Applications of Trigonometric Functions

9.2 The Law of Sines

1. $c = 5,\ \beta = 45°,\ \gamma = 95°$

$\alpha = 180° - \beta - \gamma = 180° - 45° - 95° = 40°$

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(40°)}{a} = \frac{\sin(95°)}{5} \rightarrow a = \frac{5\sin(40°)}{\sin(95°)} \approx 3.23$$

$$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(45°)}{b} = \frac{\sin(95°)}{5} \rightarrow b = \frac{5\sin(45°)}{\sin(95°)} \approx 3.55$$

2. $c = 4,\ \alpha = 45°,\ \beta = 40°$

$\gamma = 180° - \alpha - \beta = 180° - 45° - 40° = 95°$

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(45°)}{a} = \frac{\sin(95°)}{4} \rightarrow a = \frac{4\sin(45°)}{\sin(95°)} \approx 2.84$$

$$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(40°)}{b} = \frac{\sin(95°)}{4} \rightarrow b = \frac{4\sin(40°)}{\sin(95°)} \approx 2.58$$

3. $b = 3,\ \alpha = 50°,\ \gamma = 85°$

$\beta = 180° - \alpha - \gamma = 180° - 50° - 85° = 45°$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(50°)}{a} = \frac{\sin(45°)}{3} \rightarrow a = \frac{3\sin(50°)}{\sin(45°)} \approx 3.25$$

$$\frac{\sin\gamma}{c} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(85°)}{c} = \frac{\sin(45°)}{3} \rightarrow c = \frac{3\sin(85°)}{\sin(45°)} \approx 4.23$$

4. $b = 10,\ \beta = 30°,\ \gamma = 125°$

$\alpha = 180° - \beta - \gamma = 180° - 30° - 125° = 25°$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(25°)}{a} = \frac{\sin(30°)}{10} \rightarrow a = \frac{10\sin(25°)}{\sin(30°)} \approx 8.45$$

$$\frac{\sin\gamma}{c} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(125°)}{c} = \frac{\sin(30°)}{10} \rightarrow c = \frac{10\sin(125°)}{\sin(30°)} \approx 16.38$$

5. $b = 7,\ \alpha = 40^\circ,\ \beta = 45^\circ$

$$\gamma = 180^\circ - \alpha - \beta = 180^\circ - 40^\circ - 45^\circ = 95^\circ$$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(40^\circ)}{a} = \frac{\sin(45^\circ)}{7} \rightarrow a = \frac{7\sin(40^\circ)}{\sin(45^\circ)} \approx 6.36$$

$$\frac{\sin\gamma}{c} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(95^\circ)}{c} = \frac{\sin(45^\circ)}{7} \rightarrow c = \frac{7\sin(95^\circ)}{\sin(45^\circ)} \approx 9.86$$

6. $c = 5,\ \alpha = 10^\circ,\ \beta = 5^\circ$

$$\gamma = 180^\circ - \alpha - \beta = 180^\circ - 10^\circ - 5^\circ = 165^\circ$$

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(10^\circ)}{a} = \frac{\sin(165^\circ)}{5} \rightarrow a = \frac{5\sin(10^\circ)}{\sin(165^\circ)} \approx 3.35$$

$$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(5^\circ)}{b} = \frac{\sin(165^\circ)}{5} \rightarrow b = \frac{5\sin(5^\circ)}{\sin(165^\circ)} \approx 1.68$$

7. $b = 2,\ \beta = 40^\circ,\ \gamma = 100^\circ$

$$\alpha = 180^\circ - \beta - \gamma = 180^\circ - 40^\circ - 100^\circ = 40^\circ$$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(40^\circ)}{a} = \frac{\sin(40^\circ)}{2} \rightarrow a = \frac{2\sin(40^\circ)}{\sin(40^\circ)} = 2$$

$$\frac{\sin\gamma}{c} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(100^\circ)}{c} = \frac{\sin(40^\circ)}{2} \rightarrow c = \frac{2\sin(100^\circ)}{\sin(40^\circ)} \approx 3.06$$

8. $b = 6,\ \alpha = 100^\circ,\ \beta = 30^\circ$

$$\gamma = 180^\circ - \alpha - \beta = 180^\circ - 100^\circ - 30^\circ = 50^\circ$$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(100^\circ)}{a} = \frac{\sin(30^\circ)}{6} \rightarrow a = \frac{6\sin(100^\circ)}{\sin(30^\circ)} \approx 11.82$$

$$\frac{\sin\gamma}{c} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(50^\circ)}{c} = \frac{\sin(30^\circ)}{6} \rightarrow c = \frac{6\sin(50^\circ)}{\sin(30^\circ)} \approx 9.19$$

9. $\alpha = 40^\circ,\ \beta = 20^\circ,\ a = 2$

$$\gamma = 180^\circ - \alpha - \beta = 180^\circ - 40^\circ - 20^\circ = 120^\circ$$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(40^\circ)}{2} = \frac{\sin(20^\circ)}{b} \rightarrow b = \frac{2\sin(20^\circ)}{\sin(40^\circ)} \approx 1.06$$

$$\frac{\sin\gamma}{c} = \frac{\sin\alpha}{a} \rightarrow \frac{\sin(120^\circ)}{c} = \frac{\sin(40^\circ)}{2} \rightarrow c = \frac{2\sin(120^\circ)}{\sin(40^\circ)} \approx 2.69$$

