Chapter 8

Analytic Trigonometry

8.7 Trigonometric Equations (I)

1. $\sin\theta = \frac{1}{2}$

$$\theta = \frac{\pi}{6} + 2k\pi \text{ or } \theta = \frac{5\pi}{6} + 2k\pi, \text{ where } k \text{ is any integer}$$

Six solutions are $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$

2. $\tan\theta = 1$

$$\theta = \frac{\pi}{4} + k\pi, \text{ where } k \text{ is any integer}$$

Six solutions are $\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}$

3. $\tan\theta = -\frac{\sqrt{3}}{3}$

$$\theta = \frac{5\pi}{6} + k\pi, \text{ where } k \text{ is any integer}$$

Six solutions are $\theta = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}, \frac{29\pi}{6}, \frac{35\pi}{6}$

4. $\cos\theta = -\frac{\sqrt{3}}{2}$

$$\theta = \frac{5\pi}{6} + 2k\pi \text{ or } \theta = \frac{7\pi}{6} + 2k\pi, \text{ where } k \text{ is any integer}$$

Six solutions are $\theta = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}, \frac{29\pi}{6}, \frac{31\pi}{6}$

5. $\cos\theta = 0$

$$\theta = \frac{\pi}{2} + 2k\pi \text{ or } \theta = \frac{3\pi}{2} + 2k\pi, \text{ where } k \text{ is any integer}$$

Six solutions are $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$

6. $\sin\theta = \frac{\sqrt{2}}{2}$

$$\theta = \frac{\pi}{4} + 2k\pi \text{ or } \theta = \frac{3\pi}{4} + 2k\pi, \text{ where } k \text{ is any integer}$$

Six solutions are $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$

7. $\cos(2\theta) = -\frac{1}{2}$

$2\theta = \frac{2\pi}{3} + 2k\pi \quad \rightarrow \quad \theta = \frac{\pi}{3} + k\pi$, where k is any integer

$2\theta = \frac{4\pi}{3} + 2k\pi \quad \rightarrow \quad \theta = \frac{2\pi}{3} + k\pi$, where k is any integer

Six solutions are $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$

8. $\sin(2\theta) = -1$

$2\theta = \frac{5\pi}{2} + 2k\pi \quad \rightarrow \quad \theta = \frac{5\pi}{4} + k\pi$, where k is any integer

Six solutions are $\theta = \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}, \frac{25\pi}{4}$

9. $\sin\left(\frac{\theta}{2}\right) = -\frac{\sqrt{3}}{2}$

$\frac{\theta}{2} = \frac{4\pi}{3} + 2k\pi \quad \rightarrow \quad \theta = \frac{8\pi}{3} + 4k\pi$, where k is any integer

$\frac{\theta}{2} = \frac{5\pi}{3} + 2k\pi \quad \rightarrow \quad \theta = \frac{10\pi}{3} + 4k\pi$, where k is any integer

Six solutions are $\theta = \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{20\pi}{3}, \frac{22\pi}{3}, \frac{32\pi}{3}, \frac{34\pi}{3}$

10. $\tan\left(\frac{\theta}{2}\right) = -1$

$\frac{\theta}{2} = \frac{3\pi}{4} + 2k\pi \quad \rightarrow \quad \theta = \frac{3\pi}{2} + 4k\pi$, where k is any integer

$\frac{\theta}{2} = \frac{5\pi}{4} + 2k\pi \quad \rightarrow \quad \theta = \frac{5\pi}{2} + 4k\pi$, where k is any integer

Six solutions are $\theta = \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2}, \frac{19\pi}{2}, \frac{21\pi}{2}$

11. $2\sin\theta + 3 = 2$

$2\sin\theta = -1 \rightarrow \sin\theta = -\frac{1}{2}$

$\theta = \frac{7\pi}{6} + 2k\pi \quad \text{or} \quad \theta = \frac{11\pi}{6} + 2k\pi,\ \ k$ is any integer

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$.

12. $1 - \cos\theta = \dfrac{1}{2}$

$$1 - \cos\theta = \frac{1}{2} \rightarrow \frac{1}{2} = \cos\theta$$

$$\theta = \frac{\pi}{3} + 2k\pi \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2k\pi, \ k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$.

13. $4\cos^2\theta = 1$

$$\cos^2\theta = \frac{1}{4} \rightarrow \cos\theta = \pm\frac{1}{2}$$

$$\theta = \frac{\pi}{3} + k\pi \quad \text{or} \quad \theta = \frac{2\pi}{3} + k\pi, \ k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$.

