfrac{3-\sqrt{6}}{3}\right)}{2}} = \sqrt{\dfrac{3-\sqrt{6}}{6}}$

(d) $\cos\left(\dfrac{\theta}{2}\right) = -\sqrt{\dfrac{1+\cos\theta}{2}} = -\sqrt{\dfrac{\left(1+\dfrac{\sqrt{6}}{3}\right)}{2}} = -\sqrt{\dfrac{\left(\dfrac{3+\sqrt{6}}{3}\right)}{2}} = -\sqrt{\dfrac{3+\sqrt{6}}{6}}$

7. $\sec\theta = 3,\quad \sin\theta > 0;\quad 0 < \theta < \frac{\pi}{2};\quad$ thus, $0 < \frac{\theta}{2} < \frac{\pi}{4}\quad \frac{\theta}{2}$ is in quadrant I.

$x = 1,\quad r = 3$

$1^2 + y^2 = 3^2 \quad\to\quad y^2 = 9 - 1 = 8 \quad\to\quad y = 2\sqrt{2}$

$\sin\theta = \frac{2\sqrt{2}}{3},\quad \cos\theta = \frac{1}{3}$

(a) $\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{1}{3}\right) = \frac{4\sqrt{2}}{9}$

(b) $\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\frac{1}{3}\right)^2 - \left(\frac{2\sqrt{2}}{3}\right)^2 = \frac{1}{9} - \frac{8}{9} = -\frac{7}{9}$

(c) $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{\left(1-\frac{1}{3}\right)}{2}} = \sqrt{\frac{\left(\frac{2}{3}\right)}{2}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

(d) $\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{\left(1+\frac{1}{3}\right)}{2}} = \sqrt{\frac{\left(\frac{4}{3}\right)}{2}} = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$

8. $\csc\theta = -\sqrt{5},\quad \cos\theta < 0 \quad\to\quad \pi < \theta < \frac{3\pi}{2};$

thus, $\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}\quad \frac{\theta}{2}$ is in quadrant II.

$\sin\theta = \frac{-1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$

$r = \sqrt{5},\quad y = -1$

$x^2 + (-1)^2 = \left(\sqrt{5}\right)^2 \quad\to\quad x^2 = 5 - 1 = 4 \quad\to\quad x = -2$

$\cos\theta = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$

(a) $\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(-\frac{\sqrt{5}}{5}\right)\left(-\frac{2\sqrt{5}}{5}\right) = \frac{4}{5}$

(b) $\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(-\frac{2\sqrt{5}}{5}\right)^2 - \left(-\frac{\sqrt{5}}{5}\right)^2 = \frac{20}{25} - \frac{5}{25} = \frac{15}{25} = \frac{3}{5}$

(c) $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{\left(1-\left(-\frac{2\sqrt{5}}{5}\right)\right)}{2}} = \sqrt{\frac{\left(\frac{5+2\sqrt{5}}{5}\right)}{2}} = \sqrt{\frac{5+2\sqrt{5}}{10}}$

(d) $\cos\left(\frac{\theta}{2}\right)=-\sqrt{\frac{1+\cos\theta}{2}}=-\sqrt{\frac{\left(1+\left(-\frac{2\sqrt{5}}{5}\right)\right)}{2}}=-\sqrt{\frac{\left(\frac{5-2\sqrt{5}}{5}\right)}{2}}=-\sqrt{\frac{5-2\sqrt{5}}{10}}$

9. $\cot\theta=-2,\quad \sec\theta<0;\quad \frac{\pi}{2}<\theta<\pi;\quad$ thus, $\frac{\pi}{4}<\frac{\theta}{2}<\frac{\pi}{2}\qquad \frac{\theta}{2}$ is in quadrant I.

$x=-2,\ y=1$

$r^2=(-2)^2+1^2=4+1=5\quad\rightarrow\quad r=\sqrt{5}$

$\sin\theta=\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5},\quad \cos\theta=-\frac{2}{\sqrt{5}}=-\frac{2\sqrt{5}}{5}$

(a) $\sin(2\theta)=2\sin\theta\cos\theta=2\left(\frac{\sqrt{5}}{5}\right)\left(-\frac{2\sqrt{5}}{5}\right)=-\frac{20}{25}=-\frac{4}{5}$

(b) $\cos(2\theta)=\cos^2\theta-\sin^2\theta=\left(-\frac{2\sqrt{5}}{5}\right)^2-\left(\frac{\sqrt{5}}{5}\right)^2=\frac{20}{25}-\frac{5}{25}=\frac{15}{25}=\frac{3}{5}$

(c) $\sin\left(\frac{\theta}{2}\right)=\sqrt{\frac{1-\cos\theta}{2}}=\sqrt{\frac{\left(1-\left(-\frac{2\sqrt{5}}{5}\right)\right)}{2}}=\sqrt{\frac{\left(\frac{5+2\sqrt{5}}{5}\right)}{2}}=\sqrt{\frac{5+2\sqrt{5}}{10}}$

(d) $\cos\left(\frac{\theta}{2}\right)=\sqrt{\frac{1+\cos\theta}{2}}=\sqrt{\frac{\left(1+\left(-\frac{2\sqrt{5}}{5}\right)\right)}{2}}=\sqrt{\frac{\left(\frac{5-2\sqrt{5}}{5}\right)}{2}}=\sqrt{\frac{5-2\sqrt{5}}{10}}$

10. $\sec\theta=2,\quad \csc\theta<0;\quad \frac{3\pi}{2}<\theta<2\pi;\quad$ thus, $\frac{3\pi}{4}<\frac{\theta}{2}<\pi\qquad \frac{\theta}{2}$ is in quadrant II.

$x=1,\ r=2$

$1^2+y^2=2^2\quad\rightarrow\quad y^2=4-1=3\quad\rightarrow\quad y=-\sqrt{3}$

$\sin\theta=-\frac{\sqrt{3}}{2},\quad \cos\theta=\frac{1}{2}$

(a) $\sin(2\theta)=2\sin\theta\cos\theta=2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)=-\frac{\sqrt{3}}{2}$

(b) $\cos(2\theta)=\cos^2\theta-\sin^2\theta=\left(\frac{1}{2}\right)^2-\left(-\frac{\sqrt{3}}{2}\right)^2=\frac{1}{4}-\frac{3}{4}=-\frac{1}{2}$

(c) $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{\left(1-\frac{1}{2}\right)}{2}} = \sqrt{\frac{\left(\frac{1}{2}\right)}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

(d) $\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{\left(1+\frac{1}{2}\right)}{2}} = -\sqrt{\frac{\left(\frac{3}{2}\right)}{2}} = -\sqrt{\frac{3}{4}} = -\frac{\sqrt{3}}{2}$

11. $\tan\theta = -3$, $\sin\theta < 0$; $\frac{3\pi}{2} < \theta < 2\pi$; thus, $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$ or $\frac{\theta}{2}$ is in quadrant II.

