Chapter 8

Analytic Trigonometry

8.6 Product-to-Sum and Sum-to-Product Formulas

For Problems 1-10, use the formulas:

$$\sin\alpha\sin\beta=\frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right] \qquad \cos\alpha\cos\beta=\frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

$$\sin\alpha\cos\beta=\frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

1. $\sin(4\theta)\sin(2\theta)=\frac{1}{2}\left[\cos(4\theta-2\theta)-\cos(4\theta+2\theta)\right]=\frac{1}{2}\left[\cos(2\theta)-\cos(6\theta)\right]$

2. $\cos(4\theta)\cos(2\theta)=\frac{1}{2}\left[\cos(4\theta-2\theta)+\cos(4\theta+2\theta)\right]=\frac{1}{2}\left[\cos(2\theta)+\cos(6\theta)\right]$

3. $\sin(4\theta)\cos(2\theta)=\frac{1}{2}\left[\sin(4\theta+2\theta)+\sin(4\theta-2\theta)\right]=\frac{1}{2}\left[\sin(6\theta)+\sin(2\theta)\right]$

4. $\sin(3\theta)\sin(5\theta)=\frac{1}{2}\left[\cos(3\theta-5\theta)-\cos(3\theta+5\theta)\right]=\frac{1}{2}\left[\cos(-2\theta)-\cos(8\theta)\right]$
$=\frac{1}{2}\left[\cos(2\theta)-\cos(8\theta)\right]$

5. $\cos(3\theta)\cos(5\theta)=\frac{1}{2}\left[\cos(3\theta-5\theta)+\cos(3\theta+5\theta)\right]=\frac{1}{2}\left[\cos(-2\theta)+\cos(8\theta)\right]$
$=\frac{1}{2}\left[\cos(2\theta)+\cos(8\theta)\right]$

6. $\sin(4\theta)\cos(6\theta)=\frac{1}{2}\left[\sin(4\theta+6\theta)+\sin(4\theta-6\theta)\right]=\frac{1}{2}\left[\sin(10\theta)+\sin(-2\theta)\right]$
$=\frac{1}{2}\left[\sin(10\theta)-\sin(2\theta)\right]$

7. $\sin\theta\sin(2\theta)=\frac{1}{2}\left[\cos(\theta-2\theta)-\cos(\theta+2\theta)\right]=\frac{1}{2}\left[\cos(-\theta)-\cos(3\theta)\right]$
$=\frac{1}{2}\left[\cos\theta-\cos(3\theta)\right]$

8. $\cos(3\theta)\cos(4\theta) = \frac{1}{2}\left[\cos(3\theta - 4\theta) + \cos(3\theta + 4\theta)\right] = \frac{1}{2}\left[\cos(-\theta) + \cos(7\theta)\right]$
$= \frac{1}{2}\left[\cos\theta + \cos(7\theta)\right]$

9. $\sin\left(\frac{3\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \frac{1}{2}\left[\sin\left(\frac{3\theta}{2} + \frac{\theta}{2}\right) + \sin\left(\frac{3\theta}{2} - \frac{\theta}{2}\right)\right] = \frac{1}{2}\left[\sin(2\theta) + \sin\theta\right]$

10. $\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{5\theta}{2}\right) = \frac{1}{2}\left[\sin\left(\frac{\theta}{2} + \frac{5\theta}{2}\right) + \sin\left(\frac{\theta}{2} - \frac{5\theta}{2}\right)\right] = \frac{1}{2}\left[\sin(3\theta) + \sin(-2\theta)\right]$
$= \frac{1}{2}\left[\sin(3\theta) - \sin(2\theta)\right]$

For Problems 11-18, use the formulas:

$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\quad$ $\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$

$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\quad$ $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

11. $\sin(4\theta) - \sin(2\theta) = 2\sin\left(\frac{4\theta - 2\theta}{2}\right)\cos\left(\frac{4\theta + 2\theta}{2}\right) = 2\sin\theta\cos(3\theta)$

12. $\sin(4\theta) + \sin(2\theta) = 2\sin\left(\frac{4\theta + 2\theta}{2}\right)\cos\left(\frac{4\theta - 2\theta}{2}\right) = 2\sin(3\theta)\cos\theta$

13. $\cos(2\theta) + \cos(4\theta) = 2\cos\left(\frac{2\theta + 4\theta}{2}\right)\cos\left(\frac{2\theta - 4\theta}{2}\right) = 2\cos(3\theta)\cos(-\theta) = 2\cos(3\theta)\cos\theta$