14. $\tan^2\theta = \dfrac{1}{3}$

$$\tan\theta = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$$

$$\theta = \frac{\pi}{6} + k\pi \text{ or } \theta = \frac{5\pi}{6} + k\pi, \ k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}, \dfrac{11\pi}{3}$.

15. $2\sin^2\theta - 1 = 0$

$$2\sin^2\theta = 1 \rightarrow \sin^2\theta = \frac{1}{2} \rightarrow \sin\theta = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$$

$$\theta = \frac{\pi}{4} + k\pi \quad \text{or} \quad \theta = \frac{3\pi}{4} + k\pi, \ k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$.

16. $4\cos^2\theta - 3 = 0$

$$4\cos^2\theta = 3 \rightarrow \cos^2\theta = \frac{3}{4} \rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{6} + k\pi \quad \text{or} \quad \theta = \frac{5\pi}{6} + k\pi, \ k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$.

17. $\sin(3\theta) = -1$

$$3\theta = \frac{3\pi}{2} + 2k\pi \quad \rightarrow \quad \theta = \frac{\pi}{2} + \frac{2k\pi}{3}, \text{ where } k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

18. $\tan\left(\frac{\theta}{2}\right) = \sqrt{3}$

$$\frac{\theta}{2} = \frac{\pi}{3} + k\pi \quad \rightarrow \quad \theta = \frac{2\pi}{3} + 2k\pi, \text{ where } k \text{ is any integer}$$

The solution on the interval $[0, 2\pi)$ are $\theta = \frac{2\pi}{3}$

19. $\cos(2\theta) = -\frac{1}{2}$

$$2\theta = \frac{2\pi}{3} + 2k\pi \quad \rightarrow \quad \theta = \frac{\pi}{3} + k\pi, \text{ where } k \text{ is any integer}$$

$$2\theta = \frac{4\pi}{3} + 2k\pi \quad \rightarrow \quad \theta = \frac{2\pi}{3} + k\pi, \text{ where } k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

20. $\tan(2\theta) = -1$

$$2\theta = \frac{3\pi}{4} + k\pi \quad \rightarrow \quad \theta = \frac{3\pi}{8} + \frac{k\pi}{2}, \text{ where } k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

21. $\sec\left(\frac{3\theta}{2}\right) = -2$

$$\frac{3\theta}{2} = \frac{2\pi}{3} + 2k\pi \quad \rightarrow \quad \theta = \frac{4\pi}{9} + \frac{4k\pi}{3}, \text{ where } k \text{ is any integer}$$

$$\frac{3\theta}{2} = \frac{4\pi}{3} + 2k\pi \quad \rightarrow \quad \theta = \frac{8\pi}{9} + \frac{4k\pi}{3}, \text{ where } k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{16\pi}{9}$

22. $\cot\left(\frac{2\theta}{3}\right) = -\sqrt{3}$

$$\frac{2\theta}{3} = \frac{5\pi}{6} + k\pi \quad \rightarrow \quad \theta = \frac{5\pi}{4} + \frac{3k\pi}{2}, \text{ where } k \text{ is any integer}$$

The solution on the interval $[0, 2\pi)$ are $\theta = \frac{5\pi}{4}$

23. $\cos\left(2\theta - \frac{\pi}{2}\right) = -1$

$$2\theta - \frac{\pi}{2} = \pi + 2k\pi \quad \rightarrow \quad 2\theta = \frac{3\pi}{2} + 2k\pi \quad \rightarrow \quad \theta = \frac{3\pi}{4} + k\pi, \;\; k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$.

24. $\sin\left(3\theta + \frac{\pi}{18}\right) = 1$

$$3\theta + \frac{\pi}{18} = \frac{\pi}{2} + 2k\pi \quad \to \quad 3\theta = \frac{4\pi}{9} + 2k\pi \quad \to \quad \theta = \frac{4\pi}{27} + \frac{2k\pi}{3},\ k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{4\pi}{27}, \frac{22\pi}{27}, \frac{40\pi}{27}$.

25. $\tan\left(\frac{\theta}{2} + \frac{\pi}{3}\right) = 1$

$$\frac{\theta}{2} + \frac{\pi}{3} = \frac{\pi}{4} + k\pi \quad \to \quad \frac{\theta}{2} = -\frac{\pi}{12} + k\pi \quad \to \quad \theta = -\frac{\pi}{6} + 2k\pi,\ k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{11\pi}{6}$.