$x = 1$, $y = -3$

$r^2 = 1^2 + (-3)^2 = 1 + 9 = 10 \quad \rightarrow \quad r = \sqrt{10}$

$\sin\theta = \frac{-3}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$, $\cos\theta = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$

(a) $\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(-\frac{3\sqrt{10}}{10}\right)\left(\frac{\sqrt{10}}{10}\right) = -\frac{60}{100} = -\frac{3}{5}$

(b) $\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\frac{\sqrt{10}}{10}\right)^2 - \left(-\frac{3\sqrt{10}}{10}\right)^2 = \frac{10}{100} - \frac{90}{100} = -\frac{80}{100} = -\frac{4}{5}$

(c) $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{\left(1-\frac{\sqrt{10}}{10}\right)}{2}} = \sqrt{\frac{\left(\frac{10-\sqrt{10}}{10}\right)}{2}} = \sqrt{\frac{10-\sqrt{10}}{20}} = \frac{1}{2}\sqrt{\frac{10-\sqrt{10}}{5}}$

(d) $\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{\left(1+\frac{\sqrt{10}}{10}\right)}{2}} = -\sqrt{\frac{\left(\frac{10+\sqrt{10}}{10}\right)}{2}} = -\sqrt{\frac{10+\sqrt{10}}{20}}$

$= -\frac{1}{2}\sqrt{\frac{10+\sqrt{10}}{5}}$

12. $\cot\theta = 3$, $\cos\theta < 0$; $\pi < \theta < \frac{3\pi}{2}$; thus, $\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$ $\frac{\theta}{2}$ is in quadrant II.

$x = -3$, $y = -1$

$r^2 = (-3)^2 + (-1)^2 = 9 + 1 = 10 \quad \rightarrow \quad r = \sqrt{10}$

$\sin\theta = -\frac{1}{\sqrt{10}} = -\frac{\sqrt{10}}{10}$, $\cos\theta = -\frac{3}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$

(a) $\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(-\frac{\sqrt{10}}{10}\right)\left(-\frac{3\sqrt{10}}{10}\right) = \frac{60}{100} = \frac{3}{5}$

(b) $\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(-\dfrac{3\sqrt{10}}{10}\right)^2 - \left(-\dfrac{\sqrt{10}}{10}\right)^2 = \dfrac{90}{100} - \dfrac{10}{100} = \dfrac{80}{100} = \dfrac{4}{5}$

(c) $\sin\left(\dfrac{\theta}{2}\right) = \sqrt{\dfrac{1-\cos\theta}{2}} = \sqrt{\dfrac{\left(1-\left(-\dfrac{3\sqrt{10}}{10}\right)\right)}{2}} = \sqrt{\dfrac{\left(\dfrac{10+3\sqrt{10}}{10}\right)}{2}} = \sqrt{\dfrac{10+3\sqrt{10}}{20}} = \dfrac{\sqrt{10+3\sqrt{10}}}{2\sqrt{5}}$

(d) $\cos\left(\dfrac{\theta}{2}\right) = -\sqrt{\dfrac{1+\cos\theta}{2}} = -\sqrt{\dfrac{\left(1+\left(-\dfrac{3\sqrt{10}}{10}\right)\right)}{2}} = -\sqrt{\dfrac{\left(\dfrac{10-3\sqrt{10}}{10}\right)}{2}} = -\sqrt{\dfrac{10-3\sqrt{10}}{20}}$

$= -\dfrac{\sqrt{10-3\sqrt{10}}}{2\sqrt{5}}$

13. $\sin(22.5^\circ) = \sin\left(\dfrac{45^\circ}{2}\right) = \sqrt{\dfrac{1-\cos(45^\circ)}{2}} = \sqrt{\dfrac{\left(1-\dfrac{\sqrt{2}}{2}\right)}{2}} = \sqrt{\dfrac{2-\sqrt{2}}{4}} = \dfrac{\sqrt{2-\sqrt{2}}}{2}$

14. $\cos(22.5^\circ) = \cos\left(\dfrac{45^\circ}{2}\right) = \sqrt{\dfrac{1+\cos(45^\circ)}{2}} = \sqrt{\dfrac{\left(1+\dfrac{\sqrt{2}}{2}\right)}{2}} = \sqrt{\dfrac{2+\sqrt{2}}{4}} = \dfrac{\sqrt{2+\sqrt{2}}}{2}$

15. $\tan\left(\dfrac{7\pi}{8}\right) = \tan\left(\dfrac{\left(\dfrac{7\pi}{4}\right)}{2}\right) = -\sqrt{\dfrac{\left(1-\cos\left(\dfrac{7\pi}{4}\right)\right)}{\left(1+\cos\left(\dfrac{7\pi}{4}\right)\right)}} = -\sqrt{\dfrac{\left(1-\dfrac{\sqrt{2}}{2}\right)}{\left(1+\dfrac{\sqrt{2}}{2}\right)}} = -\sqrt{\left(\dfrac{2-\sqrt{2}}{2+\sqrt{2}}\right)\left(\dfrac{2-\sqrt{2}}{2-\sqrt{2}}\right)}$

$= -\sqrt{\dfrac{6-4\sqrt{2}}{2}} = -\sqrt{3-2\sqrt{2}}$

16. $\tan\left(\dfrac{9\pi}{8}\right) = \tan\left(\dfrac{\left(\dfrac{9\pi}{4}\right)}{2}\right) = \sqrt{\dfrac{\left(1-\cos\left(\dfrac{9\pi}{4}\right)\right)}{\left(1+\cos\left(\dfrac{9\pi}{4}\right)\right)}} = \sqrt{\dfrac{\left(1-\dfrac{\sqrt{2}}{2}\right)}{\left(1+\dfrac{\sqrt{2}}{2}\right)}} = \sqrt{\left(\dfrac{2-\sqrt{2}}{2+\sqrt{2}}\right)\left(\dfrac{2-\sqrt{2}}{2-\sqrt{2}}\right)}$

