# Chapter 8

# Analytic Trigonometry

## 8.R Chapter Review

1. $\sin^{-1}(1)$

   We are finding the angle $\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals 1.

   $$\sin\theta = 1 \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

   $$\theta = \frac{\pi}{2} \rightarrow \sin^{-1}(1) = \frac{\pi}{2}$$

2. $\cos^{-1}(0)$

   We are finding the angle $\theta$, $0 \le \theta \le \pi$, whose cosine equals 0.

   $$\cos\theta = 0 \qquad 0 \le \theta \le \pi$$

   $$\theta = \frac{\pi}{2} \rightarrow \cos^{-1}(0) = \frac{\pi}{2}$$

3. $\tan^{-1}(1)$

   We are finding the angle $\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals 1.

   $$\tan\theta = 1 \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

   $$\theta = \frac{\pi}{4} \rightarrow \tan^{-1}(1) = \frac{\pi}{4}$$

4. $\sin^{-1}\left(-\frac{1}{2}\right)$

   We are finding the angle $\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

   $$\sin\theta = -\frac{1}{2} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

   $$\theta = -\frac{\pi}{6} \rightarrow \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

5. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

We are finding the angle $\theta$, $0 \le \theta \le \pi$, whose cosine equals $-\frac{\sqrt{3}}{2}$.

$$\cos\theta = -\frac{\sqrt{3}}{2} \qquad 0 \le \theta \le \pi$$

$$\theta = \frac{5\pi}{6} \rightarrow \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

6. $\tan^{-1}\left(-\sqrt{3}\right)$

We are finding the angle $\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $-\sqrt{3}$.

$$\tan\theta = -\sqrt{3} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{3} \rightarrow \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

7. $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

Find the angle $\theta$, $0 \le \theta \le \pi$, whose cosine equals $\frac{\sqrt{2}}{2}$.

$$\cos\theta = \frac{\sqrt{2}}{2} \qquad 0 \le \theta \le \pi$$

$$\theta = \frac{\pi}{4} \rightarrow \sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

8. $\cos\left(\sin^{-1}(0)\right)$

Find the angle $\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals 0.

$$\sin\theta = 0 \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\theta = 0 \rightarrow \cos\left(\sin^{-1}(0)\right) = \cos(0) = 1$$

9. $\tan\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

Find the angle $\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{\sqrt{3}}{2}$.

$$\sin\theta = -\frac{\sqrt{3}}{2} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{3} \rightarrow \tan\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$$

10. $\tan\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$

Find the angle $\theta$, $0 \le \theta \le \pi$, whose cosine equals $-\frac{1}{2}$.

$$\cos\theta = -\frac{1}{2} \qquad 0 \le \theta \le \pi$$

$$\theta = \frac{2\pi}{3} \rightarrow \tan\left(\cos^{-1}\left(-\frac{1}{2}\right)\right) = \tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$$

11. $\sec\left(\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right)$

Find the angle $\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent is $\frac{\sqrt{3}}{3}$

$$\tan\theta = \frac{\sqrt{3}}{3}, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\theta = \frac{\pi}{6} \rightarrow \sec\left(\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right) = \sec\left(\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}$$

12. $\csc\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

Find the angle $\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $\frac{\sqrt{3}}{2}$.

$$\sin\theta = \frac{\sqrt{3}}{2} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\theta = \frac{\pi}{3} \rightarrow \csc\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) = \csc\left(\frac{\pi}{3}\right) = \frac{2\sqrt{3}}{3}$$

13. $\sin\left(\tan^{-1}\left(\frac{3}{4}\right)\right)$

Since $\tan\theta = \frac{3}{4}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, let $x = 4$ and $y = 3$. Solve for r:

$$16 + 9 = r^2 \rightarrow r^2 = 25 \rightarrow r = 5$$

$\theta$ is in quadrant I.

$$\sin\left(\tan^{-1}\left(\frac{3}{4}\right)\right) = \sin\theta = \frac{y}{r} = \frac{3}{5}$$

14. $\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

Since $\sin\theta = \frac{3}{5}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, let $r = 5$ and $y = 3$. Solve for x:

$$x^2 + 9 = 25 \rightarrow x^2 = 16 \rightarrow x = \pm 4$$

Since $\theta$ is in quadrant I, $x = 4$

$$\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right) = \cos\theta = \frac{x}{r} = \frac{4}{5}$$

15. $\tan\left(\sin^{-1}\left(-\frac{4}{5}\right)\right)$

Since $\sin\theta = -\frac{4}{5}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, let $y = -4$ and $r = 5$. Solve for x:

$$x^2 + 16 = 25 \rightarrow x^2 = 9 \rightarrow x = \pm 3$$

Since $\theta$ is in quadrant IV, $x = 3$.

$$\tan\left(\sin^{-1}\left(-\frac{4}{5}\right)\right) = \tan\theta = \frac{y}{x} = \frac{-4}{3} = -\frac{4}{3}$$

16. $\tan\left(\cos^{-1}\left(-\frac{3}{5}\right)\right)$

Since $\cos\theta = -\frac{3}{5}$, $0 \le \theta \le \pi$, let $x = -3$ and $r = 5$. Solve for y:

$$9 + y^2 = 25 \rightarrow y^2 = 16 \rightarrow y = \pm 4$$

Since $\theta$ is in quadrant II, $y = 4$.

$$\tan\left(\cos^{-1}\left(-\frac{3}{5}\right)\right) = \tan\theta = \frac{y}{x} = \frac{4}{-3} = -\frac{4}{3}$$

17. $\sin^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right) = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

18. $\cos^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right) = \cos^{-1}(-1) = \pi$

19. $\tan^{-1}\left(\tan\left(\frac{7\pi}{4}\right)\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$

20. $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

21. $\tan\theta\cot\theta - \sin^2\theta = \tan\theta \cdot \frac{1}{\tan\theta} - \sin^2\theta = 1 - \sin^2\theta = \cos^2\theta$

22. $\sin\theta\csc\theta - \sin^2\theta = \sin\theta \cdot \frac{1}{\sin\theta} - \sin^2\theta = 1 - \sin^2\theta = \cos^2\theta$

23. $\cos^2\theta(1+\tan^2\theta) = \cos^2\theta \cdot \sec^2\theta = \cos^2\theta \cdot \dfrac{1}{\cos^2\theta} = 1$

24. $(1-\cos^2\theta)(1+\cot^2\theta) = \sin^2\theta \cdot \csc^2\theta = \sin^2\theta \cdot \dfrac{1}{\sin^2\theta} = 1$

25. $$\begin{aligned} 4\cos^2\theta + 3\sin^2\theta &= \cos^2\theta + 3\cos^2\theta + 3\sin^2\theta = \cos^2\theta + 3(\cos^2\theta + \sin^2\theta) \\ &= \cos^2\theta + 3 \cdot 1 = \cos^2\theta + 3 = 3 + \cos^2\theta \end{aligned}$$

26. $$\begin{aligned} 4\sin^2\theta + 2\cos^2\theta &= 2\sin^2\theta + 2\sin^2\theta + 2\cos^2\theta = 2\sin^2\theta + 2(\sin^2\theta + \cos^2\theta) \\ &= 2\sin^2\theta + 2 \cdot 1 = 2(1-\cos^2\theta) + 2 = 2 - 2\cos^2\theta + 2 = 4 - 2\cos^2\theta \end{aligned}$$

27. $$\begin{aligned} \frac{1-\cos\theta}{\sin\theta} + \frac{\sin\theta}{1-\cos\theta} &= \frac{(1-\cos\theta)^2 + \sin^2\theta}{\sin\theta(1-\cos\theta)} = \frac{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}{\sin\theta(1-\cos\theta)} \\ &= \frac{1-2\cos\theta+1}{\sin\theta(1-\cos\theta)} = \frac{2-2\cos\theta}{\sin\theta(1-\cos\theta)} = \frac{2(1-\cos\theta)}{\sin\theta(1-\cos\theta)} = \frac{2}{\sin\theta} = 2\csc\theta \end{aligned}$$