26. $\cos\left(\frac{\theta}{3} - \frac{\pi}{4}\right) = \frac{1}{2}$

$$\frac{\theta}{3} - \frac{\pi}{4} = \frac{\pi}{3} + 2k\pi \quad \to \quad \frac{\theta}{3} = \frac{7\pi}{12} + 2k\pi \quad \to \quad \theta = \frac{7\pi}{4} + 6k\pi,\ k \text{ is any integer}$$

$$\frac{\theta}{3} - \frac{\pi}{4} = \frac{5\pi}{3} + 2k\pi \quad \to \quad \frac{\theta}{3} = \frac{23\pi}{12} + 2k\pi \quad \to \quad \theta = \frac{23\pi}{4} + 6k\pi,\ k \text{ is any integer}$$

The solution on the interval $[0, 2\pi)$ are $\theta = \frac{7\pi}{4}$.

27. $2\sin\theta + 1 = 0 \quad \to \quad 2\sin\theta = -1 \quad \to \quad \sin\theta = -\frac{1}{2}$

$$\theta = \frac{7\pi}{6} + 2k\pi \quad \text{or} \quad \theta = \frac{11\pi}{6} + 2k\pi,\ k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$.

28. $\cos\theta + 1 = 0 \quad \to \quad \cos\theta = -1$

$$\theta = \pi + 2k\pi,\ k \text{ is any integer}$$

The solution on the interval $[0, 2\pi)$ are $\theta = \pi$.

29. $\tan\theta + 1 = 0 \quad \to \quad \tan\theta = -1$

$$\theta = \frac{3\pi}{4} + k\pi,\ k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$.

30. $\sqrt{3}\cot\theta + 1 = 0 \quad \to \quad \sqrt{3}\cot\theta = -1 \quad \to \quad \cot\theta = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

$$\theta = \frac{2\pi}{3} + k\pi,\ k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{2\pi}{3}, \frac{5\pi}{3}$.

31. $4\sec\theta + 6 = -2 \quad \rightarrow \quad 4\sec\theta = -8 \quad \rightarrow \quad \sec\theta = -2$

$$\theta = \frac{2\pi}{3} + 2k\pi \quad \text{or} \quad \theta = \frac{4\pi}{3} + 2k\pi, \; k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$.

32. $5\csc\theta - 3 = 2 \quad \rightarrow \quad 5\csc\theta = 5 \quad \rightarrow \quad \csc\theta = 1$

$$\theta = \frac{\pi}{2} + 2k\pi, \; k \text{ is any integer}$$

The solution on the interval $[0, 2\pi)$ are $\theta = \frac{\pi}{2}$.

33. $3\sqrt{2}\cos\theta + 2 = -1 \quad \rightarrow \quad 3\sqrt{2}\cos\theta = -3 \quad \rightarrow \quad \cos\theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

$$\theta = \frac{3\pi}{4} + 2k\pi \quad \text{or} \quad \theta = \frac{5\pi}{4} + 2k\pi, \; k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}$.

34. $4\sin\theta + 3\sqrt{3} = \sqrt{3} \quad \rightarrow \quad 4\sin\theta = -2\sqrt{3} \quad \rightarrow \quad \sin\theta = -\frac{2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$

$$\theta = \frac{4\pi}{3} + 2k\pi \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2k\pi, \; k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{4\pi}{3}, \frac{5\pi}{3}$.

35. $\sin\theta = 0.4$

$\theta \approx 0.4115168$ or $\theta \approx \pi - 0.4115168 = 2.7300758$
$\theta \approx 0.41, 2.73$

36. $\cos\theta = 0.6$

$\theta \approx 0.92729522$ or $\theta \approx 2\pi - 0.92729522 = 5.35589009$
$\theta \approx 0.93, 5.36$

37. $\tan\theta = 5$

$\theta \approx 1.3734008$ or $\theta \approx \pi + 1.3734008 = 4.5149934$
$\theta \approx 1.37, \; 4.51$

38. $\cot\theta = 2 \quad \rightarrow \quad \tan\theta = \frac{1}{2}$

$\theta \approx 0.46364761$ or $\theta \approx \pi + 0.46364761 = 3.60524026$
$\theta \approx 0.46, 3.61$

39. $\cos\theta = -0.9$

$\theta \approx 2.6905658$ or $\theta \approx 2\pi - 2.6905658 = 3.5926195$
$\theta \approx 2.69, 3.59$