$= \sqrt{\dfrac{6-4\sqrt{2}}{2}} = \sqrt{3-2\sqrt{2}}$

17. $\cos(165^\circ) = \cos\left(\dfrac{330^\circ}{2}\right) = -\sqrt{\dfrac{1+\cos(330^\circ)}{2}} = -\sqrt{\dfrac{\left(1+\dfrac{\sqrt{3}}{2}\right)}{2}} = -\sqrt{\dfrac{2+\sqrt{3}}{4}} = -\dfrac{\sqrt{2+\sqrt{3}}}{2}$

18. $\sin(195^\circ) = \sin\left(\dfrac{390^\circ}{2}\right) = -\sqrt{\dfrac{1-\cos(390^\circ)}{2}} = -\sqrt{\dfrac{\left(1-\dfrac{\sqrt{3}}{2}\right)}{2}} = -\sqrt{\dfrac{2-\sqrt{3}}{4}} = -\dfrac{\sqrt{2-\sqrt{3}}}{2}$

19.

$$\sec\left(\frac{15\pi}{8}\right) = \frac{1}{\cos\left(\frac{15\pi}{8}\right)} = \frac{1}{\cos\left(\frac{\left(\frac{15\pi}{4}\right)}{2}\right)} = \frac{1}{\sqrt{\dfrac{\left(1+\cos\left(\frac{15\pi}{4}\right)\right)}{2}}} = \frac{1}{\sqrt{\dfrac{\left(1+\frac{\sqrt{2}}{2}\right)}{2}}} = \frac{1}{\sqrt{\dfrac{2+\sqrt{2}}{4}}}$$

$$= \left(\frac{2}{\sqrt{2+\sqrt{2}}}\right)\left(\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2+\sqrt{2}}}\right) = \left(\frac{2\sqrt{2+\sqrt{2}}}{2+\sqrt{2}}\right)\left(\frac{2-\sqrt{2}}{2-\sqrt{2}}\right)$$

$$= \frac{2\left(2-\sqrt{2}\right)\sqrt{2+\sqrt{2}}}{2} = \left(2-\sqrt{2}\right)\sqrt{2+\sqrt{2}}$$

20.

$$\csc\left(\frac{7\pi}{8}\right) = \frac{1}{\sin\left(\frac{7\pi}{8}\right)} = \frac{1}{\sin\left(\frac{\left(\frac{7\pi}{4}\right)}{2}\right)} = \frac{1}{\sqrt{\dfrac{\left(1-\cos\left(\frac{7\pi}{4}\right)\right)}{2}}} = \frac{1}{\sqrt{\dfrac{\left(1-\frac{\sqrt{2}}{2}\right)}{2}}} = \frac{1}{\sqrt{\dfrac{2-\sqrt{2}}{4}}}$$

$$= \left(\frac{2}{\sqrt{2-\sqrt{2}}}\right)\left(\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2-\sqrt{2}}}\right) = \left(\frac{2\sqrt{2-\sqrt{2}}}{2-\sqrt{2}}\right)\left(\frac{2+\sqrt{2}}{2+\sqrt{2}}\right)$$

$$= \frac{2\left(2+\sqrt{2}\right)\sqrt{2-\sqrt{2}}}{2} = \left(2+\sqrt{2}\right)\sqrt{2-\sqrt{2}}$$

21. $\sin\left(-\dfrac{\pi}{8}\right) = \sin\left(\dfrac{\left(-\frac{\pi}{4}\right)}{2}\right) = -\sqrt{\dfrac{1-\cos\left(-\frac{\pi}{4}\right)}{2}} = -\sqrt{\dfrac{\left(1-\frac{\sqrt{2}}{2}\right)}{2}} = -\sqrt{\dfrac{2-\sqrt{2}}{4}} = -\dfrac{\sqrt{2-\sqrt{2}}}{2}$

22. $\cos\left(-\dfrac{3\pi}{8}\right) = \cos\left(\dfrac{\left(-\frac{3\pi}{4}\right)}{2}\right) = \sqrt{\dfrac{1+\cos\left(-\frac{3\pi}{4}\right)}{2}} = \sqrt{\dfrac{\left(1+\left(-\frac{\sqrt{2}}{2}\right)\right)}{2}} = \sqrt{\dfrac{2-\sqrt{2}}{4}} = \dfrac{\sqrt{2-\sqrt{2}}}{2}$

23. $\sin^4\theta = \left(\sin^2\theta\right)^2 = \left(\dfrac{1-\cos(2\theta)}{2}\right)^2 = \dfrac{1}{4}\left(1-2\cos(2\theta)+\cos^2(2\theta)\right)$

$$= \frac{1}{4}-\frac{1}{2}\cos(2\theta)+\frac{1}{4}\cos^2(2\theta) = \frac{1}{4}-\frac{1}{2}\cos(2\theta)+\frac{1}{4}\left(\frac{1+\cos(4\theta)}{2}\right)$$

$$= \frac{1}{4}-\frac{1}{2}\cos(2\theta)+\frac{1}{8}+\frac{1}{8}\cos(4\theta) = \frac{3}{8}-\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta)$$

24. $\cos(3\theta) = \cos(2\theta+\theta) = \cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta$

$$= \left(2\cos^2\theta - 1\right)\cos\theta - 2\sin\theta\cos\theta\sin\theta = 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$

$$= 2\cos^3\theta - \cos\theta - 2\left(1-\cos^2\theta\right)\cos\theta = 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$$

$$= 4\cos^3\theta - 3\cos\theta$$

25. $\sin(4\theta) = \sin(2(2\theta)) = 2\sin(2\theta)\cos(2\theta) = 2(2\sin\theta\cos\theta)\left(1-2\sin^2\theta\right)$

$$= \cos\theta\left(4\sin\theta - 8\sin^3\theta\right)$$

26. $\cos(4\theta) = \cos(2(2\theta)) = 2\cos^2(2\theta) - 1 = 2\left(2\cos^2\theta - 1\right)^2 - 1$

$$= 2\left(4\cos^4\theta - 4\cos^2\theta + 1\right) - 1 = 8\cos^4\theta - 8\cos^2\theta + 2 - 1$$