14. $\cos(5\theta) - \cos(3\theta) = -2\sin\left(\frac{5\theta + 3\theta}{2}\right)\sin\left(\frac{5\theta - 3\theta}{2}\right) = -2\sin(4\theta)\sin\theta$

15. $\sin\theta + \sin(3\theta) = 2\sin\left(\frac{\theta + 3\theta}{2}\right)\cos\left(\frac{\theta - 3\theta}{2}\right) = 2\sin(2\theta)\cos(-\theta) = 2\sin(2\theta)\cos\theta$

16. $\cos\theta + \cos(3\theta) = 2\cos\left(\frac{\theta + 3\theta}{2}\right)\cos\left(\frac{\theta - 3\theta}{2}\right) = 2\cos(2\theta)\cos(-\theta) = 2\cos(2\theta)\cos\theta$

17. $\cos\left(\frac{\theta}{2}\right) - \cos\left(\frac{3\theta}{2}\right)$
$= -2\sin\left(\frac{\left(\frac{\theta}{2} + \frac{3\theta}{2}\right)}{2}\right)\sin\left(\frac{\left(\frac{\theta}{2} - \frac{3\theta}{2}\right)}{2}\right) = -2\sin\theta\sin\left(-\frac{\theta}{2}\right) = -2\sin\theta\left(-\sin\left(\frac{\theta}{2}\right)\right) = 2\sin\theta\sin\left(\frac{\theta}{2}\right)$

18. $\sin\left(\frac{\theta}{2}\right)-\sin\left(\frac{3\theta}{2}\right)$

$= 2\sin\left(\frac{\frac{\theta}{2}-\frac{3\theta}{2}}{2}\right)\cos\left(\frac{\frac{\theta}{2}+\frac{3\theta}{2}}{2}\right) = 2\sin\left(-\frac{\theta}{2}\right)\cos\theta = -2\sin\left(\frac{\theta}{2}\right)\cos\theta$

19. $\frac{\sin\theta+\sin(3\theta)}{2\sin(2\theta)} = \frac{2\sin(2\theta)\cos(-\theta)}{2\sin(2\theta)} = \cos(-\theta) = \cos\theta$

20. $\frac{\cos\theta+\cos(3\theta)}{2\cos(2\theta)} = \frac{2\cos(2\theta)\cos(-\theta)}{2\cos(2\theta)} = \cos(-\theta) = \cos\theta$

21. $\frac{\sin(4\theta)+\sin(2\theta)}{\cos(4\theta)+\cos(2\theta)} = \frac{2\sin(3\theta)\cos\theta}{2\cos(3\theta)\cos\theta} = \frac{\sin(3\theta)}{\cos(3\theta)} = \tan(3\theta)$

22. $\frac{\cos\theta-\cos(3\theta)}{\sin(3\theta)-\sin\theta} = \frac{-2\sin(2\theta)\sin(-\theta)}{2\sin\theta\cos(2\theta)} = \frac{-(-\sin\theta)\sin(2\theta)}{\sin\theta\cos(2\theta)} = \tan(2\theta)$

23. $\frac{\cos\theta-\cos(3\theta)}{\sin\theta+\sin(3\theta)} = \frac{-2\sin(2\theta)\sin(-\theta)}{2\sin(2\theta)\cos(-\theta)} = \frac{-(-\sin\theta)}{\cos\theta} = \tan\theta$

24. $\frac{\cos\theta-\cos(5\theta)}{\sin\theta+\sin(5\theta)} = \frac{-2\sin(3\theta)\sin(-2\theta)}{2\sin(3\theta)\cos(-2\theta)} = \frac{-(-\sin 2\theta)}{\cos(2\theta)} = \tan(2\theta)$

25. $\sin\theta[\sin\theta+\sin(3\theta)] = \sin\theta[2\sin(2\theta)\cos(-\theta)] = \cos\theta[2\sin(2\theta)\sin\theta]$

$= \cos\theta\left[2\cdot\frac{1}{2}(\cos\theta-\cos(3\theta))\right] = \cos\theta(\cos\theta-\cos(3\theta))$

26. $\sin\theta[\sin 3\theta+\sin(5\theta)] = \sin\theta[2\sin(4\theta)\cos(-\theta)] = \cos\theta[2\sin(4\theta)\sin\theta]$

$= \cos\theta\left[2\cdot\frac{1}{2}(\cos 3\theta-\cos(5\theta))\right] = \cos\theta(\cos 3\theta-\cos(5\theta))$