10. $\alpha = 50^\circ, \gamma = 20^\circ, a = 3$

$\beta = 180^\circ - \alpha - \gamma = 180^\circ - 50^\circ - 20^\circ = 110^\circ$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(50^\circ)}{3} = \frac{\sin(110^\circ)}{b} \rightarrow b = \frac{3\sin(110^\circ)}{\sin(50^\circ)} \approx 3.68$$

$$\frac{\sin\gamma}{c} = \frac{\sin\alpha}{a} \rightarrow \frac{\sin(20^\circ)}{c} = \frac{\sin(50^\circ)}{3} \rightarrow c = \frac{3\sin(20^\circ)}{\sin(50^\circ)} \approx 1.34$$

11. $\beta = 70^\circ, \gamma = 10^\circ, \quad b = 5$

$\alpha = 180^\circ - \beta - \gamma = 180^\circ - 70^\circ - 10^\circ = 100^\circ$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(100^\circ)}{a} = \frac{\sin(70^\circ)}{5} \rightarrow a = \frac{5\sin(100^\circ)}{\sin(70^\circ)} \approx 5.24$$

$$\frac{\sin\gamma}{c} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(10^\circ)}{c} = \frac{\sin(70^\circ)}{5} \rightarrow c = \frac{5\sin(10^\circ)}{\sin(70^\circ)} \approx 0.92$$

12. $\alpha = 70^\circ, \beta = 60^\circ, \quad c = 4$

$\gamma = 180^\circ - \alpha - \beta = 180^\circ - 70^\circ - 60^\circ = 50^\circ$

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(70^\circ)}{a} = \frac{\sin(50^\circ)}{4} \rightarrow a = \frac{4\sin(70^\circ)}{\sin(50^\circ)} \approx 4.91$$

$$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(60^\circ)}{b} = \frac{\sin(50^\circ)}{4} \rightarrow b = \frac{4\sin(60^\circ)}{\sin(50^\circ)} \approx 4.52$$

13. $\alpha = 110^\circ, \gamma = 30^\circ, c = 3$

$\beta = 180^\circ - \alpha - \gamma = 180^\circ - 110^\circ - 30^\circ = 40^\circ$

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(110^\circ)}{a} = \frac{\sin(30^\circ)}{3} \rightarrow a = \frac{3\sin(110^\circ)}{\sin(30^\circ)} \approx 5.64$$

$$\frac{\sin\gamma}{c} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(30^\circ)}{3} = \frac{\sin(40^\circ)}{b} \rightarrow b = \frac{3\sin(40^\circ)}{\sin(30^\circ)} \approx 3.86$$

14. $\beta = 10^\circ, \quad \gamma = 100^\circ, b = 2$

$\alpha = 180^\circ - \beta - \gamma = 180^\circ - 10^\circ - 100^\circ = 70^\circ$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(70^\circ)}{a} = \frac{\sin(10^\circ)}{2} \rightarrow a = \frac{2\sin(70^\circ)}{\sin(10^\circ)} \approx 10.82$$

$$\frac{\sin\gamma}{c} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(100^\circ)}{c} = \frac{\sin(10^\circ)}{2} \rightarrow c = \frac{2\sin(100^\circ)}{\sin(10^\circ)} \approx 11.34$$

15. $\alpha = 40°$, $\beta = 40°$, $c = 2$
$\gamma = 180° - \alpha - \beta = 180° - 40° - 40° = 100°$
$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(40°)}{a} = \frac{\sin(100°)}{2} \rightarrow a = \frac{2\sin(40°)}{\sin(100°)} \approx 1.31$$
$$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(40°)}{b} = \frac{\sin(100°)}{2} \rightarrow b = \frac{2\sin(40°)}{\sin(100°)} \approx 1.31$$

16. $\beta = 20°$, $\gamma = 70°$, $a = 1$
$\alpha = 180° - \beta - \gamma = 180° - 20° - 70° = 90°$
$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} \rightarrow \frac{\sin(90°)}{1} = \frac{\sin(20°)}{b} \rightarrow b = \frac{1\sin(20°)}{\sin(90°)} \approx 0.34$$
$$\frac{\sin\gamma}{c} = \frac{\sin\alpha}{a} \rightarrow \frac{\sin(70°)}{c} = \frac{\sin(90°)}{1} \rightarrow c = \frac{1\sin(70°)}{\sin(90°)} \approx 0.94$$

17. $a = 3$, $b = 2$, $\alpha = 50°$
$$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a} \rightarrow \frac{\sin\beta}{2} = \frac{\sin(50°)}{3} \rightarrow \sin\beta = \frac{2\sin(50°)}{3} \approx 0.5107$$
$\rightarrow \beta = 30.7°$ or $\beta = 149.3°$
The second value is discarded because $\alpha + \beta > 180°$.
$\gamma = 180° - \alpha - \beta = 180° - 50° - 30.7° = 99.3°$
$$\frac{\sin\gamma}{c} = \frac{\sin\alpha}{a} \rightarrow \frac{\sin(99.3°)}{c} = \frac{\sin(50°)}{3} \rightarrow c = \frac{3\sin(99.3°)}{\sin(50°)} \approx 3.86$$
One triangle: $\beta \approx 30.7°$, $\gamma \approx 99.3°$, $c \approx 3.86$