$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

27. Use the result of problem 25 to help solve the problem:

$$\sin(5\theta) = \sin(4\theta+\theta) = \sin(4\theta)\cos\theta + \cos(4\theta)\sin\theta$$

$$= \cos\theta\left(4\sin\theta - 8\sin^3\theta\right)\cos\theta + \cos(2(2\theta))\sin\theta$$

$$= \cos^2\theta\left(4\sin\theta - 8\sin^3\theta\right) + \left(1-2\sin^2(2\theta)\right)\sin\theta$$

$$= \left(1-\sin^2\theta\right)\left(4\sin\theta - 8\sin^3\theta\right) + \sin\theta\left(1-2(2\sin\theta\cos\theta)^2\right)$$

$$= 4\sin\theta - 12\sin^3\theta + 8\sin^5\theta + \sin\theta\left(1-8\sin^2\theta\cos^2\theta\right)$$

$$= 4\sin\theta - 12\sin^3\theta + 8\sin^5\theta + \sin\theta - 8\sin^3\theta\left(1-\sin^2\theta\right)$$

$$= 5\sin\theta - 12\sin^3\theta + 8\sin^5\theta - 8\sin^3\theta + 8\sin^5\theta$$

$$= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$$

28. Use the result from problems 25 and 26 to help solve the problem:

$$\begin{aligned}
\cos(5\theta) &= \cos(4\theta+\theta) = \cos(4\theta)\cos\theta - \sin(4\theta)\sin\theta \\
&= \left(8\cos^4\theta - 8\cos^2\theta + 1\right)\cos\theta - \left(\cos\theta\left(4\sin\theta - 8\sin^3\theta\right)\sin\theta\right) \\
&= 8\cos^5\theta - 8\cos^3\theta + \cos\theta - 4\cos\theta\sin^2\theta + 8\cos\theta\sin^4\theta \\
&= 8\cos^5\theta - 8\cos^3\theta + \cos\theta - 4\cos\theta(1-\cos^2\theta) + 8\cos\theta(1-\cos^2\theta)^2 \\
&= 8\cos^5\theta - 8\cos^3\theta + \cos\theta - 4\cos\theta + 4\cos^3\theta + 8\cos\theta(1-2\cos^2\theta+\cos^4\theta) \\
&= 8\cos^5\theta - 4\cos^3\theta - 3\cos\theta + 8\cos\theta - 16\cos^3\theta + 8\cos^5\theta \\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta
\end{aligned}$$

29. $\cos^4\theta - \sin^4\theta = \left(\cos^2\theta + \sin^2\theta\right)\left(\cos^2\theta - \sin^2\theta\right) = 1\ \cos(2\theta) = \cos(2\theta)$

30. $$\frac{\cot\theta - \tan\theta}{\cot\theta + \tan\theta} = \frac{\left(\dfrac{\cos\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta}\right)}{\left(\dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}\right)} = \frac{\left(\dfrac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}\right)}{\left(\dfrac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}\right)} = \frac{\cos(2\theta)}{\sin\theta\cos\theta}\ \ \frac{\sin\theta\cos\theta}{1} = \cos(2\theta)$$

31. $$\cot(2\theta) = \frac{1}{\tan(2\theta)} = \frac{1}{\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)} = \frac{1-\tan^2\theta}{2\tan\theta} = \frac{\left(1 - \dfrac{1}{\cot^2\theta}\right)}{\left(\dfrac{2}{\cot\theta}\right)} = \frac{\left(\dfrac{\cot^2\theta - 1}{\cot^2\theta}\right)}{\left(\dfrac{2}{\cot\theta}\right)} = \frac{\cot^2\theta - 1}{2\cot\theta}$$

32. $$\cot(2\theta) = \frac{1}{\tan(2\theta)} = \frac{1}{\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)} = \frac{1-\tan^2\theta}{2\tan\theta} = \frac{1}{2}\ \ \cot\theta\left(1-\tan^2\theta\right) = \frac{1}{2}\ \ \left(\cot\theta - \tan\theta\right)$$

33. $$\sec(2\theta) = \frac{1}{\cos(2\theta)} = \frac{1}{2\cos^2\theta - 1} = \frac{1}{\left(\dfrac{2}{\sec^2\theta} - 1\right)} = \frac{1}{\left(\dfrac{2-\sec^2\theta}{\sec^2\theta}\right)} = \frac{\sec^2\theta}{2-\sec^2\theta}$$

34. $$\csc(2\theta) = \frac{1}{\sin(2\theta)} = \frac{1}{2\sin\theta\cos\theta} = \frac{1}{2}\ \ \csc\theta\sec\theta = \frac{1}{2}\ \ \sec\theta\csc\theta$$

35. $\cos^2(2\theta) - \sin^2(2\theta) = \cos\left(2(2\theta)\right) = \cos(4\theta)$

36. $(4\sin\theta\cos\theta)(1-2\sin^2\theta) = \left(2\sin(2\theta)\right)(\cos 2\theta) = \sin(4\theta)$

37. $$\frac{\cos(2\theta)}{1+\sin(2\theta)} = \frac{\cos^2\theta - \sin^2\theta}{1+2\sin\theta\cos\theta} = \frac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta}$$

$$= \frac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{(\cos\theta + \sin\theta)(\cos\theta + \sin\theta)} = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \frac{\left(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\sin\theta}\right)}{\left(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\sin\theta}\right)} = \frac{\cot\theta - 1}{\cot\theta + 1}$$

38. $$\sin^2\theta\cos^2\theta = \frac{1}{4}(2\sin\theta\cos\theta)^2 = \frac{1}{4}\left(\sin(2\theta)\right)^2 = \frac{1}{4}\left(\frac{1-\cos(4\theta)}{2}\right) = \frac{1}{8}\left(1-\cos(4\theta)\right)$$

39. $$\sec^2\left(\frac{\theta}{2}\right) = \frac{1}{\cos^2\left(\frac{\theta}{2}\right)} = \frac{1}{\left(\frac{1+\cos\theta}{2}\right)} = \frac{2}{1+\cos\theta}$$

40. $$\csc^2\left(\frac{\theta}{2}\right) = \frac{1}{\sin^2\left(\frac{\theta}{2}\right)} = \frac{1}{\left(\frac{1-\cos\theta}{2}\right)} = \frac{2}{1-\cos\theta}$$