27. $\frac{\sin(4\theta)+\sin(8\theta)}{\cos(4\theta)+\cos(8\theta)} = \frac{2\sin(6\theta)\cos(-2\theta)}{2\cos(6\theta)\cos(-2\theta)} = \frac{\sin(6\theta)}{\cos(6\theta)} = \tan(6\theta)$

28. $\frac{\sin(4\theta)-\sin(8\theta)}{\cos(4\theta)-\cos(8\theta)} = \frac{2\sin(-2\theta)\cos(6\theta)}{-2\sin(6\theta)\sin(-2\theta)} = \frac{\cos(6\theta)}{-\sin(6\theta)} = -\cot(6\theta)$

29. $\frac{\sin(4\theta)+\sin(8\theta)}{\sin(4\theta)-\sin(8\theta)} = \frac{2\sin(6\theta)\cos(-2\theta)}{2\sin(-2\theta)\cos(6\theta)} = \frac{\sin(6\theta)\cos(2\theta)}{-\sin(2\theta)\cos(6\theta)}$

$= -\tan(6\theta)\cot(2\theta) = -\frac{\tan(6\theta)}{\tan(2\theta)}$

30. $\frac{\cos(4\theta)-\cos(8\theta)}{\cos(4\theta)+\cos(8\theta)} = \frac{-2\sin(6\theta)\sin(-2\theta)}{2\cos(6\theta)\cos(-2\theta)} = -\tan(6\theta)\tan(-2\theta) = \tan(2\theta)\tan(6\theta)$

31. $$\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}=\frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}=\tan\left(\frac{\alpha+\beta}{2}\right)\cot\left(\frac{\alpha-\beta}{2}\right)$$

32. $$\frac{\cos\alpha+\cos\beta}{\cos\alpha-\cos\beta}=\frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}=-\cot\left(\frac{\alpha+\beta}{2}\right)\cot\left(\frac{\alpha-\beta}{2}\right)$$

33. $$\frac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta}=\frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}=\tan\left(\frac{\alpha+\beta}{2}\right)$$

34. $$\frac{\sin\alpha-\sin\beta}{\cos\alpha-\cos\beta}=\frac{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}=-\cot\left(\frac{\alpha+\beta}{2}\right)$$

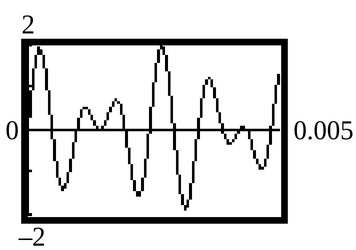
35. $$\begin{aligned}1+\cos(2\theta)+\cos(4\theta)+\cos(6\theta)&=\cos 0+\cos(6\theta)+\cos(2\theta)+\cos(4\theta)\\&=2\cos(3\theta)\cos(-3\theta)+2\cos(3\theta)\cos(-\theta)=2\cos^2(3\theta)+2\cos(3\theta)\cos\theta\\&=2\cos(3\theta)\big(\cos(3\theta)+\cos\theta\big)=2\cos(3\theta)2\cos(2\theta)\cos\theta\\&=4\cos\theta\cos(2\theta)\cos(3\theta)\end{aligned}$$

36. $$\begin{aligned}1-\cos(2\theta)+\cos(4\theta)-\cos(6\theta)&=\cos 0-\cos(6\theta)+\cos(4\theta)-\cos(2\theta)\\&=-2\sin(3\theta)\sin(-3\theta)-2\sin(3\theta)\sin(\theta)=2\sin^2(3\theta)-2\sin(3\theta)\sin\theta\\&=2\sin(3\theta)\big(\sin(3\theta)-\sin\theta\big)=2\sin(3\theta)2\sin\theta\cos(2\theta)\\&=4\sin\theta\cos(2\theta)\sin(3\theta)\end{aligned}$$

37. (a) $$\begin{aligned}y&=\sin\left[2\pi(852)t\right]+\sin\left[2\pi(1209)t\right]\\&=2\sin\left(\frac{2\pi(852)t+2\pi(1209)t}{2}\right)\cos\left(\frac{2\pi(852)t-2\pi(1209)t}{2}\right)\\&=2\sin(2061\pi t)\cos(357\pi t)\end{aligned}$$

(b) The maximum value of y is 2.