18. $b = 4$, $c = 3$, $\beta = 40°$
$$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(40°)}{4} = \frac{\sin\gamma}{3} \rightarrow \sin\gamma = \frac{3\sin(40°)}{4} = 0.4821$$
$\rightarrow \gamma = 28.8°$ or $\gamma = 151.2°$
The second value is discarded because $\beta + \gamma > 180°$.
$\alpha = 180° - \beta - \gamma = 180° - 40° - 28.8° = 111.2°$
$$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a} \rightarrow \frac{\sin(40°)}{4} = \frac{\sin(111.2°)}{a} \rightarrow a = \frac{4\sin(111.2°)}{\sin(40°)} \approx 5.80$$
One triangle: $\alpha \approx 111.2°$, $\gamma \approx 28.8°$, $a \approx 5.80$

19. $b = 5$, $c = 3$, $\beta = 100°$
$$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(100°)}{5} = \frac{\sin\gamma}{3} \rightarrow \sin\gamma = \frac{3\sin(100°)}{5} = 0.5909$$
$\rightarrow \gamma = 36.2°$ or $\gamma = 143.8°$
The second value is discarded because $\beta + \gamma > 180°$.

$\alpha = 180^\circ - \beta - \gamma = 180^\circ - 100^\circ - 36.2^\circ = 43.8^\circ$

$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a} \quad \rightarrow \quad \frac{\sin(100^\circ)}{5} = \frac{\sin(43.8^\circ)}{a} \quad \rightarrow \quad a = \frac{5\sin(43.8^\circ)}{\sin(100^\circ)} \approx 3.51$

One triangle: $\alpha \approx 43.8^\circ$, $\gamma \approx 36.2^\circ$, $a \approx 3.51$

20. $a = 2$, $c = 1$, $\alpha = 120^\circ$

$\frac{\sin\gamma}{c} = \frac{\sin\alpha}{a} \quad \rightarrow \quad \frac{\sin\gamma}{1} = \frac{\sin(120^\circ)}{2} \quad \rightarrow \quad \sin\gamma = \frac{1\sin(120^\circ)}{2} = 0.4330$

$\rightarrow \quad \gamma = 25.7^\circ$ or $\gamma = 154.3^\circ$

The second value is discarded because $\alpha + \gamma > 180^\circ$.

$\beta = 180^\circ - \alpha - \gamma = 180^\circ - 120^\circ - 25.7^\circ = 34.3^\circ$

$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a} \quad \rightarrow \quad \frac{\sin(34.3^\circ)}{b} = \frac{\sin(120^\circ)}{2} \quad \rightarrow \quad b = \frac{2\sin(34.3^\circ)}{\sin(120^\circ)} \approx 1.30$

One triangle: $\beta \approx 34.3^\circ$, $\gamma \approx 25.7^\circ$, $b \approx 1.30$

21. $a = 4$, $b = 5$, $\alpha = 60^\circ$

$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a} \quad \rightarrow \quad \frac{\sin\beta}{5} = \frac{\sin(60^\circ)}{4} \quad \rightarrow \quad \sin\beta = \frac{5\sin(60^\circ)}{4} = 1.0825$

There is no angle β for which $\sin\beta > 1$. Therefore, there is no triangle with the given measurements.

22. $b = 2$, $c = 3$, $\beta = 40^\circ$

$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \quad \rightarrow \quad \frac{\sin(40^\circ)}{2} = \frac{\sin\gamma}{3} \quad \rightarrow \quad \sin\gamma = \frac{3\sin(40^\circ)}{2} = 0.9642$

$\rightarrow \quad \gamma_1 = 74.6^\circ$ or $\gamma_2 = 105.4^\circ$

For both values, $\beta + \gamma < 180^\circ$. Therefore, there are two triangles.

$\alpha_1 = 180^\circ - \beta - \gamma_1 = 180^\circ - 40^\circ - 74.6^\circ = 65.4^\circ$

$\frac{\sin\beta}{b} = \frac{\sin\alpha_1}{a_1} \quad \rightarrow \quad \frac{\sin(40^\circ)}{2} = \frac{\sin(65.4^\circ)}{a_1} \quad \rightarrow \quad a_1 = \frac{2\sin(65.4^\circ)}{\sin(40^\circ)} \approx 2.83$

$\alpha_2 = 180^\circ - \beta - \gamma_2 = 180^\circ - 40^\circ - 105.4^\circ = 34.6^\circ$

$\frac{\sin\beta}{b} = \frac{\sin\alpha_2}{a_2} \quad \rightarrow \quad \frac{\sin(40^\circ)}{2} = \frac{\sin(34.6^\circ)}{a_2} \quad \rightarrow \quad a_2 = \frac{2\sin(34.6^\circ)}{\sin(40^\circ)} \approx 1.77$

Two triangles: $\alpha_1 \approx 65.4^\circ$, $\gamma_1 \approx 74.6^\circ$, $a_1 \approx 2.83$
or $\alpha_2 \approx 34.6^\circ$, $\gamma_2 \approx 105.4^\circ$, $a_2 \approx 1.77$

23. $b = 4$, $c = 6$, $\beta = 20^\circ$

$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \quad \rightarrow \quad \frac{\sin(20^\circ)}{4} = \frac{\sin\gamma}{6} \quad \rightarrow \quad \sin\gamma = \frac{6\sin(20^\circ)}{4} \approx 0.5130$

$\rightarrow \quad \gamma_1 = 30.9^\circ$ or $\gamma_2 = 149.1^\circ$

For both values, $\beta + \gamma < 180^\circ$. Therefore, there are two triangles.