28. $$\begin{aligned} \frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} &= \frac{\sin^2\theta + (1+\cos\theta)^2}{\sin\theta(1+\cos\theta)} = \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)} \\ &= \frac{1+2\cos\theta+1}{\sin\theta(1+\cos\theta)} = \frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)} = \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = \frac{2}{\sin\theta} = 2\csc\theta \end{aligned}$$

29. $$\frac{\cos\theta}{\cos\theta-\sin\theta} = \frac{\cos\theta}{\cos\theta-\sin\theta} \cdot \frac{\left(\dfrac{1}{\cos\theta}\right)}{\left(\dfrac{1}{\cos\theta}\right)} = \frac{1}{\left(1-\dfrac{\sin\theta}{\cos\theta}\right)} = \frac{1}{1-\tan\theta}$$

30. $$\begin{aligned} 1 - \frac{\cos^2\theta}{1+\sin\theta} &= \frac{1+\sin\theta-\cos^2\theta}{1+\sin\theta} = \frac{1+\sin\theta-(1-\sin^2\theta)}{1+\sin\theta} = \frac{\sin\theta+\sin^2\theta}{1+\sin\theta} \\ &= \frac{\sin\theta(1+\sin\theta)}{1+\sin\theta} = \sin\theta \end{aligned}$$

31. $$\frac{\csc\theta}{1+\csc\theta} = \frac{\left(\dfrac{1}{\sin\theta}\right)}{\left(1+\dfrac{1}{\sin\theta}\right)} = \frac{\left(\dfrac{1}{\sin\theta}\right)}{\left(\dfrac{\sin\theta+1}{\sin\theta}\right)} = \left(\frac{1}{1+\sin\theta}\right)\left(\frac{1-\sin\theta}{1-\sin\theta}\right) = \frac{1-\sin\theta}{1-\sin^2\theta} = \frac{1-\sin\theta}{\cos^2\theta}$$

32. $$\frac{1+\sec\theta}{\sec\theta} = \frac{\left(1+\dfrac{1}{\cos\theta}\right)}{\left(\dfrac{1}{\cos\theta}\right)} = \frac{\left(\dfrac{\cos\theta+1}{\cos\theta}\right)}{\left(\dfrac{1}{\cos\theta}\right)} = \left(\frac{1+\cos\theta}{1}\right)\left(\frac{1-\cos\theta}{1-\cos\theta}\right) = \frac{1-\cos^2\theta}{1-\cos\theta} = \frac{\sin^2\theta}{1-\cos\theta}$$

33. $$\csc\theta - \sin\theta = \frac{1}{\sin\theta} - \sin\theta = \frac{1-\sin^2\theta}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta} = \cos\theta \cdot \frac{\cos\theta}{\sin\theta} = \cos\theta\cot\theta$$

34. $$\frac{\csc\theta}{1-\cos\theta} = \left(\frac{\left(\frac{1}{\sin\theta}\right)}{1-\cos\theta}\right)\left(\frac{1+\cos\theta}{1+\cos\theta}\right) = \left(\frac{1}{\sin\theta}\right)\left(\frac{1+\cos\theta}{1-\cos^2\theta}\right) = \frac{1+\cos\theta}{\sin\theta\sin^2\theta} = \frac{1+\cos\theta}{\sin^3\theta}$$

35. $$\frac{1-\sin\theta}{\sec\theta} = \cos\theta(1-\sin\theta) = \cos\theta(1-\sin\theta)\ \frac{1+\sin\theta}{1+\sin\theta} = \frac{\cos\theta(1-\sin^2\theta)}{1+\sin\theta}$$
$$= \frac{\cos\theta(\cos^2\theta)}{1+\sin\theta} = \frac{\cos^3\theta}{1+\sin\theta}$$

36. $$\frac{1-\cos\theta}{1+\cos\theta} = \frac{\left(\frac{1-\cos\theta}{\sin\theta}\right)}{\left(\frac{1+\cos\theta}{\sin\theta}\right)} = \left(\frac{\csc\theta-\cot\theta}{\csc\theta+\cot\theta}\right)\left(\frac{\csc\theta-\cot\theta}{\csc\theta-\cot\theta}\right) = \frac{(\csc\theta-\cot\theta)^2}{\csc^2\theta-\cot^2\theta}$$
$$= \frac{(\csc\theta-\cot\theta)^2}{1} = (\csc\theta-\cot\theta)^2$$

37. $$\cot\theta-\tan\theta = \frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta-\sin^2\theta}{\sin\theta\cos\theta} = \frac{1-\sin^2\theta-\sin^2\theta}{\sin\theta\cos\theta} = \frac{1-2\sin^2\theta}{\sin\theta\cos\theta}$$

38. $$\frac{\left(2\sin^2\theta-1\right)^2}{\sin^4\theta-\cos^4\theta} = \frac{\left(-\left(1-2\sin^2\theta\right)\right)^2}{\left(\sin^2\theta-\cos^2\theta\right)\left(\sin^2\theta+\cos^2\theta\right)} = \frac{\left(-\cos(2\theta)\right)^2}{-1\left(\cos^2\theta-\sin^2\theta\right)\ 1}$$
$$= \frac{\cos^2(2\theta)}{-\cos\left(2\theta\right)} = -\cos(2\theta) = -1\left(2\cos^2\theta-1\right) = 1-2\cos^2\theta$$

39. $$\frac{\cos(\alpha+\beta)}{\cos\alpha\sin\beta} = \frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\sin\beta} = \frac{\cos\alpha\cos\beta}{\cos\alpha\sin\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\sin\beta} = \cot\beta-\tan\alpha$$

40. $$\frac{\sin(\alpha-\beta)}{\sin\alpha\cos\beta} = \frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\sin\alpha\cos\beta} = \frac{\sin\alpha\cos\beta}{\sin\alpha\cos\beta}-\frac{\cos\alpha\sin\beta}{\sin\alpha\cos\beta} = 1-\cot\alpha\tan\beta$$

41. $$\frac{\cos(\alpha-\beta)}{\cos\alpha\cos\beta} = \frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\cos\beta} = \frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta} = 1+\tan\alpha\tan\beta$$

42. $$\frac{\cos(\alpha+\beta)}{\sin\alpha\cos\beta} = \frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\cos\beta} = \frac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\sin\alpha\cos\beta} = \cot\alpha-\tan\beta$$

43. $$(1+\cos\theta)\left(\tan\left(\frac{\theta}{2}\right)\right) = (1+\cos\theta)\ \frac{\sin\theta}{1+\cos\theta} = \sin\theta$$

44. $$\sin\theta\tan\left(\frac{\theta}{2}\right) = \sin\theta\ \frac{1-\cos\theta}{\sin\theta} = 1-\cos\theta$$

45. $2\cot\theta\cot(2\theta) = 2\,\frac{\cos\theta}{\sin\theta}\,\frac{\cos 2\theta}{\sin 2\theta} = \frac{2\cos\theta(\cos^2\theta - \sin^2\theta)}{\sin\theta\, 2\sin\theta\cos\theta} = \frac{\cos^2\theta - \sin^2\theta}{\sin^2\theta}$
$= \cot^2\theta - 1$

46. $2\sin(2\theta)\left(1 - 2\sin^2\theta\right) = 2\sin(2\theta)\cos(2\theta) = \sin(2(2\theta)) = \sin(4\theta)$

47. $1 - 8\sin^2\theta\cos^2\theta = 1 - 2\left(2\sin\theta\cos\theta\right)^2 = 1 - 2\sin^2(2\theta) = \cos(4\theta)$

48. $\frac{\sin(3\theta)\cos\theta - \cos(3\theta)\sin\theta}{\sin(2\theta)} = \frac{\sin(3\theta - \theta)}{\sin(2\theta)} = \frac{\sin(2\theta)}{\sin(2\theta)} = 1$