(c)



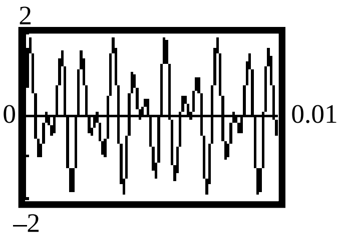
38. (a) $y = \sin\left[2\pi(941)t\right] + \sin\left[2\pi(1477)t\right]$

$$= 2\sin\left(\frac{2\pi(941)\,t + 2\pi(1477)\,t}{2}\right)\cos\left(\frac{2\pi(941)\,t - 2\pi(14779)\,t}{2}\right)$$

$$= 2\sin(2418\pi t)\cos(-536\pi t) = 2\sin(2418\pi t)\cos(536\pi t)$$

(b) The maximum value of y is 2.

(c)



39. $\sin(2\alpha) + \sin(2\beta) + \sin(2\gamma)$

$$= 2\sin\left(\frac{2\alpha + 2\beta}{2}\right)\cos\left(\frac{2\alpha - 2\beta}{2}\right) + \sin(2\gamma)$$

$$= 2\sin(\alpha + \beta)\cos(\alpha - \beta) + 2\sin\gamma\cos\gamma$$

$$= 2\sin(\pi - \gamma)\cos(\alpha - \beta) + 2\sin\gamma\cos\gamma$$

$$= 2\sin\gamma\cos(\alpha - \beta) + 2\sin\gamma\cos\gamma = 2\sin\gamma\left[\cos(\alpha - \beta) + \cos\gamma\right]$$

$$= 2\sin\gamma\left(2\cos\left(\frac{\alpha - \beta + \gamma}{2}\right)\cos\left(\frac{\alpha - \beta - \gamma}{2}\right)\right)$$

$$= 4\sin\gamma\cos\left(\frac{\pi}{2} - \beta\right)\cos\left(\alpha - \frac{\pi}{2}\right) = 4\sin\gamma\sin\beta\sin\alpha$$

$$= 4\sin\alpha\sin\beta\sin\gamma$$

40. $\tan\alpha + \tan\beta + \tan\gamma = \dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta} + \dfrac{\sin\gamma}{\cos\gamma}$

$$= \frac{\sin\alpha\cos\beta\cos\gamma + \sin\beta\cos\alpha\cos\gamma + \sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma}$$

$$= \frac{\cos\gamma(\sin\alpha\cos\beta + \cos\alpha\sin\beta) + \sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma}$$

$$= \frac{\cos\gamma\sin(\alpha + \beta) + \sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma} = \frac{\cos\gamma\sin(\pi - \gamma) + \sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma}$$

$$= \frac{\cos\gamma\sin\gamma + \sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma} = \frac{\sin\gamma(\cos\gamma + \cos\alpha\cos\beta)}{\cos\alpha\cos\beta\cos\gamma}$$

$$= \frac{\sin\gamma(\cos(\pi - (\alpha + \beta)) + \cos\alpha\cos\beta)}{\cos\alpha\cos\beta\cos\gamma} = \frac{\sin\gamma(-\cos(\alpha + \beta) + \cos\alpha\cos\beta)}{\cos\alpha\cos\beta\cos\gamma}$$

$$= \frac{\sin\gamma(-\cos\alpha\cos\beta + \sin\alpha\sin\beta + \cos\alpha\cos\beta)}{\cos\alpha\cos\beta\cos\gamma} = \frac{\sin\gamma(\sin\alpha\sin\beta)}{\cos\alpha\cos\beta\cos\gamma}$$

$$= \tan\alpha\tan\beta\tan\gamma$$

41. Add the two sum formulas for $\sin(\alpha+\beta)$ and $\sin(\alpha-\beta)$ and solve:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$$
$$\sin\alpha\cos\beta=\frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

42. $$2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)=2\cdot\frac{1}{2}\left[\sin\left(\frac{\alpha-\beta}{2}+\frac{\alpha+\beta}{2}\right)+\sin\left(\frac{\alpha-\beta}{2}-\frac{\alpha+\beta}{2}\right)\right]$$
$$=\sin\alpha+\sin(-\beta)=\sin\alpha-\sin\beta$$

Therefore, $\sin\alpha-\sin\beta=2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$

43. $$2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)=2\cdot\frac{1}{2}\left[\cos\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)\right]$$
$$=\cos\left(\frac{2\beta}{2}\right)+\cos\left(\frac{2\alpha}{2}\right)=\cos\beta+\cos\alpha$$

Therefore, $\cos\alpha+\cos\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

44. $$-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)=-2\cdot\frac{1}{2}\left[\cos\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)\right]$$
$$=-\left(\cos\left(\frac{2\beta}{2}\right)-\cos\left(\frac{2\alpha}{2}\right)\right)=\cos\alpha-\cos\beta$$

Therefore, $\cos\alpha-\cos\beta=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$