41. $$\cot^2\left(\frac{\theta}{2}\right) = \frac{1}{\tan^2\left(\frac{\theta}{2}\right)} = \frac{1}{\left(\frac{1-\cos\theta}{1+\cos\theta}\right)} = \frac{1+\cos\theta}{1-\cos\theta} = \frac{\left(1+\frac{1}{\sec\theta}\right)}{\left(1-\frac{1}{\sec\theta}\right)} = \frac{\left(\frac{\sec\theta+1}{\sec\theta}\right)}{\left(\frac{\sec\theta-1}{\sec\theta}\right)} = \frac{\sec\theta+1}{\sec\theta-1}$$

42. $$\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \csc\theta - \cot\theta$$

43. $$\frac{\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)}{\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)} = \frac{\left(1-\frac{1-\cos\theta}{1+\cos\theta}\right)}{\left(1+\frac{1-\cos\theta}{1+\cos\theta}\right)} = \frac{\left(\frac{1+\cos\theta-(1-\cos\theta)}{1+\cos\theta}\right)}{\left(\frac{1+\cos\theta+1-\cos\theta}{1+\cos\theta}\right)} = \frac{2\cos\theta}{2} = \cos\theta$$

44. $$\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} = \frac{(\sin\theta+\cos\theta)\left(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta\right)}{\sin\theta + \cos\theta}$$

$$= 1 - \sin\theta\cos\theta = 1 - \frac{1}{2}\sin(2\theta)$$

45. $$\frac{\sin(3\theta)}{\sin\theta} - \frac{\cos(3\theta)}{\cos\theta} = \frac{\sin(3\theta)\cos\theta - \cos(3\theta)\sin\theta}{\sin\theta\cos\theta} = \frac{\sin(3\theta-\theta)}{\sin\theta\cos\theta}$$

$$= \frac{\sin 2\theta}{\sin\theta\cos\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2$$

46. $$\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} - \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \frac{(\cos\theta + \sin\theta)^2 - (\cos\theta - \sin\theta)^2}{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}$$

$$= \frac{\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta - \left(\cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta\right)}{\cos^2\theta - \sin^2\theta}$$

$$= \frac{1 + 2\cos\theta\sin\theta - 1 + 2\cos\theta\sin\theta}{\cos 2\theta} = \frac{2(2\sin\theta\cos\theta)}{\cos 2\theta} = \frac{2\sin(2\theta)}{\cos(2\theta)} = 2\tan(2\theta)$$

47. $$\tan(3\theta) = \tan(2\theta + \theta) = \frac{\tan(2\theta) + \tan\theta}{1 - \tan(2\theta)\tan\theta} = \frac{\left(\frac{2\tan\theta}{1 - \tan^2\theta} + \tan\theta\right)}{\left(1 - \frac{2\tan\theta}{1 - \tan^2\theta}\quad\tan\theta\right)}$$

$$= \frac{\left(\frac{2\tan\theta + \tan\theta - \tan^3\theta}{1 - \tan^2\theta}\right)}{\left(\frac{1 - \tan^2\theta - 2\tan^2\theta}{1 - \tan^2\theta}\right)} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

48. $$\tan\theta + \tan(\theta + 120^\circ) + \tan(\theta + 240^\circ) = \tan\theta + \frac{\tan\theta + \tan(120^\circ)}{1 - \tan\theta\tan(120^\circ)} + \frac{\tan\theta + \tan(240^\circ)}{1 - \tan\theta\tan(240^\circ)}$$

$$= \tan\theta + \frac{\tan\theta - \sqrt{3}}{1 - \tan\theta\left(-\sqrt{3}\right)} + \frac{\tan\theta + \sqrt{3}}{1 - \tan\theta\left(\sqrt{3}\right)} = \tan\theta + \frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta} + \frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta}$$

$$= \frac{\tan\theta\left(1 - 3\tan^2\theta\right) + \left(\tan\theta - \sqrt{3}\right)\left(1 - \sqrt{3}\tan\theta\right) + \left(\tan\theta + \sqrt{3}\right)\left(1 + \sqrt{3}\tan\theta\right)}{1 - 3\tan^2\theta}$$

$$= \frac{\tan\theta - 3\tan^3\theta + \tan\theta - \sqrt{3}\tan^2\theta - \sqrt{3} + 3\tan\theta + \tan\theta + \sqrt{3}\tan^2\theta + \sqrt{3} + 3\tan\theta}{1 - 3\tan^2\theta}$$

$$= \frac{-3\tan^3\theta + 9\tan\theta}{1 - 3\tan^2\theta} = \frac{3\left(3\tan\theta - \tan^3\theta\right)}{1 - 3\tan^2\theta} = 3\tan(3\theta)\ \text{(by Problem 47)}$$

49. $$\frac{1}{2}\left(\ln\left|1 - \cos(2\theta)\right| - \ln 2\right) = \frac{1}{2}\ln\left|\frac{1 - \cos 2\theta}{2}\right| = \ln\left(\left|\frac{1 - \cos(2\theta)}{2}\right|^{1/2}\right) = \ln\left(\left|\sin^2\theta\right|^{1/2}\right) = \ln\left|\sin\theta\right|$$

50. $$\frac{1}{2}\left(\ln\left|1 + \cos(2\theta)\right| - \ln 2\right) = \frac{1}{2}\ln\left|\frac{1 + \cos 2\theta}{2}\right| = \ln\left(\left|\frac{1 + \cos(2\theta)}{2}\right|^{1/2}\right) = \ln\left(\left|\cos^2\theta\right|^{1/2}\right) = \ln\left|\cos\theta\right|$$

51. $$\sin\left(2\sin^{-1}\left(\frac{1}{2}\right)\right) = \sin\left(2\left(\frac{\pi}{6}\right)\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

52. $$\sin\left(2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) = \sin\left(2\left(\frac{\pi}{3}\right)\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

53. $$\cos\left(2\sin^{-1}\left(\frac{3}{5}\right)\right) = 1 - 2\sin^2\left(\sin^{-1}\left(\frac{3}{5}\right)\right) = 1 - 2\left(\frac{3}{5}\right)^2 = 1 - 2\left(\frac{9}{25}\right) = 1 - \frac{18}{25} = \frac{7}{25}$$

54. $\cos\left(2\cos^{-1}\left(\frac{4}{5}\right)\right) = 2\cos^2\left(\cos^{-1}\left(\frac{4}{5}\right)\right) - 1 = 2\left(\frac{4}{5}\right)^2 - 1 = 2\left(\frac{16}{25}\right) - 1 = \frac{32}{25} - 1 = \frac{7}{25}$

55. $\tan\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]$

Let $\alpha = \cos^{-1}\left(-\frac{3}{5}\right)$. α is in quadrant II.