49. $\frac{\sin(2\theta) + \sin(4\theta)}{\cos(2\theta) + \cos(4\theta)} = \frac{2\sin(3\theta)\cos(-\theta)}{2\cos(3\theta)\cos(-\theta)} = \frac{\sin(3\theta)}{\cos(3\theta)} = \tan(3\theta)$

50. $\frac{\sin(2\theta) + \sin(4\theta)}{\sin(2\theta) - \sin(4\theta)} + \frac{\tan(3\theta)}{\tan\theta} = \frac{2\sin(3\theta)\cos(-\theta)}{2\sin(-\theta)\cos(3\theta)} + \frac{\tan(3\theta)}{\tan\theta}$
$= \tan(3\theta)\cot(-\theta) + \frac{\tan(3\theta)}{\tan\theta} = -\frac{\tan(3\theta)}{\tan\theta} + \frac{\tan(3\theta)}{\tan\theta} = 0$

51. $\frac{\cos(2\theta) - \cos(4\theta)}{\cos(2\theta) + \cos(4\theta)} - \tan\theta\tan(3\theta) = \frac{-2\sin(3\theta)\sin(-\theta)}{2\cos(3\theta)\cos(-\theta)} - \tan\theta\tan(3\theta)$
$= \frac{2\sin(3\theta)\sin\theta}{2\cos(3\theta)\cos\theta} - \tan\theta\tan(3\theta) = \tan(3\theta)\tan\theta - \tan\theta\tan(3\theta) = 0$

52. $\cos(2\theta) - \cos(10\theta) = -2\sin(6\theta)\sin(-4\theta) = 2\sin(6\theta)\sin(4\theta)$
$= \frac{\sin(4\theta)}{\cos(4\theta)}\left(2\sin 6\theta\cos 4\theta\right) = \tan 4\theta\left(\sin(10\theta) + \sin(2\theta)\right)$
$= \tan(4\theta)\left(\sin(2\theta) + \sin(10\theta)\right)$

53. $\sin(165^\circ) = \sin(120^\circ + 45^\circ) = \sin(120^\circ)\cos(45^\circ) + \cos(120^\circ)\sin(45^\circ)$
$= \frac{\sqrt{3}}{2}\,\frac{\sqrt{2}}{2} + \frac{1}{2}\,\frac{\sqrt{2}}{2} = \frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)$

54. $\tan(105^\circ) = \sin(60^\circ + 45^\circ) = \frac{\tan(60^\circ) + \tan(45^\circ)}{1 - \tan(60^\circ)\tan(45^\circ)} = \frac{\sqrt{3} + 1}{1 - \sqrt{3}\;1} = \left(\frac{1 + \sqrt{3}}{1 - \sqrt{3}}\right)\left(\frac{1 + \sqrt{3}}{1 + \sqrt{3}}\right)$
$= \frac{1 + 2\sqrt{3} + 3}{1 - 3} = \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3}$

55. $\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}\,\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\,\frac{1}{2}$
$= \frac{1}{4}\left(\sqrt{6} - \sqrt{2}\right)$

56. $\sin\left(-\frac{\pi}{12}\right) = \sin\left(\frac{2\pi}{12} - \frac{3\pi}{12}\right) = \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) = \frac{1}{2}\,\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\,\frac{\sqrt{2}}{2}$

$$= \frac{\sqrt{2}-\sqrt{6}}{4}$$

57. $\cos(80^\circ)\cos(20^\circ)+\sin(80^\circ)\sin(20^\circ)=\cos(80^\circ-20^\circ)=\cos(60^\circ)=\frac{1}{2}$

58. $\sin(70^\circ)\cos(40^\circ)-\cos(70^\circ)\sin(40^\circ)=\sin(70^\circ-40^\circ)=\sin(30^\circ)=\frac{1}{2}$

59. $\tan\left(\frac{\pi}{8}\right)=\tan\left(\frac{\left(\frac{\pi}{4}\right)}{2}\right)=\sqrt{\frac{\left(1-\cos\left(\frac{\pi}{4}\right)\right)}{\left(1+\cos\left(\frac{\pi}{4}\right)\right)}}=\sqrt{\frac{\left(1-\frac{\sqrt{2}}{2}\right)}{\left(1+\frac{\sqrt{2}}{2}\right)}}=\sqrt{\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\left(\frac{2-\sqrt{2}}{2-\sqrt{2}}\right)}$

$$=\sqrt{\frac{6-4\sqrt{2}}{2}}=\sqrt{3-2\sqrt{2}}$$

60. $\sin\left(\frac{5\pi}{8}\right)=\sin\left(\frac{\left(\frac{5\pi}{4}\right)}{2}\right)=\sqrt{\frac{\left(1-\cos\left(\frac{5\pi}{4}\right)\right)}{2}}=\sqrt{\frac{\left(1-\left(-\frac{\sqrt{2}}{2}\right)\right)}{2}}=\sqrt{\frac{2+\sqrt{2}}{4}}=\frac{\sqrt{2+\sqrt{2}}}{2}$

61. $\sin\alpha=\frac{4}{5},\ 0<\alpha<\frac{\pi}{2};\qquad \sin\beta=\frac{5}{13},\ \frac{\pi}{2}<\beta<\pi$

$\cos\alpha=\frac{3}{5},\ \tan\alpha=\frac{4}{3},\ \cos\beta=-\frac{12}{13},\ \tan\beta=-\frac{5}{12},\quad 0<\frac{\alpha}{2}<\frac{\pi}{4},\quad \frac{\pi}{4}<\frac{\beta}{2}<\frac{\pi}{2}$

(a) $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta=\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)+\left(\frac{3}{5}\right)\left(\frac{5}{13}\right)=\frac{-48+15}{65}=-\frac{33}{65}$

(b) $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta=\left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)-\left(\frac{4}{5}\right)\left(\frac{5}{13}\right)=\frac{-36-20}{65}=-\frac{56}{65}$

(c) $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta=\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)-\left(\frac{3}{5}\right)\left(\frac{5}{13}\right)=\frac{-48-15}{65}=-\frac{63}{65}$

(d) $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\left(\frac{4}{3}\right)+\left(-\frac{5}{12}\right)}{\left(1-\left(\frac{4}{3}\right)\left(-\frac{5}{12}\right)\right)}=\frac{\left(\frac{11}{12}\right)}{\left(\frac{14}{9}\right)}=\frac{11}{12}\ \frac{9}{14}=\frac{33}{56}$

(e) $\sin(2\alpha)=2\sin\alpha\cos\alpha=2\ \frac{4}{5}\ \frac{3}{5}=\frac{24}{25}$

(f) $\cos(2\beta)=\cos^2\beta-\sin^2\beta=\left(-\frac{12}{13}\right)^2-\left(\frac{5}{13}\right)^2=\frac{144}{169}-\frac{25}{169}=\frac{119}{169}$

(g) $\sin\left(\frac{\beta}{2}\right)=\sqrt{\frac{1-\cos\beta}{2}}=\sqrt{\frac{1-\left(-\frac{12}{13}\right)}{2}}=\sqrt{\frac{\left(\frac{25}{13}\right)}{2}}=\sqrt{\frac{25}{26}}=\frac{5}{\sqrt{26}}=\frac{5\sqrt{26}}{26}$

(h) $\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{\left(1+\frac{3}{5}\right)}{2}} = \sqrt{\frac{\left(\frac{8}{5}\right)}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

62. $\cos\alpha = \frac{4}{5},\ 0 < \alpha < \frac{\pi}{2};\qquad \cos\beta = \frac{5}{13},\ -\frac{\pi}{2} < \beta < 0$

$\sin\alpha = \frac{3}{5},\ \tan\alpha = \frac{3}{4},\ \sin\beta = -\frac{12}{13},\ \tan\beta = -\frac{12}{5},\quad 0 < \frac{\alpha}{2} < \frac{\pi}{4},\quad -\frac{\pi}{4} < \frac{\beta}{2} < 0$

(a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) = \frac{15-48}{65} = -\frac{33}{65}$

(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) = \frac{20+36}{65} = \frac{56}{65}$