$\alpha_1 = 180^\circ - \beta - \gamma_1 = 180^\circ - 20^\circ - 30.9^\circ = 129.1^\circ$

$$\frac{\sin\beta}{b} = \frac{\sin\alpha_1}{a_1} \rightarrow \frac{\sin(20^\circ)}{4} = \frac{\sin(129.1^\circ)}{a_1} \rightarrow a_1 = \frac{4\sin(129.1^\circ)}{\sin(20^\circ)} \approx 9.08$$

$\alpha_2 = 180^\circ - \beta - \gamma_2 = 180^\circ - 20^\circ - 149.1^\circ = 10.9^\circ$

$$\frac{\sin\beta}{b} = \frac{\sin\alpha_2}{a_2} \rightarrow \frac{\sin(20^\circ)}{4} = \frac{\sin(10.9^\circ)}{a_2} \rightarrow a_2 = \frac{4\sin(10.9^\circ)}{\sin(20^\circ)} \approx 2.21$$

Two triangles: $\alpha_1 \approx 129.1^\circ$, $\gamma_1 \approx 30.9^\circ$, $a_1 \approx 9.08$
or $\alpha_2 \approx 10.9^\circ$, $\gamma_2 \approx 149.1^\circ$, $a_2 \approx 2.21$

24. $a = 3$, $b = 7$, $\alpha = 70^\circ$

$$\frac{\sin\beta}{b} = \frac{\sin\alpha}{a} \rightarrow \frac{\sin\beta}{7} = \frac{\sin(70^\circ)}{3} \rightarrow \sin\beta = \frac{7\sin(70^\circ)}{3} \approx 2.1926$$

There is no angle β for which $\sin\beta > 1$. Therefore, there is no triangle with the given measurements.

25. $a = 2$, $c = 1$, $\gamma = 100^\circ$

$$\frac{\sin\gamma}{c} = \frac{\sin\alpha}{a} \rightarrow \frac{\sin(100^\circ)}{1} = \frac{\sin\alpha}{2} \rightarrow \sin\alpha = \frac{2\sin(100^\circ)}{1} \approx 1.9696$$

There is no angle α for which $\sin\alpha > 1$. Therefore, there is no triangle with the given measurements.

26. $b = 4$, $c = 5$, $\beta = 95^\circ$

$$\frac{\sin\gamma}{c} = \frac{\sin\beta}{b} \rightarrow \frac{\sin\gamma}{5} = \frac{\sin(95^\circ)}{4} \rightarrow \sin\gamma = \frac{5\sin(95^\circ)}{4} \approx 1.2452$$

There is no angle γ for which $\sin\gamma > 1$. Therefore, there is no triangle with the given measurements.

27. $a = 2$, $c = 1$, $\gamma = 25^\circ$

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin\alpha}{2} = \frac{\sin(25^\circ)}{1} \rightarrow \sin\alpha = \frac{2\sin(25^\circ)}{1} \approx 0.8452$$

$$\rightarrow \alpha_1 = 57.7^\circ \text{ or } \alpha_2 = 122.3^\circ$$

For both values, $\alpha + \gamma < 180^\circ$. Therefore, there are two triangles.

$\beta_1 = 180^\circ - \alpha_1 - \gamma = 180^\circ - 57.7^\circ - 25^\circ = 97.3^\circ$

$$\frac{\sin\beta_1}{b_1} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(97.3^\circ)}{b_1} = \frac{\sin(25^\circ)}{1} \rightarrow b_1 = \frac{1\sin(97.3^\circ)}{\sin(25^\circ)} \approx 2.35$$

$\beta_2 = 180^\circ - \alpha_2 - \gamma = 180^\circ - 122.3^\circ - 25^\circ = 32.7^\circ$

$$\frac{\sin\beta_2}{b_2} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(32.7^\circ)}{b_2} = \frac{\sin(25^\circ)}{1} \rightarrow b_2 = \frac{1\sin(32.7^\circ)}{\sin(25^\circ)} \approx 1.28$$

Two triangles: $\alpha_1 \approx 57.7^\circ$, $\beta_1 \approx 97.3^\circ$, $b_1 \approx 2.35$
or $\alpha_2 \approx 122.3^\circ$, $\beta_2 \approx 32.7^\circ$, $b_2 \approx 1.28$

28. $b = 4,\ c = 5,\ \beta = 40^\circ$

$$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \rightarrow \frac{\sin(40^\circ)}{4} = \frac{\sin\gamma}{5} \rightarrow \sin\gamma = \frac{5\sin(40^\circ)}{4} \approx 0.8035$$

$$\rightarrow \gamma_1 = 53.5^\circ \text{ or } \gamma_2 = 126.5^\circ$$

For both values, $\beta + \gamma < 180^\circ$. Therefore, there are two triangles.

$$\alpha_1 = 180^\circ - \beta - \gamma_1 = 180^\circ - 40^\circ - 53.5^\circ = 86.5^\circ$$

$$\frac{\sin\beta}{b} = \frac{\sin\alpha_1}{a_1} \rightarrow \frac{\sin(40^\circ)}{4} = \frac{\sin(86.5^\circ)}{a_1} \rightarrow a_1 = \frac{4\sin(86.5^\circ)}{\sin(40^\circ)} \approx 6.21$$

$$\alpha_2 = 180^\circ - \beta - \gamma_2 = 180^\circ - 40^\circ - 126.5^\circ = 13.5^\circ$$

$$\frac{\sin\beta}{b} = \frac{\sin\alpha_2}{a_2} \rightarrow \frac{\sin(40^\circ)}{4} = \frac{\sin(13.5^\circ)}{a_2} \rightarrow a_2 = \frac{4\sin(13.5^\circ)}{\sin(40^\circ)} \approx 1.45$$