40. $\sin\theta = -0.2$

$\theta \approx -0.20135792 = 6.08182739$ or $\theta \approx \pi + 0.20135792 = 3.34295057$
$\theta \approx 3.34, \; 6.08$

41. $\sec\theta = -4 \quad \rightarrow \quad \cos\theta = -\frac{1}{4}$

$$\theta \approx 1.8234766 \text{ or } \theta \approx 2\pi - 1.8234766 = 4.4597087$$

$\theta \approx 1.82,\ 4.46$

42. $\csc\theta = -3 \quad \rightarrow \quad \sin\theta = -\frac{1}{3}$

$$\theta \approx -0.33983691 = 5.94334840 \text{ or } \theta \approx \pi + 0.33983691 = 3.48142956$$

$\theta \approx 3.48,\ 5.94$

43. Use Snell's Law to solve:

$$\frac{\sin(40^\circ)}{\sin\theta_2} = 1.33 \rightarrow \sin(40^\circ) = 1.33\sin\theta_2 \rightarrow \sin\theta_2 = \frac{\sin(40^\circ)}{1.33} \approx 0.4833$$

$$\theta_2 = \sin^{-1}(0.4833) \approx 28.9^\circ$$

44. Use Snell's Law to solve:

$$\frac{\sin(50^\circ)}{\sin\theta_2} = 1.66 \rightarrow \sin(50^\circ) = 1.66\sin\theta_2 \rightarrow \sin\theta_2 = \frac{\sin(50^\circ)}{1.66} \approx 0.4615$$

$$\theta_2 = \sin^{-1}(0.4615) \approx 27.5^\circ$$

45. Calculate the index of refraction for each:

$\theta_1 = 10^\circ,\ \theta_2 = 7^\circ 45' = 7.75^\circ \qquad \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin(10^\circ)}{\sin(7.75^\circ)} \approx 1.2877$

$\theta_1 = 20^\circ,\ \theta_2 = 15^\circ 30' = 15.5^\circ \qquad \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin(20^\circ)}{\sin(15.5^\circ)} \approx 1.2798$

$\theta_1 = 30^\circ,\ \theta_2 = 22^\circ 30' = 22.5^\circ \qquad \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin(30^\circ)}{\sin(22.5^\circ)} \approx 1.3066$

$\theta_1 = 40^\circ,\ \theta_2 = 29^\circ 0' = 29^\circ \qquad \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin(40^\circ)}{\sin(29^\circ)} \approx 1.3259$

$\theta_1 = 50^\circ,\ \theta_2 = 35^\circ 0' = 35^\circ \qquad \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin(50^\circ)}{\sin(35^\circ)} \approx 1.3356$

$\theta_1 = 60^\circ,\ \theta_2 = 40^\circ 30' = 40.5^\circ \qquad \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin(60^\circ)}{\sin(40.5^\circ)} \approx 1.3335$

$\theta_1 = 70^\circ,\ \theta_2 = 45^\circ 30' = 45.5^\circ \qquad \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin(70^\circ)}{\sin(45.5^\circ)} \approx 1.3175$

$\theta_1 = 80^\circ,\ \theta_2 = 50^\circ 0' = 50^\circ \qquad \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin(80)^\circ}{\sin(50^\circ)} \approx 1.2856$

The results range from 1.28 to 1.34 and are surprisingly close to Snell's Law.

46. $\frac{v_1}{v_2} = \frac{2.99 \times 10^8}{1.92 \times 10^8} \approx 1.56$

The index of refraction for this liquid is 1.56.

47. Calculate the index of refraction:

$\theta_1 = 40^\circ,\ \theta_2 = 26^\circ \qquad \dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{\sin(40^\circ)}{\sin(26^\circ)} \approx 1.47$

48. The index of refraction of crown glass is 1.52.

$$\frac{\sin(30^\circ)}{\sin\theta_2} = 1.52 \rightarrow \sin\theta_2 = \frac{\sin(30^\circ)}{1.52} = 0.3289$$

$$\theta_2 = 19.2^\circ$$

The angle of refraction is 19.2˚.

49. If θ is the original angle of incidence and ϕ is the angle of refraction, then $\dfrac{\sin\theta}{\sin\phi} = n_2$. The angle of incidence of the emerging beam is also ϕ, and the index of refraction is $\dfrac{1}{n_2}$. Thus, θ is the angle of refraction of the emerging beam. The two beams are parallel since the original angle of incidence and the angle of refraction of the emerging beam are equal.