(c) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{3}{5}\cdot\frac{5}{13} - \frac{4}{5}\cdot\frac{-12}{13} = \frac{15+48}{65} = \frac{63}{65}$

(d) $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \frac{\frac{3}{4} + \left(-\frac{12}{5}\right)}{1 - \left(\frac{3}{4}\right)\left(-\frac{12}{5}\right)} = \frac{\left(-\frac{33}{20}\right)}{\left(\frac{56}{20}\right)} = -\frac{33}{20}\cdot\frac{20}{56} = -\frac{33}{56}$

(e) $\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$

(f) $\cos(2\beta) = \cos^2\beta - \sin^2\beta = \left(\frac{5}{13}\right)^2 - \left(\frac{-12}{13}\right)^2 = \frac{25}{169} - \frac{144}{169} = -\frac{119}{169}$

(g) $\sin\left(\frac{\beta}{2}\right) = -\sqrt{\frac{1-\cos\beta}{2}} = -\sqrt{\frac{\left(1-\left(\frac{5}{13}\right)\right)}{2}} = -\sqrt{\frac{\left(\frac{8}{13}\right)}{2}} = -\sqrt{\frac{4}{13}} = -\frac{2}{\sqrt{13}} = -\frac{2\sqrt{13}}{13}$

(h) $\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{\left(1+\frac{4}{5}\right)}{2}} = \sqrt{\frac{\left(\frac{9}{5}\right)}{2}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$

63. $\sin\alpha = -\frac{3}{5},\ \pi < \alpha < \frac{3\pi}{2};\qquad \cos\beta = \frac{12}{13},\ \frac{3\pi}{2} < \beta < 2\pi$

$\cos\alpha = -\frac{4}{5},\ \tan\alpha = \frac{3}{4},\ \sin\beta = -\frac{5}{13},\ \tan\beta = -\frac{5}{12},\quad \frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4},\quad \frac{3\pi}{4} < \frac{\beta}{2} < \pi$

(a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right) + \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right) = \frac{-36+20}{65} = -\frac{16}{65}$

(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) - \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) = \frac{-48-15}{65} = -\frac{63}{65}$

(c) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right) - \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right) = \frac{-36-20}{65} = -\frac{56}{65}$

(d) $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \frac{\frac{3}{4} + \left(-\frac{5}{12}\right)}{1 - \left(\frac{3}{4}\right)\left(-\frac{5}{12}\right)} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{21}{16}\right)} = \frac{1}{3}\cdot\frac{16}{21} = \frac{16}{63}$

(e) $\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2\left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right) = \frac{24}{25}$

(f) $\cos(2\beta) = \cos^2\beta - \sin^2\beta = \left(\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 = \frac{144}{169} - \frac{25}{169} = \frac{119}{169}$

(g) $\sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{1-\cos\beta}{2}} = \sqrt{\frac{1-\left(\frac{12}{13}\right)}{2}} = \sqrt{\frac{\left(\frac{1}{13}\right)}{2}} = \sqrt{\frac{1}{26}} = \frac{1}{\sqrt{26}} = \frac{\sqrt{26}}{26}$

(h) $\cos\left(\frac{\alpha}{2}\right) = -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1+\left(-\frac{4}{5}\right)}{2}} = -\sqrt{\frac{\left(\frac{1}{5}\right)}{2}} = -\sqrt{\frac{1}{10}} = -\frac{1}{\sqrt{10}} = -\frac{\sqrt{10}}{10}$

64. $\sin\alpha = -\frac{4}{5},\ -\frac{\pi}{2} < \alpha < 0; \qquad \cos\beta = -\frac{5}{13},\ \frac{\pi}{2} < \beta < \pi$

$\cos\alpha = \frac{3}{5}, \tan\alpha = -\frac{4}{3}, \sin\beta = \frac{12}{13}, \tan\beta = -\frac{12}{5}, \quad -\frac{\pi}{4} < \frac{\alpha}{2} < 0, \quad \frac{\pi}{4} < \frac{\beta}{2} < \frac{\pi}{2}$

(a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \frac{3}{5}\cdot\frac{12}{13} = \frac{20+36}{65} = \frac{56}{65}$

(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) - \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) = \frac{-15+48}{65} = \frac{33}{65}$

(c) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) = \frac{20-36}{65} = -\frac{16}{65}$

(d) $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \frac{\left(-\frac{4}{3}\right) + \left(-\frac{12}{5}\right)}{1-\left(-\frac{4}{3}\right)\left(-\frac{12}{5}\right)} = \frac{\left(-\frac{56}{15}\right)}{\left(-\frac{33}{15}\right)} = \left(-\frac{56}{15}\right)\left(-\frac{15}{33}\right) = \frac{56}{33}$

(e) $\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) = -\frac{24}{25}$

(f) $\cos(2\beta) = \cos^2\beta - \sin^2\beta = \left(-\frac{5}{13}\right)^2 - \left(\frac{12}{13}\right)^2 = \frac{25}{169} - \frac{144}{169} = -\frac{119}{169}$

(g) $\sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{1-\cos\beta}{2}} = \sqrt{\frac{1-\left(-\frac{5}{13}\right)}{2}} = \sqrt{\frac{\left(\frac{18}{13}\right)}{2}} = \sqrt{\frac{9}{13}} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$

(h) $\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{\left(1+\frac{3}{5}\right)}{2}} = \sqrt{\frac{\left(\frac{8}{5}\right)}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

65. $\tan\alpha = \frac{3}{4},\ \pi < \alpha < \frac{3\pi}{2}; \qquad \tan\beta = \frac{12}{5},\ 0 < \beta < \frac{\pi}{2}$

$\sin\alpha = -\frac{3}{5}, \cos\alpha = -\frac{4}{5}, \sin\beta = \frac{12}{13}, \cos\beta = \frac{5}{13}, \quad \frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}, \quad 0 < \frac{\beta}{2} < \frac{\pi}{4}$

(a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) = \frac{-15-48}{65} = -\frac{63}{65}$

(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(-\frac{4}{5}\right)\left(\frac{5}{13}\right) - \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right) = \frac{-20+36}{65} = \frac{16}{65}$

(c) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) = \frac{-15+48}{65} = \frac{33}{65}$

(d) $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \dfrac{\left(\frac{3}{4}+\left(\frac{12}{5}\right)\right)}{\left(1-\left(\frac{3}{4}\right)\left(\frac{12}{5}\right)\right)} = \dfrac{\left(\frac{15+48}{20}\right)}{\left(-\frac{4}{5}\right)} = \left(\frac{63}{20}\right)\left(-\frac{5}{4}\right) = -\frac{63}{16}$

(e) $\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2\left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right) = \frac{24}{25}$

(f) $\cos(2\beta) = \cos^2\beta - \sin^2\beta = \left(\frac{5}{13}\right)^2 - \left(\frac{12}{13}\right)^2 = \frac{25}{169} - \frac{144}{169} = -\frac{119}{169}$

(g) $\sin\left(\frac{\beta}{2}\right) = \sqrt{\dfrac{1-\cos\beta}{2}} = \sqrt{\dfrac{\left(1-\left(\frac{5}{13}\right)\right)}{2}} = \sqrt{\dfrac{\left(\frac{8}{13}\right)}{2}} = \sqrt{\frac{4}{13}} = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$

(h) $\cos\left(\frac{\alpha}{2}\right) = -\sqrt{\dfrac{1+\cos\alpha}{2}} = -\sqrt{\dfrac{\left(1+\left(-\frac{4}{5}\right)\right)}{2}} = -\sqrt{\dfrac{\left(\frac{1}{5}\right)}{2}} = -\sqrt{\frac{1}{10}} = -\frac{1}{\sqrt{10}} = -\frac{\sqrt{10}}{10}$

66. $\tan\alpha = -\frac{4}{3},\ \frac{\pi}{2} < \alpha < \pi;\qquad \cot\beta = \frac{12}{5},\ \pi < \beta < \frac{3\pi}{2}$