Two triangles: $\alpha_1 \approx 86.5^\circ,\ \gamma_1 \approx 53.5^\circ,\ a_1 \approx 6.21$

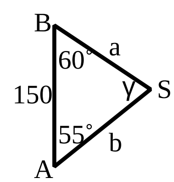
or $\alpha_2 \approx 13.5^\circ,\ \gamma_2 \approx 126.5^\circ,\ a_2 \approx 1.45$

29. (a) Find γ; then use the Law of Sines:

$$\gamma = 180^\circ - 60^\circ - 55^\circ = 65^\circ$$

$$\frac{\sin(55^\circ)}{a} = \frac{\sin(65^\circ)}{150} \rightarrow a = \frac{150\sin(55^\circ)}{\sin(65^\circ)} \approx 135.6 \text{ miles}$$

$$\frac{\sin(60^\circ)}{b} = \frac{\sin(65^\circ)}{150} \rightarrow b = \frac{150\sin(60^\circ)}{\sin(65^\circ)} \approx 143.3 \text{ miles}$$

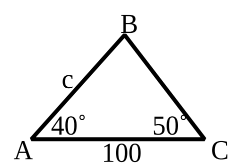


(b) $t = \dfrac{a}{r} = \dfrac{135.6}{200} \approx 0.68$ hours or ≈ 41 minutes

30. Find β; then use the Law of Sines:

$$\beta = 180^\circ - 40^\circ - 50^\circ = 90^\circ$$

$$\frac{\sin 50^\circ}{c} = \frac{\sin(90^\circ)}{100} \rightarrow c = \frac{100\sin(50^\circ)}{\sin(90^\circ)} \approx 76.6 \text{ feet}$$



31. $CAB = 180^\circ - 25^\circ = 155^\circ \qquad ABC = 180^\circ - 155^\circ - 15^\circ = 10^\circ$

Let c represent the distance from A to B.

$$\frac{\sin(15^\circ)}{c} = \frac{\sin(10^\circ)}{1000} \rightarrow c = \frac{1000\sin(15^\circ)}{\sin(10^\circ)} \approx 1490.5 \text{ feet}$$

The length of the proposed ski lift is approximately 1490 feet.

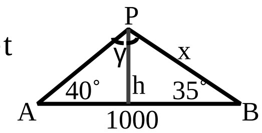
32. Use the results of Problem 31 that the distance from A to B is 1490 feet.
Let h represent the distance from B to D.

$$\sin(25^\circ) = \frac{h}{1490} \rightarrow h = 1490\sin(25^\circ) \approx 1490(0.4226) \approx 630 \text{ feet}$$

33. Find the distance from B to the plane:

$$\gamma = 180^\circ - 40^\circ - 35^\circ = 105^\circ \qquad (\gamma = \quad APB)$$

$$\frac{\sin(40^\circ)}{x} = \frac{\sin(105^\circ)}{1000} \quad \to \quad x = \frac{1000\sin(40^\circ)}{\sin(105^\circ)} \approx 665.5 \text{ feet}$$



Find the height:

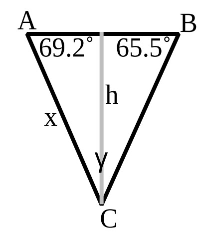
$$\sin(35^\circ) = \frac{h}{x} = \frac{h}{665.5} \quad \to \quad h = (665.5)\sin(35^\circ) \approx 381.7 \text{ fee}$$

The plane is 381.7 feet high.

34. Find the distance from C to the bridge:

$$\gamma = 180^\circ - 69.2^\circ - 65.5^\circ = 45.3^\circ \qquad (\gamma = \quad ACB)$$

$$\frac{\sin(65.5^\circ)}{x} = \frac{\sin(45.3^\circ)}{880} \quad \to \quad x = \frac{880\sin(65.5^\circ)}{\sin(45.3^\circ)} \approx 1127 \text{ feet}$$



Find the height :

$$\sin(69.2^\circ) = \frac{h}{x} = \frac{h}{1127} \quad \to \quad h = (1127)\sin(69.2^\circ) \approx 1054 \text{ feet}$$

The bridge is 1054 feet high.

35. (a) $ABC = 180^\circ - 40^\circ = 140^\circ$

Find the angle at city C:

$$\frac{\sin C}{150} = \frac{\sin(140^\circ)}{300} \quad \to \quad \sin C = \frac{150\sin(140^\circ)}{300} \approx 0.3214 \quad \to \quad C \approx 18.7^\circ$$

Find the angle at city A:

$$A = 180^\circ - 140^\circ - 18.7^\circ = 21.3^\circ$$

$$\frac{\sin(21.3^\circ)}{y} = \frac{\sin(140^\circ)}{300} \quad \to \quad y = \frac{300\sin(21.3^\circ)}{\sin(140^\circ)} \approx 170 \text{ miles}$$

The distance from city B to city C is approximately 170 miles.

(b) To find the angle to turn, subtract angle C from 180˚:

$$180^\circ - 18.7^\circ = 161.3^\circ$$

The pilot needs to turn through an angle of 161.3˚ to return to city A.