Then $\cos\alpha = -\frac{3}{5}$, $0 \le \alpha \le \pi$.

$$\sec\alpha = -\frac{5}{3};\quad \tan\alpha = -\sqrt{\sec^2\alpha - 1} = -\sqrt{\left(-\frac{5}{3}\right)^2 - 1} = -\sqrt{\frac{25}{9} - 1} = -\sqrt{\frac{16}{9}} = -\frac{4}{3}$$

$$\tan\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right] = \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(-\frac{4}{3}\right)}{\left(1-\left(-\frac{4}{3}\right)^2\right)} = \frac{\left(-\frac{8}{3}\right)}{\left(1-\frac{16}{9}\right)} = \frac{\left(-\frac{8}{3}\right)}{\left(-\frac{7}{9}\right)} = \left(-\frac{8}{3}\right)\left(-\frac{9}{7}\right) = \frac{24}{7}$$

56.

$$\tan\left(2\tan^{-1}\left(\frac{3}{4}\right)\right) = \frac{2\tan\left(\tan^{-1}\left(\frac{3}{4}\right)\right)}{1-\tan^2\left(\tan^{-1}\left(\frac{3}{4}\right)\right)} = \frac{2\left(\frac{3}{4}\right)}{\left(1-\left(\frac{3}{4}\right)^2\right)} = \frac{\left(\frac{3}{2}\right)}{\left(1-\frac{9}{16}\right)} = \frac{\left(\frac{3}{2}\right)}{\left(\frac{7}{16}\right)} = \frac{3}{2}\cdot\frac{16}{7} = \frac{24}{7}$$

57. $\sin\left(2\cos^{-1}\left(\frac{4}{5}\right)\right)$

Let $\alpha = \cos^{-1}\left(\frac{4}{5}\right)$. α is in quadrant I.

Then $\cos\alpha = \frac{4}{5}$, $0 \le \alpha \le \pi$.

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\sin\left[2\cos^{-1}\left(\frac{4}{5}\right)\right] = \sin 2\alpha = 2\sin\alpha\cos\alpha = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$$

58. $\cos\left(2\tan^{-1}\left(-\frac{4}{3}\right)\right)$

Let $\alpha = \tan^{-1}\left(-\frac{4}{3}\right)$. α is in quadrant IV.

Then $\tan\alpha = -\frac{4}{3}$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

$$\sec\alpha = \sqrt{\tan^2\alpha + 1} = \sqrt{\left(-\frac{4}{3}\right)^2 + 1} = \sqrt{\frac{16}{9} + 1} = \sqrt{\frac{25}{9}} = \frac{5}{3}; \quad \cos\alpha = \frac{3}{5}$$

$$\cos\left[2\tan^{-1}\left(-\frac{4}{3}\right)\right] = \cos 2\alpha = 2\cos^2\alpha - 1 = 2\left(\frac{3}{5}\right)^2 - 1 = 2\left(\frac{9}{25}\right) - 1 = -\frac{7}{25}$$

59. $\sin^2\left[\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)\right] = \dfrac{1 - \cos\left(\cos^{-1}\left(\frac{3}{5}\right)\right)}{2} = \dfrac{\left(1 - \frac{3}{5}\right)}{2} = \dfrac{\left(\frac{2}{5}\right)}{2} = \dfrac{1}{5}$

60. $\cos^2\left(\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)\right)$

Let $\alpha = \sin^{-1}\left(\frac{3}{5}\right)$. α is in quadrant I.

Then $\sin\alpha = \frac{3}{5}, \ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

$$\cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\cos^2\left(\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)\right) = \cos^2\left(\frac{1}{2}\alpha\right) = \frac{1 + \cos\alpha}{2} = \frac{\left(1 + \frac{4}{5}\right)}{2} = \frac{\left(\frac{9}{5}\right)}{2} = \frac{9}{10}$$

61. $\sec\left(2\tan^{-1}\left(\frac{3}{4}\right)\right)$

Let $\alpha = \tan^{-1}\left(\frac{3}{4}\right)$. α is in quadrant I.

Then $\tan\alpha = \frac{3}{4}, \ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

$$\sec\alpha = \sqrt{\tan^2\alpha + 1} = \sqrt{\left(\frac{3}{4}\right)^2 + 1} = \sqrt{\frac{9}{16} + 1} = \sqrt{\frac{25}{16}} = \frac{5}{4}; \quad \cos\alpha = \frac{4}{5}$$

$$\sec\left[2\tan^{-1}\left(\frac{3}{4}\right)\right] = \sec(2\alpha) = \frac{1}{\cos(2\alpha)} = \frac{1}{2\cos^2\alpha - 1} = \frac{1}{2\left(\frac{4}{5}\right)^2 - 1} = \frac{1}{2\left(\frac{16}{25}\right) - 1} = \frac{1}{\left(\frac{7}{25}\right)} = \frac{25}{7}$$

62. $\csc\left(2\sin^{-1}\left(-\frac{3}{5}\right)\right)$

Let $\alpha = \sin^{-1}\left(-\frac{3}{5}\right)$. α is in quadrant IV.

Then $\sin\alpha = -\frac{3}{5}, \ -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\csc\left(2\sin^{-1}\left(-\frac{3}{5}\right)\right) = \csc(2\alpha) = \frac{1}{\sin(2\alpha)} = \frac{1}{2\sin\alpha\cos\alpha} = \frac{1}{2\left(-\frac{3}{5}\right)\left(\frac{4}{5}\right)} = \frac{1}{\left(-\frac{24}{25}\right)} = -\frac{25}{24}$$