$\sin\alpha = \frac{4}{5}, \cos\alpha = -\frac{3}{5}, \sin\beta = -\frac{5}{13}, \cos\beta = -\frac{12}{13},\quad \frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2},\quad \frac{\pi}{2} < \frac{\beta}{2} < \frac{3\pi}{4}$

(a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) = \frac{-48+15}{65} = -\frac{33}{65}$

(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(-\frac{3}{5}\right)\left(-\frac{12}{13}\right) - \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) = \frac{36+20}{65} = \frac{56}{65}$

(c) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{4}{5}\ \frac{-12}{13} - \frac{-3}{5}\ \frac{-5}{13} = \frac{-48-15}{65} = -\frac{63}{65}$

(d) $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \dfrac{\left(-\frac{4}{3}+\frac{5}{12}\right)}{\left(1-\left(-\frac{4}{3}\right)\left(\frac{5}{12}\right)\right)} = \dfrac{\left(\frac{-16+5}{12}\right)}{\left(\frac{56}{36}\right)} = \left(-\frac{11}{12}\right)\left(\frac{36}{56}\right) = -\frac{33}{56}$

(e) $\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$

(f) $\cos(2\beta) = \cos^2\beta - \sin^2\beta = \left(-\frac{12}{13}\right)^2 - \left(-\frac{5}{13}\right)^2 = \frac{144}{169} - \frac{25}{169} = \frac{119}{169}$

(g) $\sin\left(\frac{\beta}{2}\right) = \sqrt{\dfrac{1-\cos\beta}{2}} = \sqrt{\dfrac{\left(1-\left(-\frac{12}{13}\right)\right)}{2}} = \sqrt{\dfrac{\left(\frac{25}{13}\right)}{2}} = \sqrt{\frac{25}{26}} = \frac{5}{\sqrt{26}} = \frac{5\sqrt{26}}{26}$

(h) $\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{\left(\frac{2}{5}\right)}{2}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

47. $\sec\alpha = 2,\ -\frac{\pi}{2} < \alpha < 0; \qquad \sec\beta = 3,\ \frac{3\pi}{2} < \beta < 2\pi$

$\sin\alpha = -\frac{\sqrt{3}}{2},\ \cos\alpha = \frac{1}{2},\ \tan\alpha = -\sqrt{3},\ \sin\beta = -\frac{2\sqrt{2}}{3},\ \cos\beta = \frac{1}{3},\ \tan\beta = -2\sqrt{2},$

$-\frac{\pi}{4} < \frac{\alpha}{2} < 0, \quad \frac{3\pi}{4} < \frac{\beta}{2} < \pi$

(a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(-\frac{2\sqrt{2}}{3}\right) = \frac{-\sqrt{3}-2\sqrt{2}}{6}$

(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) - \left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{2\sqrt{2}}{3}\right) = \frac{1-2\sqrt{6}}{6}$

(c) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{2}\right)\left(-\frac{2\sqrt{2}}{3}\right) = \frac{-\sqrt{3}+2\sqrt{2}}{6}$

(d) $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{\left(-\sqrt{3}+\left(-2\sqrt{2}\right)\right)}{\left(1-\left(-\sqrt{3}\right)\left(-2\sqrt{2}\right)\right)} = \left(\frac{-\sqrt{3}-2\sqrt{2}}{1-2\sqrt{6}}\right)\left(\frac{1+2\sqrt{6}}{1+2\sqrt{6}}\right)$

$= \frac{-9\sqrt{3}-8\sqrt{2}}{-23} = \frac{9\sqrt{3}+8\sqrt{2}}{23}$

(e) $\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = -\frac{\sqrt{3}}{2}$

(f) $\cos(2\beta) = \cos^2\beta - \sin^2\beta = \left(\frac{1}{3}\right)^2 - \left(-\frac{2\sqrt{2}}{3}\right)^2 = \frac{1}{9} - \frac{8}{9} = -\frac{7}{9}$

(g) $\sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{1-\cos\beta}{2}} = \sqrt{\frac{1-\left(\frac{1}{3}\right)}{2}} = \sqrt{\frac{\left(\frac{2}{3}\right)}{2}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

(h) $\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{\left(1+\frac{1}{2}\right)}{2}} = \sqrt{\frac{\left(\frac{3}{2}\right)}{2}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

68. $\csc\alpha = 2,\ \frac{\pi}{2} < \alpha < \pi; \qquad \sec\beta = -3,\ \frac{\pi}{2} < \beta < \pi$

$\sin\alpha = \frac{1}{2},\ \cos\alpha = -\frac{\sqrt{3}}{2},\ \tan\alpha = -\frac{\sqrt{3}}{3},\ \sin\beta = \frac{2\sqrt{2}}{3},\ \cos\beta = -\frac{1}{3},\ \tan\beta = -2\sqrt{2},$

$\frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2}, \quad \frac{\pi}{4} < \frac{\beta}{2} < \frac{\pi}{2}$

(a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \left(\frac{1}{2}\right)\left(-\frac{1}{3}\right) + \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{2\sqrt{2}}{3}\right) = \frac{-1-2\sqrt{6}}{6}$

(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{3}\right) - \left(\frac{1}{2}\right)\left(\frac{2\sqrt{2}}{3}\right) = \frac{\sqrt{3}-2\sqrt{2}}{6}$

(c) $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta=\left(\frac{1}{2}\right)\left(-\frac{1}{3}\right)-\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{2\sqrt{2}}{3}\right)=\frac{-1+2\sqrt{6}}{6}$

(d) $$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\left(-\frac{\sqrt{3}}{3}+\left(-2\sqrt{2}\right)\right)}{\left(1-\left(-\frac{\sqrt{3}}{3}\right)\left(-2\sqrt{2}\right)\right)}=\frac{\left(\frac{-\sqrt{3}-6\sqrt{2}}{3}\right)}{\left(\frac{3-2\sqrt{6}}{3}\right)}$$

$$=\left(\frac{-\sqrt{3}-6\sqrt{2}}{3-2\sqrt{6}}\right)\left(\frac{3+2\sqrt{6}}{3+2\sqrt{6}}\right)=\frac{-3\sqrt{3}-2\sqrt{18}-18\sqrt{2}-12\sqrt{12}}{9-24}$$

$$=\frac{-27\sqrt{3}-24\sqrt{2}}{-15}=\frac{9\sqrt{3}+8\sqrt{2}}{5}$$

(e) $\sin(2\alpha)=2\sin\alpha\cos\alpha=2\left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)=-\frac{\sqrt{3}}{2}$

(f) $\cos(2\beta)=\cos^2\beta-\sin^2\beta=\left(-\frac{1}{3}\right)^2-\left(\frac{2\sqrt{2}}{3}\right)^2=\frac{1}{9}-\frac{8}{9}=-\frac{7}{9}$

(g) $\sin\left(\frac{\beta}{2}\right)=\sqrt{\frac{1-\cos\beta}{2}}=\sqrt{\frac{\left(1-\left(-\frac{1}{3}\right)\right)}{2}}=\sqrt{\frac{\left(\frac{4}{3}\right)}{2}}=\sqrt{\frac{4}{6}}=\frac{2}{\sqrt{6}}=\frac{2\sqrt{6}}{6}=\frac{\sqrt{6}}{3}$

(h) $\cos\left(\frac{\alpha}{2}\right)=\sqrt{\frac{1+\cos\alpha}{2}}=\sqrt{\frac{\left(1+\left(-\frac{\sqrt{3}}{2}\right)\right)}{2}}=\sqrt{\frac{\left(\frac{2-\sqrt{3}}{3}\right)}{2}}=\sqrt{\frac{2-\sqrt{3}}{4}}=\frac{\sqrt{2-\sqrt{3}}}{2}$

69. $\sin\alpha=-\frac{2}{3},\ \pi<\alpha<\frac{3\pi}{2};\qquad \cos\beta=-\frac{2}{3},\ \pi<\beta<\frac{3\pi}{2}$