36. The time of the actual trip was:

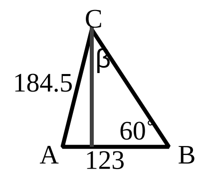
$$t = \frac{50 + 70}{250} = \frac{120}{250} = 0.48 \text{ hour}$$

$a = 70,\ b = 50,\ \alpha = 10^\circ$ Solve the triangle:

$$\frac{\sin(10^\circ)}{70}=\frac{\sin\beta}{50}\rightarrow\sin\beta=\frac{50\sin(10^\circ)}{70}\approx 0.1240$$
$$\beta\approx 7.1^\circ$$
$$\gamma=180^\circ-10^\circ-7.1^\circ=162.9^\circ$$
$$\frac{\sin(10^\circ)}{70}=\frac{\sin(162.9^\circ)}{c}\rightarrow c=\frac{70\sin(162.9^\circ)}{\sin(10^\circ)}\approx 118.5$$
$$t=\frac{118.5}{250}\approx 0.474\text{ hour}$$

The trip should have taken 0.474 hour but because of the incorrect course took 0.48 hour. Thus the trip took 0.006 hour or 0.36 minutes longer.

37. Find angle β (ACB):



$$\frac{\sin\beta}{123}=\frac{\sin(60^\circ)}{184.5}\quad\rightarrow\quad\sin\beta=\frac{123\sin(60^\circ)}{184.5}\approx 0.5774$$
$$\beta\approx 35.3^\circ$$
$$CAB=180^\circ-60^\circ-35.3^\circ\approx 84.7^\circ$$

Find the perpendicular distance:

$$\sin(84.7^\circ)=\frac{h}{184.5}\rightarrow h=184.5\sin 84.7^\circ\approx 183.7\text{ feet}$$

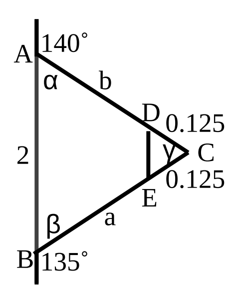
38. Let θ be AOP.

$$\frac{\sin\theta}{9}=\frac{\sin(15^\circ)}{3}\rightarrow\sin\theta=\frac{9\sin(15^\circ)}{3}\approx 0.7765$$
$$\theta\approx 51^\circ\text{ or }\theta\approx 180^\circ-51^\circ=129^\circ$$
$$A=114^\circ\quad\text{or } A=36^\circ$$
$$\frac{\sin(114^\circ)}{a}=\frac{\sin(15^\circ)}{3}\qquad\qquad\frac{\sin(36^\circ)}{a}=\frac{\sin(15^\circ)}{3}$$
$$a=\frac{3\sin(114^\circ)}{\sin(15^\circ)}\approx 10.6\text{ inches}\quad\text{or}\quad a=\frac{3\sin(36^\circ)}{\sin(15^\circ)}\approx 6.8\text{ inches}$$

The distance from the piston to the center of the crankshaft is either 6.8 inches or 10.6 inches.

39. $\alpha=180^\circ-140^\circ=40^\circ\qquad\beta=180^\circ-135^\circ=45^\circ$

$\gamma=180^\circ-40^\circ-45^\circ=95^\circ$



$$\frac{\sin(40^\circ)}{a}=\frac{\sin(95^\circ)}{2}\quad\rightarrow\quad a=\frac{2\sin(40^\circ)}{\sin 95^\circ}\approx 1.290\text{ mi}$$
$$\frac{\sin(45^\circ)}{b}=\frac{\sin(95^\circ)}{2}\quad\rightarrow\quad b=\frac{2\sin(45^\circ)}{\sin(95^\circ)}\approx 1.420\text{ mi}$$
$$\overline{BE}=1.290-0.125=1.165\text{ mi}$$
$$\overline{AD}=1.420-0.125=1.295\text{ mi}$$

$$CDE = \quad CED = \frac{180^\circ - 95^\circ}{2} = 42.5^\circ$$

For the isosceles triangle, $\dfrac{\sin(95^\circ)}{DE} = \dfrac{\sin(42.5^\circ)}{0.125} \quad \rightarrow \quad DE = \dfrac{0.125\sin(95^\circ)}{\sin(42.5^\circ)} \approx 0.184$ miles

The length of the highway is 1.165 + 1.295 + 0.184 = 2.644 miles.

40. From the diagram, $ABC = 55^\circ$ and $BAC = 75^\circ$ (where point C is the ship)

(a) Use the Law of Sines:

$$\frac{\sin(50^\circ)}{3} = \frac{\sin(55^\circ)}{x} \rightarrow x = \frac{3\sin(55^\circ)}{\sin(50^\circ)} \approx 3.2 \text{ miles}$$

(x is the distance from the ship to lighthouse A.)

(b) Use the Law of Sines:

$$\frac{\sin(50^\circ)}{3} = \frac{\sin(75^\circ)}{y} \rightarrow y = \frac{3\sin(75^\circ)}{\sin(50^\circ)} \approx 3.8 \text{ miles}$$

(y is the distance from the ship to lighthouse B.)

(c) Use the Law of Sines:

$$\frac{\sin(90^\circ)}{3.2} = \frac{\sin(75^\circ)}{z} \rightarrow z = \frac{3.2\sin(75^\circ)}{\sin(90^\circ)} \approx 3.1 \text{ miles}$$

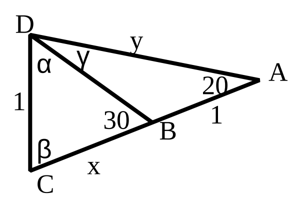
(z is the distance from the ship to the shore.)