63. If $x = 2\tan\theta$, then:

$$\sin(2\theta) = 2\sin\theta\cos\theta = \frac{2\sin\theta}{\cos\theta}\,\frac{\cos^2\theta}{1} = \frac{\left(2\frac{\sin\theta}{\cos\theta}\right)}{\left(\frac{1}{\cos^2\theta}\right)} = \frac{2\tan\theta}{\sec^2\theta} = \frac{2\tan\theta}{1+\tan^2\theta}\,\frac{4}{4}$$

$$= \frac{4(2\tan\theta)}{4+(2\tan\theta)^2} = \frac{4x}{4+x^2}$$

64. $$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta} = \frac{\left(\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}\right)}{\left(\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right)} = \frac{1-\tan^2\theta}{1+\tan^2\theta}\,\frac{4}{4}$$

$$= \frac{4-4\tan^2\theta}{4+4\tan^2\theta} = \frac{4-x^2}{4+x^2}$$

65. Solve for C:

$$\frac{1}{2}\sin^2\theta + C = -\frac{1}{4}\cos(2\theta)$$

$$C = -\frac{1}{4}\cos(2\theta) - \frac{1}{2}\sin^2\theta = -\frac{1}{4}\left(\cos(2\theta) + 2\sin^2\theta\right)$$

$$= -\frac{1}{4}\left(1 - 2\sin^2\theta + 2\sin^2\theta\right) = -\frac{1}{4}(1) = -\frac{1}{4}$$

66. $$\frac{1}{2}\cos^2\theta + C = \frac{1}{4}\cos(2\theta)$$

$$C = \frac{1}{4}\cos(2\theta) - \frac{1}{2}\cos^2\theta = \frac{1}{4}\left(2\cos^2\theta - 1\right) - \frac{1}{2}\cos^2\theta = \frac{1}{2}\cos^2\theta - \frac{1}{4} - \frac{1}{2}\cos^2\theta = -\frac{1}{4}$$

67. $$z = \tan\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{\sin\alpha}$$

$$z\sin\alpha = 1 - \cos\alpha$$

$$z\sin\alpha = 1 - \sqrt{1-\sin^2\alpha}$$

$$z\sin\alpha - 1 = -\sqrt{1-\sin^2\alpha}$$

$$z^2\sin^2\alpha - 2z\sin\alpha + 1 = 1 - \sin^2\alpha \rightarrow z^2\sin^2\alpha + \sin^2\alpha = 2z\sin\alpha$$

$$\sin^2\alpha(z^2+1) = 2z\sin\alpha \rightarrow \sin\alpha(z^2+1) = 2z \rightarrow \sin\alpha = \frac{2z}{z^2+1}$$

68. $$z = \tan\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

$$z^2 = \frac{1-\cos\alpha}{1+\cos\alpha} \rightarrow z^2 + z^2\cos\alpha = 1-\cos\alpha$$

$$z^2\cos\alpha + \cos\alpha = 1 - z^2 \rightarrow \cos\alpha\left(z^2+1\right) = 1 - z^2 \rightarrow \cos\alpha = \frac{1-z^2}{1+z^2}$$

69. Let b represent the base of the triangle.

$$\cos\left(\frac{\theta}{2}\right) = \frac{h}{s} \quad \rightarrow \quad h = s\cos\left(\frac{\theta}{2}\right) \qquad \sin\left(\frac{\theta}{2}\right) = \frac{\left(\frac{1}{2}b\right)}{s} \quad \rightarrow \quad b = 2s\sin\left(\frac{\theta}{2}\right)$$

$$A = \frac{1}{2}b\;h = \frac{1}{2}\left(2s\sin\left(\frac{\theta}{2}\right)\right)\left(s\cos\left(\frac{\theta}{2}\right)\right) = s^2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \frac{1}{2}s^2\sin\theta$$

70. $\sin\theta = \frac{y}{1} = y;\qquad \cos\theta = \frac{x}{1} = x$

(a) $A = 2xy = 2\cos\theta\sin\theta$

(b) $2\cos\theta\sin\theta = 2\sin\theta\cos\theta = \sin(2\theta)$

(c) The largest value of the sine function is 1. Solve:

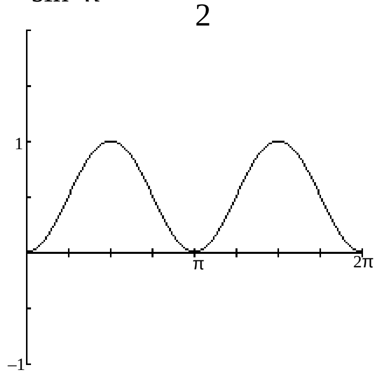
$$\sin(2\theta) = 1$$

$$2\theta = \frac{\pi}{2} \rightarrow \theta = \frac{\pi}{4}$$

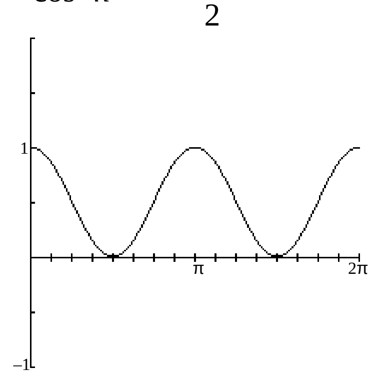
(d) $x = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \qquad y = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

The dimensions are $\sqrt{2}$ by $\frac{\sqrt{2}}{2}$.

71. $f(x) = \sin^2 x = \frac{1-\cos(2x)}{2}$



72. $g(x) = \cos^2 x = \frac{1+\cos(2x)}{2}$



73.

$$\sin\left(\frac{\pi}{24}\right) = \sin\left(\frac{\left(\frac{\pi}{12}\right)}{2}\right) = \sqrt{\frac{\left(1-\cos\left(\frac{\pi}{12}\right)\right)}{2}} = \sqrt{\frac{1-\left(\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)\right)}{2}} = \sqrt{\frac{1}{2}-\frac{1}{8}\left(\sqrt{6}+\sqrt{2}\right)}$$

$$= \sqrt{\frac{8-2\left(\sqrt{6}+\sqrt{2}\right)}{16}} = \frac{\sqrt{8-2\left(\sqrt{6}+\sqrt{2}\right)}}{4}$$

$$\cos\left(\frac{\pi}{24}\right) = \cos\left(\frac{\left(\frac{\pi}{12}\right)}{2}\right) = \sqrt{\frac{\left(1+\cos\left(\frac{\pi}{12}\right)\right)}{2}} = \sqrt{\frac{1+\left(\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)\right)}{2}} = \sqrt{\frac{1}{2}+\frac{1}{8}\left(\sqrt{6}+\sqrt{2}\right)}$$

$$= \sqrt{\frac{8+2\left(\sqrt{6}+\sqrt{2}\right)}{16}} = \frac{\sqrt{8+2\left(\sqrt{6}+\sqrt{2}\right)}}{4}$$

74.