$\cos\alpha=-\frac{\sqrt{5}}{3},\ \tan\alpha=\frac{2\sqrt{5}}{5},\ \sin\beta=-\frac{\sqrt{5}}{3},\ \tan\beta=\frac{\sqrt{5}}{2},\ \frac{\pi}{2}<\frac{\alpha}{2}<\frac{3\pi}{4},\ \frac{\pi}{2}<\frac{\beta}{2}<\frac{3\pi}{4}$

(a) $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta=\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)+\left(-\frac{\sqrt{5}}{3}\right)\left(-\frac{\sqrt{5}}{3}\right)=\frac{4+5}{9}=1$

(b) $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta=\left(-\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right)-\left(-\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right)=\frac{2\sqrt{5}-2\sqrt{5}}{9}=0$

(c) $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta=\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)-\left(-\frac{\sqrt{5}}{3}\right)\left(-\frac{\sqrt{5}}{3}\right)=\frac{4-5}{9}=-\frac{1}{9}$

(d) $$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\left(\frac{2\sqrt{5}}{5}+\frac{\sqrt{5}}{2}\right)}{\left(1-\left(\frac{2\sqrt{5}}{5}\right)\left(\frac{\sqrt{5}}{2}\right)\right)}=\frac{\left(\frac{4\sqrt{5}+5\sqrt{5}}{10}\right)}{\left(\frac{10-10}{10}\right)}=\frac{\left(\frac{9\sqrt{5}}{10}\right)}{0}\ \text{Undefined}$$

(e) $\sin(2\alpha)=2\sin\alpha\cos\alpha=2\left(-\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right)=\frac{4\sqrt{5}}{9}$

(f) $\cos(2\beta) = \cos^2\beta - \sin^2\beta = \left(\frac{2}{3}\right)^2 - \left(-\frac{\sqrt{5}}{3}\right)^2 = \frac{4}{9} - \frac{5}{9} = -\frac{1}{9}$

(g) $\sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{1-\cos\beta}{2}} = \sqrt{\frac{\left(1-\left(-\frac{2}{3}\right)\right)}{2}} = \sqrt{\frac{\left(\frac{5}{3}\right)}{2}} = \sqrt{\frac{5}{6}} = \frac{\sqrt{30}}{6}$

(h) $\cos\left(\frac{\alpha}{2}\right) = -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{\left(1+\left(-\frac{\sqrt{5}}{3}\right)\right)}{2}} = -\sqrt{\frac{\left(\frac{3-\sqrt{5}}{3}\right)}{2}} = -\sqrt{\frac{3-\sqrt{5}}{6}} = -\frac{\sqrt{18-6\sqrt{5}}}{6}$

70. $\tan\alpha = -2,\ \frac{\pi}{2} < \alpha < \pi;\qquad \cot\beta = -2,\ \frac{\pi}{2} < \beta < \pi \qquad \tan\beta = -\frac{1}{2}$

$\sin\alpha = \frac{2}{\sqrt{5}},\ \cos\alpha = -\frac{1}{\sqrt{5}},\ \sin\beta = \frac{1}{\sqrt{5}},\ \cos\beta = -\frac{2}{\sqrt{5}},\quad \frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2},\quad \frac{\pi}{4} < \frac{\beta}{2} < \frac{\pi}{2}$

(a) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \left(\frac{2}{\sqrt{5}}\right)\left(-\frac{2}{\sqrt{5}}\right) + \left(-\frac{1}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) = \frac{-4-1}{5} = -\frac{5}{5} = -1$

(b) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(-\frac{1}{\sqrt{5}}\right)\left(-\frac{2}{\sqrt{5}}\right) - \left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) = \frac{2-2}{5} = 0$

(c) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \left(\frac{2}{\sqrt{5}}\right)\left(-\frac{2}{\sqrt{5}}\right) - \left(-\frac{1}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) = \frac{-4+1}{5} = -\frac{3}{5}$

(d) $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \frac{\left(-2+\left(-\frac{1}{2}\right)\right)}{\left(1-(-2)\left(-\frac{1}{2}\right)\right)} = \frac{\left(-\frac{5}{2}\right)}{0} = \text{undefined}$

(e) $\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2\left(\frac{2}{\sqrt{5}}\right)\left(-\frac{1}{\sqrt{5}}\right) = -\frac{4}{5}$

(f) $\cos(2\beta) = \cos^2\beta - \sin^2\beta = \left(\frac{-2}{\sqrt{5}}\right)^2 - \left(\frac{1}{\sqrt{5}}\right)^2 = \frac{4}{5} - \frac{1}{5} = \frac{3}{5}$

(g) $\sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{1-\cos\beta}{2}} = \sqrt{\frac{\left(1-\left(-\frac{2}{\sqrt{5}}\right)\right)}{2}} = \sqrt{\frac{\left(\frac{\sqrt{5}+2}{\sqrt{5}}\right)}{2}} = \sqrt{\frac{\sqrt{5}+2}{2\sqrt{5}}} = \sqrt{\frac{5+2\sqrt{5}}{10}}$

(h) $\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{\left(1+\left(-\frac{1}{\sqrt{5}}\right)\right)}{2}} = \sqrt{\frac{\left(\frac{\sqrt{5}-1}{\sqrt{5}}\right)}{2}} = \sqrt{\frac{\sqrt{5}-1}{2\sqrt{5}}} = \sqrt{\frac{5-\sqrt{5}}{10}}$

71. $\cos\left(\sin^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{1}{2}\right)\right)$

Let $\alpha = \sin^{-1}\left(\frac{3}{5}\right)$ and $\beta = \cos^{-1}\left(\frac{1}{2}\right)$. $\alpha$ is in quadrant I; $\beta$ is in quadrant I.

Then $\sin\alpha = \frac{3}{5},\ -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, and $\cos\beta = \frac{1}{2},\ 0 \le \beta \le \pi$.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{1}{2}\right)^2} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$\cos\left(\sin^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{1}{2}\right)\right) = \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$= \left(\frac{4}{5}\right)\left(\frac{1}{2}\right) + \left(\frac{3}{5}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{4+3\sqrt{3}}{10}$$

72. $\sin\left(\cos^{-1}\left(\frac{5}{13}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right)$

Let $\alpha = \cos^{-1}\left(\frac{5}{13}\right)$ and $\beta = \cos^{-1}\left(\frac{4}{5}\right)$. $\alpha$ is in quadrant I; $\beta$ is in quadrant I.

Then $\cos\alpha = \frac{5}{13}$, $0 \le \alpha \le \pi$, and $\cos\beta = \frac{4}{5}$, $0 \le \beta \le \pi$.

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{5}{13}\right)^2} = \sqrt{1-\frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\sin\left(\cos^{-1}\left(\frac{5}{13}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right) = \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \left(\frac{12}{13}\right)\left(\frac{4}{5}\right) - \left(\frac{5}{13}\right)\left(\frac{3}{5}\right) = \frac{48-15}{65} = \frac{33}{65}$$

73. $\tan\left[\sin^{-1}\left(-\frac{1}{2}\right) - \tan^{-1}\left(\frac{3}{4}\right)\right]$

Let $\alpha = \sin^{-1}\left(-\frac{1}{2}\right)$ and $\beta = \tan^{-1}\left(\frac{3}{4}\right)$. $\alpha$ is in quadrant IV; $\beta$ is in quadrant I.

Then $\sin\alpha = -\frac{1}{2}$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, and $\tan\beta = \frac{3}{4}$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(-\frac{1}{2}\right)^2} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}; \quad \tan\alpha = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\tan\left[\sin^{-1}\left(\frac{1}{2}\right)-\tan^{-1}\left(\frac{3}{4}\right)\right]=\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\frac{\left(\frac{\sqrt{3}}{3}-\frac{3}{4}\right)}{\left(1+\left(\frac{\sqrt{3}}{3}\right)\left(\frac{3}{4}\right)\right)}$$

$$=\frac{\left(\frac{4\sqrt{3}-9}{12}\right)}{\left(1+\frac{3\sqrt{3}}{12}\right)}=\left(\frac{-9+4\sqrt{3}}{12+3\sqrt{3}}\right)\left(\frac{12-3\sqrt{3}}{12-3\sqrt{3}}\right)$$

$$=\frac{-144+75\sqrt{3}}{117}=\frac{-48+25\sqrt{3}}{39}$$

74. $\cos\left(\tan^{-1}(-1)+\cos^{-1}\left(-\frac{4}{5}\right)\right)$

Let $\alpha=\tan^{-1}(-1)$ and $\beta=\cos^{-1}\left(-\frac{4}{5}\right)$. $\alpha$ is in quadrant IV; $\beta$ is in quadrant II.