41. $ABD = 180^\circ - 30^\circ = 150^\circ \qquad \gamma = 180^\circ - 150^\circ - 20^\circ = 10^\circ$

$$\frac{\sin(150^\circ)}{y} = \frac{\sin(10^\circ)}{1} \quad \rightarrow \quad y = \frac{1\sin(150^\circ)}{\sin(10^\circ)} \approx 2.88 \text{ mi}$$

$$\frac{\sin\beta}{2.88} = \frac{\sin(20^\circ)}{1} \quad \rightarrow \quad \sin\beta = \frac{2.88\sin(20^\circ)}{1} \approx 0.9850$$

$$\beta \approx 80^\circ$$

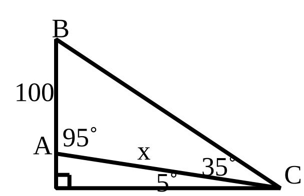


$\alpha = 180^\circ - 80^\circ - 30^\circ = 70^\circ$

$$\frac{\sin 70^\circ}{x} = \frac{\sin 30^\circ}{1} \quad \rightarrow \quad x = \frac{\sin 70^\circ}{\sin 30^\circ} \approx 1.88 \text{ mi}$$

The ship is about 1.88 miles from the harbor.

42. The tower forms an angle of 95˚ with the ground. Let x be the distance from the ranger to the tower.



$ABC = 180^\circ - 95^\circ - 35^\circ = 50^\circ$

$$\frac{\sin(35^\circ)}{100} = \frac{\sin(50^\circ)}{x} \rightarrow x = \frac{100\sin(50^\circ)}{\sin(35^\circ)} \approx 133.6 \text{ feet}$$

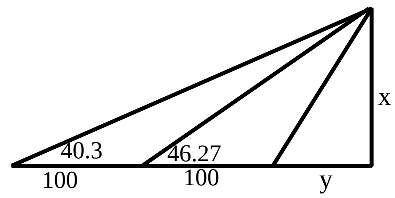
43. Using the Law of Sines:

$$\frac{\sin(46.27^\circ)}{x} = \frac{\sin(90^\circ - 46.27^\circ)}{y+100}$$

$$(y+100)\sin(46.27^\circ) = x\sin(43.73^\circ)$$

$$y\sin(46.27^\circ) + 100\sin(46.27^\circ) = x\sin(43.73^\circ)$$

$$y = \frac{x\sin(43.73^\circ) - 100\sin(46.27^\circ)}{\sin(46.27^\circ)}$$



$$\frac{\sin(40.3^\circ)}{x} = \frac{\sin(90^\circ - 40.3^\circ)}{y+200}$$

$$(y+200)\sin(40.3^\circ) = x\sin(49.7^\circ)$$

$$y\sin(40.3^\circ) + 200\sin(40.3^\circ) = x\sin(49.7^\circ)$$

$$y = \frac{x\sin(49.7^\circ) - 200\sin(40.3^\circ)}{\sin(40.3^\circ)}$$

Set the two equations equal to each other and solve:

$$\frac{x\sin(43.73^\circ) - 100\sin(46.27^\circ)}{\sin(46.27^\circ)} = \frac{x\sin(49.7^\circ) - 200\sin(40.3^\circ)}{\sin(40.3^\circ)}$$

$$x\sin(43.73^\circ)\ \sin(40.3^\circ) - 100\sin(46.27^\circ)\ \sin(40.3^\circ)$$
$$= x\sin(49.7^\circ)\ \sin(46.27^\circ) - 200\sin(40.3^\circ)\ \sin(46.27^\circ)$$

$$x\sin(43.73^\circ)\ \sin(40.3^\circ) - x\sin(49.7^\circ)\ \sin(46.27^\circ)$$
$$= 100\sin(46.27^\circ)\ \sin(40.3^\circ) - 200\sin(40.3^\circ)\ \sin(46.27^\circ)$$

$$x = \frac{100\sin(46.27^\circ)\ \sin(40.3^\circ) - 200\sin(40.3^\circ)\ \sin(46.27^\circ)}{\sin(43.73^\circ)\ \sin(40.3^\circ) - \sin(49.7^\circ)\ \sin(46.27^\circ)} \approx 449.36 \text{ feet}$$

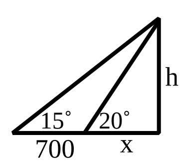
44. Use the Law of Sines:

$$\frac{\sin(20^\circ)}{h} = \frac{\sin(70^\circ)}{x} \quad \rightarrow \quad x = \frac{h\sin(70^\circ)}{\sin(20^\circ)}$$

$$\frac{\sin(15^\circ)}{h} = \frac{\sin(75^\circ)}{x+700} \quad \rightarrow \quad x = \frac{h\sin(75^\circ)}{\sin(15^\circ)} - 700$$

$$\frac{h\sin(70^\circ)}{\sin(20^\circ)} = \frac{h\sin(75^\circ)}{\sin(15^\circ)} - 700$$

$$h\left(\frac{\sin(70^\circ)}{\sin(20^\circ)} - \frac{\sin(75^\circ)}{\sin(15^\circ)}\right) = -700 \rightarrow h = \frac{-700}{\left(\frac{\sin(70^\circ)}{\sin(20^\circ)} - \frac{\sin(75^\circ)}{\sin(15^\circ)}\right)} \approx 711 \text{ feet}$$

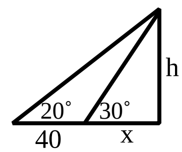


45. Using the Law of Sines:

$$\frac{\sin(30^\circ)}{h} = \frac{\sin(60^\circ)}{x} \quad \rightarrow \quad x = \frac{h\sin(60^\circ)}{\sin(30^\circ)}$$

$$\frac{\sin(20^\circ)}{h} = \frac{\sin(70^\circ)}{x+40} \quad \rightarrow \quad x = \frac{h\sin(70^\circ)}{\sin(20^\circ)} - 40$$