$$\cos\left(\frac{\pi}{8}\right) = \cos\left(\frac{\left(\frac{\pi}{4}\right)}{2}\right) = \sqrt{\frac{\left(1+\cos\left(\frac{\pi}{4}\right)\right)}{2}} = \sqrt{\frac{\left(1+\frac{\sqrt{2}}{2}\right)}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\sin\left(\frac{\pi}{16}\right) = \sin\left(\frac{\left(\frac{\pi}{8}\right)}{2}\right) = \sqrt{\frac{\left(1-\cos\left(\frac{\pi}{8}\right)\right)}{2}} = \sqrt{\frac{\left(1-\frac{\sqrt{2+\sqrt{2}}}{2}\right)}{2}} = \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{4}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$$

$$\cos\left(\frac{\pi}{16}\right) = \cos\left(\frac{\left(\frac{\pi}{8}\right)}{2}\right) = \sqrt{\frac{\left(1+\cos\left(\frac{\pi}{8}\right)\right)}{2}} = \sqrt{\frac{\left(1+\frac{\sqrt{2+\sqrt{2}}}{2}\right)}{2}} = \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{4}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$$

75. $\sin^3\theta + \sin^3(\theta + 120°) + \sin^3(\theta + 240°)$

$$= \sin^3\theta + \left(\sin\theta\cos(120^\circ) + \cos\theta\sin(120^\circ)\right)^3 + \left(\sin\theta\cos(240^\circ) + \cos\theta\sin(240^\circ)\right)^3$$

$$= \sin^3\theta + \left(-\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right)^3 + \left(-\frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta\right)^3$$

$$= \sin^3\theta + \frac{1}{8}\left(-\sin^3\theta + 3\sqrt{3}\sin^2\theta\cos\theta - 9\sin\theta\cos^2\theta + 3\sqrt{3}\cos^3\theta\right)$$

$$-\frac{1}{8}\left(\sin^3\theta + 3\sqrt{3}\sin^2\theta\cos\theta + 9\sin\theta\cos^2\theta + 3\sqrt{3}\cos^3\theta\right)$$

$$= \sin^3\theta - \frac{1}{8}\sin^3\theta + \frac{3\sqrt{3}}{8}\sin^2\theta\cos\theta - \frac{9}{8}\sin\theta\cos^2\theta + \frac{3\sqrt{3}}{8}\cos^3\theta$$

$$-\frac{1}{8}\sin^3\theta - \frac{3\sqrt{3}}{8}\sin^2\theta\cos\theta - \frac{9}{8}\sin\theta\cos^2\theta - \frac{3\sqrt{3}}{8}\cos^3\theta$$

$$= \frac{3}{4}\sin^3\theta - \frac{9}{4}\sin\theta\cos^2\theta = \frac{3}{4}\left(\sin^3\theta - 3\sin\theta(1 - \sin^2\theta)\right)$$

$$= \frac{3}{4}\left(\sin^3\theta - 3\sin\theta + 3\sin^3\theta\right) = \frac{3}{4}\left(4\sin^3\theta - 3\sin\theta\right) = -\frac{3}{4}\sin(3\theta)$$

(See the formula for $\sin(3\theta)$ on page *** of the text.)

76. $\tan\theta = \tan\left(3\left(\frac{\theta}{3}\right)\right) = \dfrac{3\tan\left(\frac{\theta}{3}\right) - \tan^3\left(\frac{\theta}{3}\right)}{1 - 3\tan^2\left(\frac{\theta}{3}\right)} = a\tan\left(\frac{\theta}{3}\right)$ (from Problem 47)

$$3\tan\left(\frac{\theta}{3}\right) - \tan^3\left(\frac{\theta}{3}\right) = a\tan\left(\frac{\theta}{3}\right)\left(1 - 3\tan^2\left(\frac{\theta}{3}\right)\right)$$

$$3 - \tan^2\left(\frac{\theta}{3}\right) = a\left(1 - 3\tan^2\left(\frac{\theta}{3}\right)\right)$$

$$3 - \tan^2\left(\frac{\theta}{3}\right) = a - 3a\tan^2\left(\frac{\theta}{3}\right)$$

$$(3a - 1)\tan^2\left(\frac{\theta}{3}\right) = a - 3$$

$$\tan^2\left(\frac{\theta}{3}\right) = \frac{a - 3}{3a - 1}$$

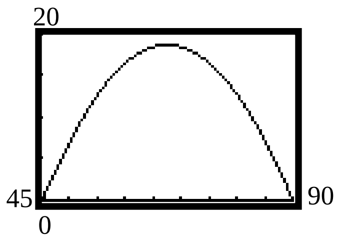
$$\tan\left(\frac{\theta}{3}\right) = \pm\sqrt{\frac{a - 3}{3a - 1}}$$

77. (a) $R(\theta) = \dfrac{v_0^2\sqrt{2}}{16}\cos\theta(\sin\theta - \cos\theta) = \dfrac{v_0^2\sqrt{2}}{16}(\cos\theta\sin\theta - \cos^2\theta)$

$$= \frac{v_0^2\sqrt{2}}{16} \cdot \frac{1}{2}(2\cos\theta\sin\theta - 2\cos^2\theta) = \frac{v_0^2\sqrt{2}}{32}\left(\sin 2\theta - 2\left(\frac{1+\cos 2\theta}{2}\right)\right)$$

$$= \frac{v_0^2\sqrt{2}}{32}\left(\sin(2\theta) - 1 - \cos(2\theta)\right) = \frac{v_0^2\sqrt{2}}{32}\left(\sin(2\theta) - \cos(2\theta) - 1\right)$$

(b)



(c) Using the MAXIMUM feature on the calculator:
R has the largest value when $\theta \approx 67.5^{\circ}$.

78. $f(x) = \frac{1}{2}\sin(2\pi x) + \frac{1}{4}\sin(4\pi x) = \frac{1}{2}\sin(2\pi x) + \frac{1}{4}\left(2\sin(2\pi x)\cos(2\pi x)\right)$

$$= \frac{1}{2}\sin(2\pi x)\left(1 + \cos(2\pi x)\right) = \frac{1}{2}\sin(2\pi x)\left(1 + (2\cos^2(\pi x) - 1)\right)$$

$$= \frac{1}{2}\sin(2\pi x)\left(2\cos^2(\pi x)\right) = \sin(2\pi x)\cos^2(\pi x)$$