Then $\tan\alpha=-1$, $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, and $\cos\beta=-\frac{4}{5}$, $0\le\beta\le\pi$.

$$\sec\alpha=\sqrt{1+\tan^2\alpha}=\sqrt{1+(-1)^2}=\sqrt{2};\quad \cos\alpha=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

$$\sin\alpha=-\sqrt{1-\cos^2\alpha}=-\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}=-\sqrt{1-\frac{1}{2}}=-\sqrt{\frac{1}{2}}=-\frac{\sqrt{2}}{2}$$

$$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-\left(-\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

$$\cos\left(\tan^{-1}(-1)+\cos^{-1}\left(-\frac{4}{5}\right)\right)=\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$=\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{4}{5}\right)-\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{3}{5}\right)=\frac{-4\sqrt{2}+3\sqrt{2}}{10}=-\frac{\sqrt{2}}{10}$$

75. $\sin\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]$

Let $\alpha=\cos^{-1}\left(-\frac{3}{5}\right)$. $\alpha$ is in quadrant II.

Then $\cos\alpha=-\frac{3}{5}$, $0\le\alpha\le\pi$.

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\left(-\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

$$\sin\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]=\sin2\alpha=2\sin\alpha\cos\alpha=2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)=-\frac{24}{25}$$

76. $\cos\left(2\tan^{-1}\left(\frac{4}{5}\right)\right)$

Let $\alpha = \tan^{-1}\left(\frac{4}{3}\right)$. $\alpha$ is in quadrant I.

Then $\tan\alpha = \frac{4}{3}$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

$$\sec\alpha = \sqrt{\tan^2\alpha + 1} = \sqrt{\left(\frac{4}{3}\right)^2 + 1} = \sqrt{\frac{16}{9} + 1} = \sqrt{\frac{25}{9}} = \frac{5}{3}; \quad \cos\alpha = \frac{3}{5}$$

$$\cos\left[2\tan^{-1}\left(\frac{4}{3}\right)\right] = \cos 2\alpha = 2\cos^2\alpha - 1 = 2\left(\frac{3}{5}\right)^2 - 1 = 2\left(\frac{9}{25}\right) - 1 = -\frac{7}{25}$$

77. $\cos\theta = \frac{1}{2}$

$$\theta = \frac{\pi}{3} + 2k\pi \text{ or } \theta = \frac{5\pi}{3} + 2k\pi, \text{ where } k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

78. $\sin\theta = -\frac{\sqrt{3}}{2}$

$$\theta = \frac{4\pi}{3} + 2k\pi \text{ or } \theta = \frac{5\pi}{3} + 2k\pi, \text{ where } k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{4\pi}{3}, \frac{5\pi}{3}$

79. $2\cos\theta + \sqrt{2} = 0 \quad \rightarrow \quad 2\cos\theta = -\sqrt{2} \quad \rightarrow \quad \cos\theta = -\frac{\sqrt{2}}{2}$

$$\theta = \frac{3\pi}{4} + 2k\pi \quad \text{or} \quad \theta = \frac{5\pi}{4} + 2k\pi, \ k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}$.

80. $\tan\theta + \sqrt{3} = 0 \quad \rightarrow \quad \tan\theta = -\sqrt{3}$

$$\theta = \frac{2\pi}{3} + k\pi, \ k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{2\pi}{3}, \frac{5\pi}{3}$.

81. $\sin(2\theta) + 1 = 0 \quad \rightarrow \quad \sin(2\theta) = -1$

$$2\theta = \frac{3\pi}{2} + 2k\pi \quad \rightarrow \quad \theta = \frac{3\pi}{4} + k\pi, \text{ where } k \text{ is any integer}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

82. $\cos(2\theta) = 0$

$$\left.\begin{aligned} 2\theta &= \frac{\pi}{2} + 2k\pi \quad \rightarrow \quad \theta = \frac{\pi}{4} + k\pi \\ 2\theta &= \frac{3\pi}{2} + 2k\pi \quad \rightarrow \quad \theta = \frac{3\pi}{4} + k\pi \end{aligned}\right\}\text{, where } k \text{ is any integer}$$

The solutions on the interval $[0,\ 2\pi)$ are $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

83. $\tan(2\theta) = 0$

$2\theta = 0 + k\pi \quad \rightarrow \quad \theta = \frac{k\pi}{2}$, where $k$ is any integer

The solutions on the interval $[0,\ 2\pi)$ are $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

84. $\sin(3\theta) = 1$

$3\theta = \frac{\pi}{2} + 2k\pi \quad \rightarrow \quad \theta = \frac{\pi}{6} + \frac{2k\pi}{3}$, where $k$ is any integer

The solutions on the interval $[0,\ 2\pi)$ are $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

85. $\sec^2\theta = 4$

$\sec\theta = \pm 2 \rightarrow \cos\theta = \pm\frac{1}{2}$

$\theta = \frac{\pi}{3} + k\pi$, where $k$ is any integer

$\theta = \frac{2\pi}{3} + k\pi$, where $k$ is any integer

The solutions on the interval $[0,\ 2\pi)$ are $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

86. $\csc^2\theta = 1$

$\csc\theta = \pm 1 \rightarrow \sin\theta = \pm 1$

$\theta = \frac{\pi}{2} + k\pi$, where $k$ is any integer

The solutions on the interval $[0,\ 2\pi)$ are $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

87. $\sin\theta = \tan\theta$

$\sin\theta = \frac{\sin\theta}{\cos\theta} \rightarrow \sin\theta\cos\theta = \sin\theta$

$\sin\theta\cos\theta - \sin\theta = 0 \rightarrow \sin\theta(\cos\theta - 1) = 0$

$\cos\theta - 1 = 0 \rightarrow \cos\theta = 1 \rightarrow \theta = \pi$

or $\sin\theta = 0 \rightarrow \theta = 0$

The solutions on the interval $[0,\ 2\pi)$ are $\theta = 0, \pi$.

88. $\cos\theta = \sec\theta$

$$\cos\theta = \frac{1}{\cos\theta} \rightarrow \cos^2\theta = 1$$

$$\cos\theta = 1 \rightarrow \theta = 0$$

or $\cos\theta = -1 \rightarrow \theta = \pi$

The solutions on the interval $[0, 2\pi)$ are $\theta = 0, \pi$.

89. $\sin\theta + \sin(2\theta) = 0$

$$\sin\theta + 2\sin\theta\cos\theta = 0 \sin\theta(1 + 2\cos\theta) = 0$$

$$1 + 2\cos\theta = 0 \rightarrow \cos\theta = -\frac{1}{2} \rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

or $\sin\theta = 0 \rightarrow \theta = 0, \pi$

The solutions on the interval $[0, 2\pi)$ are $\theta = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$.

90. $\cos(2\theta) = \sin\theta$

$$1 - 2\sin^2\theta = \sin\theta \rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$$

$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$

$$2\sin\theta - 1 = 0 \rightarrow \sin\theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

or $\sin\theta + 1 = 0 \rightarrow \sin\theta = -1 \rightarrow \theta = \frac{3\pi}{2}$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

91. $\sin(2\theta) - \cos\theta - 2\sin\theta + 1 = 0$

$$2\sin\theta\cos\theta - \cos\theta - 2\sin\theta + 1 = 0 \rightarrow \cos\theta(2\sin\theta - 1) - 1(2\sin\theta - 1) = 0$$

$$(2\sin\theta - 1)(\cos\theta - 1) = 0$$

$$\sin\theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

or $\cos\theta = 1 \rightarrow \theta = 0$

The solutions on the interval $[0, 2\pi)$ are $\theta = 0, \frac{\pi}{6}, \frac{5\pi}{6}$.