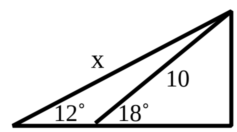
$$\frac{h\sin(60^\circ)}{\sin(30^\circ)} = \frac{h\sin(70^\circ)}{\sin(20^\circ)} - 40$$



$$h\left(\frac{\sin(60^\circ)}{\sin(30^\circ)} - \frac{\sin(70^\circ)}{\sin(20^\circ)}\right) = -40 \rightarrow h = \frac{-40}{\left(\frac{\sin(60^\circ)}{\sin(30^\circ)} - \frac{\sin(70^\circ)}{\sin(20^\circ)}\right)} \approx 39.4 \text{ feet}$$

46. Using the Law of Sines:

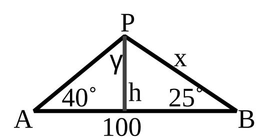
$$\frac{\sin(162^\circ)}{x} = \frac{\sin(12^\circ)}{10} \rightarrow x = \frac{10\sin(162^\circ)}{\sin(12^\circ)} \approx 14.9 \text{ feet}$$



47. Find the distance from B to the helicopter:

$$\gamma = 180^\circ - 40^\circ - 25^\circ = 115^\circ \qquad (\gamma = APB)$$

$$\frac{\sin(40^\circ)}{x} = \frac{\sin(115^\circ)}{100} \quad \rightarrow \quad x = \frac{100\sin(40^\circ)}{\sin(115^\circ)} \approx 70.9 \text{ feet}$$



Find the height:

$$\sin(25^\circ) = \frac{h}{x} \approx \frac{h}{70.9} \quad \rightarrow \quad h \approx 70.9(\sin 25^\circ) \approx 30 \text{ feet}$$

The helicopter is about 30 feet high.

48. $$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} = \frac{\sin\alpha}{\sin\gamma} + \frac{\sin\beta}{\sin\gamma} = \frac{\sin\alpha + \sin\beta}{\sin\gamma} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right)}$$

$$= \frac{\sin\left(\frac{\pi}{2} - \frac{\gamma}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right)} = \frac{\cos\left(\frac{\gamma}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right)} = \frac{\cos\left(\frac{1}{2}(\alpha-\beta)\right)}{\sin\left(\frac{1}{2}\gamma\right)}$$

49. $$\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c} = \frac{\sin\alpha}{\sin\gamma} - \frac{\sin\beta}{\sin\gamma} = \frac{\sin\alpha - \sin\beta}{\sin\gamma} = \frac{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(2\,\frac{\gamma}{2}\right)}$$

$$= \frac{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}{2\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right)} = \frac{\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\pi}{2}-\frac{\gamma}{2}\right)}{\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right)} = \frac{\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)}{\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right)}$$

$$= \frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\gamma}{2}\right)} = \frac{\sin\left(\frac{1}{2}(\alpha-\beta)\right)}{\cos\left(\frac{1}{2}\gamma\right)}$$

50. $$a = \frac{b\sin\alpha}{\sin\beta} = \frac{b\sin\left(180^\circ - (\beta+\gamma)\right)}{\sin\beta} = \frac{b}{\sin\beta}\sin(\beta+\gamma)$$
$$= \frac{b}{\sin\beta}\left(\sin\beta\cos\gamma + \cos\beta\sin\gamma\right) = b\cos\gamma + \frac{b\sin\gamma}{\sin\beta}\cos\beta = b\cos\gamma + c\cos\beta$$

51. Derive the Law of Tangents:

$$\frac{a-b}{a+b} = \frac{\left(\frac{a-b}{c}\right)}{\left(\frac{a+b}{c}\right)} = \frac{\left(\frac{\sin\left(\frac{1}{2}(\alpha-\beta)\right)}{\cos\left(\frac{1}{2}\gamma\right)}\right)}{\left(\frac{\cos\left(\frac{1}{2}(\alpha-\beta)\right)}{\sin\left(\frac{1}{2}\gamma\right)}\right)} = \frac{\sin\left(\frac{1}{2}(\alpha-\beta)\right)}{\cos\left(\frac{1}{2}\gamma\right)}\cdot\frac{\sin\left(\frac{1}{2}\gamma\right)}{\cos\left(\frac{1}{2}(\alpha-\beta)\right)} = \tan\left(\frac{1}{2}(\alpha-\beta)\right)\tan\left(\frac{1}{2}\gamma\right)$$

$$= \tan\left(\frac{1}{2}(\alpha-\beta)\right)\tan\left(\frac{1}{2}\left(\pi-(\alpha+\beta)\right)\right) = \tan\left(\frac{1}{2}(\alpha-\beta)\right)\tan\left(\frac{\pi}{2}-\left(\frac{\alpha+\beta}{2}\right)\right)$$

$$= \tan\left(\frac{1}{2}(\alpha-\beta)\right)\cot\left(\frac{\alpha+\beta}{2}\right) = \frac{\tan\left(\frac{1}{2}(\alpha-\beta)\right)}{\tan\left(\frac{1}{2}(\alpha+\beta)\right)}$$

52. $\sin\beta = \sin(\ ABC) = \sin(\ AB\ C) = \dfrac{b}{2r}$

$\dfrac{\sin\beta}{b} = \dfrac{1}{2r}$

The result follows from the Law of Sines.