92. $\sin(2\theta) - \sin\theta - 2\cos\theta + 1 = 0$

$$2\sin\theta\cos\theta - \sin\theta - 2\cos\theta + 1 = 0 \rightarrow \sin\theta(2\cos\theta - 1) - 1(2\cos\theta - 1) = 0$$

$$(2\cos\theta - 1)(\sin\theta - 1) = 0$$

$$\cos\theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

or $\sin\theta = 1 \rightarrow \theta = \frac{\pi}{2}$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{\pi}{3}, \frac{\pi}{2}, \frac{5\pi}{3}$.

93. $2\sin^2\theta - 3\sin\theta + 1 = 0$

$(2\sin\theta - 1)(\sin\theta - 1) = 0$

$$2\sin\theta - 1 = 0 \rightarrow \sin\theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\text{or } \sin\theta - 1 = 0 \rightarrow \sin\theta = 1 \rightarrow \theta = \frac{\pi}{2}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.

94. $2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$$2\cos\theta - 1 = 0 \rightarrow \cos\theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\text{or } \cos\theta + 1 = 0 \rightarrow \cos\theta = -1 \rightarrow \theta = \pi$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

95. $4\sin^2\theta = 1 + 4\cos\theta$

$$4\left(1 - \cos^2\theta\right) = 1 + 4\cos\theta \rightarrow 4 - 4\cos^2\theta = 1 + 4\cos\theta$$

$$4\cos^2\theta + 4\cos\theta - 3 = 0 \rightarrow \left(2\cos\theta - 1\right)\left(2\cos\theta + 3\right) = 0$$

$$2\cos\theta - 1 = 0 \rightarrow \cos\theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\text{or } 2\cos\theta + 3 = 0 \rightarrow \cos\theta = -\frac{3}{2}, \text{ which is impossible}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$.

96. $9 - 12\sin^2\theta = 4\cos^2\theta$

$$9 - 12\sin^2\theta = 4\left(1 - \sin^2\theta\right) \rightarrow 9 - 12\sin^2\theta = 4 - 4\sin^2\theta$$

$$5 = 8\sin^2\theta \rightarrow \sin^2\theta = \frac{5}{8} \rightarrow \sin\theta = \pm\sqrt{\frac{5}{8}} = \pm\frac{\sqrt{10}}{4}$$

The solutions on the interval $[0, 2\pi)$ are

97. $\sin(2\theta) = \sqrt{2}\cos\theta$

$2\sin\theta\cos\theta = \sqrt{2}\cos\theta \to 2\sin\theta\cos\theta - \sqrt{2}\cos\theta = 0 \to \cos\theta\left(2\sin\theta - \sqrt{2}\right) = 0$

$$\cos\theta = 0 \to \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\text{or } 2\sin\theta - \sqrt{2} = 0 \to \sin\theta = \frac{\sqrt{2}}{2} \to \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}$.

98. $1 + \sqrt{3}\cos\theta + \cos(2\theta) = 0$

$1 + \sqrt{3}\cos\theta + 2\cos^2(\theta) - 1 = 0 \to 2\cos^2(\theta) + \sqrt{3}\cos\theta = 0 \to \cos\theta\left(2\cos\theta + \sqrt{3}\right) = 0$

$$\cos\theta = 0 \to \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\text{or } 2\cos\theta + \sqrt{3} \to \cos\theta = -\frac{\sqrt{3}}{2} \to \theta = \frac{5\pi}{6}, \frac{7\pi}{6}$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}$.

99. $\sin\theta - \cos\theta = 1$

Divide each side by $\sqrt{2}$:

$$\frac{1}{\sqrt{2}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta = \frac{1}{\sqrt{2}}$$

Rewrite in the difference of two angles form where

$$\cos\phi = \frac{1}{\sqrt{2}} \text{ and } \sin\phi = \frac{1}{\sqrt{2}} \text{ and } \phi = \frac{\pi}{4}:$$

$$\sin\theta\cos\phi - \cos\theta\sin\phi = \frac{1}{\sqrt{2}}$$

$$\sin(\theta - \phi) = \frac{\sqrt{2}}{2}$$

$$\theta - \phi = \frac{\pi}{4} \text{ or } \theta - \phi = \frac{3\pi}{4}$$

$$\theta - \frac{\pi}{4} = \frac{\pi}{4} \text{ or } \theta - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\theta = \frac{\pi}{2} \text{ or } \theta = \pi$$

The solutions on the interval $[0, 2\pi)$ are $\theta = \frac{\pi}{2}, \pi$.

100. $\sin\theta + 2\cos\theta = 1$

$$2\cos\theta = 1 - \sin\theta \rightarrow 4\cos^2\theta = 1 - 2\sin\theta + \sin^2\theta$$

$$4(1 - \sin^2\theta) = 1 - 2\sin\theta + \sin^2\theta \rightarrow 4 - 4\sin^2\theta = 1 - 2\sin\theta + \sin^2\theta$$

$$5\sin^2\theta - 2\sin\theta - 3 = 0 \rightarrow (5\sin\theta + 3)(\sin\theta - 1) = 0$$

$$5\sin\theta + 3 = 0 \rightarrow \sin\theta = -\frac{3}{5} \rightarrow \theta = -0.6435, 3.79, 5.64$$

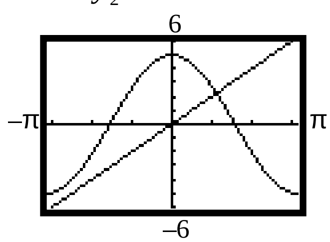
$$\text{or } \sin\theta - 1 = 0 \rightarrow \sin\theta = 1 \rightarrow \theta = \frac{\pi}{2}$$

The solutions on the interval $[0, 2\pi)$ are

101. $2x = 5\cos x$

Find the intersection of

$y_1 = 2x$ and $y_2 = 5\cos x$:

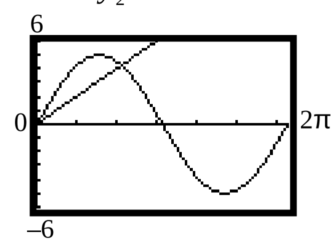



$x \approx 1.11$

102. $2x = 5\sin x$

Find the intersection of

$y_1 = 2x$ and $y_2 = 5\sin x$:

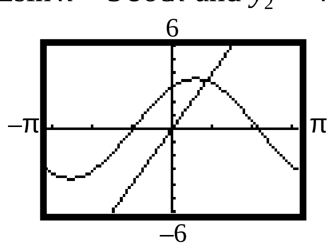



$x \approx 0, 2.13$

103. $2\sin x + 3\cos x = 4x$

Find the intersection of

$y_1 = 2\sin x + 3\cos x$ and $y_2 = 4x$:




$x \approx 0.87$

104. $3\cos x + x = \sin x$

Find the intersection of

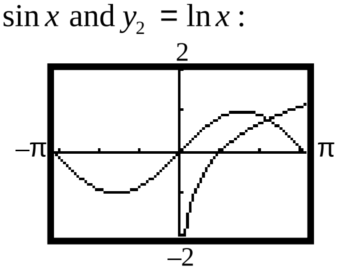
$y_1 = 3\cos x + x$ and $y_2 = \sin x$:


$x \approx 1.89, 3.07$

105. $\sin x = \ln x$

Find the intersection of

$y_1 = \sin x$ and $y_2 = \ln x$:

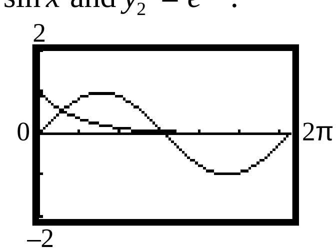



$x \approx 2.22$

106. $\sin x = e^{-x}$

Find the intersection of

$y_1 = \sin x$ and $y_2 = e^{-x}$:




$x \approx 0.59, 3.10$