Chapter 9

Applications of Trigonometric Functions

9.3 The Law of Cosines

1. $a = 2,\ \ c = 4,\ \ \beta = 45^{\circ} \qquad b^2 = a^2 + c^2 - 2ac\cos\beta$

$$b^2 = 2^2 + 4^2 - 2\ \ 2\ \ 4\cos\left(45^{\circ}\right) = 20 - 16\ \frac{\sqrt{2}}{2} = 20 - 8\sqrt{2} \approx 8.6863$$

$$b \approx 2.95$$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad 2bc\cos\alpha = b^2 + c^2 - a^2 \quad \rightarrow \quad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\alpha = \frac{2.95^2 + 4^2 - 2^2}{2(2.95)(4)} = \frac{20.6863}{23.6} \approx 0.8765 \rightarrow \alpha \approx 28.8^{\circ}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\gamma = \frac{2^2 + 2.95^2 - 4^2}{2(2)(2.95)} = \frac{-3.2975}{11.8} \approx -0.2794 \rightarrow \gamma = 106.2^{\circ}$$

2. $b = 3,\ \ c = 4,\ \ \alpha = 30^{\circ} \qquad a^2 = b^2 + c^2 - 2bc\cos\alpha$

$$a^2 = 3^2 + 4^2 - 2\ \ 3\ \ 4\cos\left(30^{\circ}\right) = 25 - 24\ \ (0.8660) \approx 4.216$$

$$a \approx 2.05$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\gamma = \frac{2.05^2 + 3^2 - 4^2}{2(2.05)(3)} \approx -0.2274 \rightarrow \gamma \approx 103.1^{\circ}$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos\beta = \frac{2.05^2 + 4^2 - 3^2}{2(2.05)(4)} \approx 0.6831 \rightarrow \beta \approx 46.9^{\circ}$$

3. $a = 2,\ b = 3,\ \gamma = 95^\circ \qquad c^2 = a^2 + b^2 - 2ab\cos\gamma$

$c^2 = 2^2 + 3^2 - 2\ 2\ 3\cos(95^\circ) = 13 - 12\ (-0.0872) \approx 14.0459$

$c \approx 3.75$

$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos\alpha = \dfrac{3^2 + 3.75^2 - 2^2}{2(3)(3.75)} \approx 0.8472 \rightarrow \alpha \approx 32.1^\circ$

$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\cos\beta = \dfrac{2^2 + 3.75^2 - 3^2}{2(2)(3.75)} \approx 0.6042 \rightarrow \beta = 52.8^\circ$

4. $a = 2,\ c = 5,\ \beta = 20^\circ \qquad b^2 = a^2 + c^2 - 2ac\cos\beta$

$b^2 = 2^2 + 5^2 - 2\ 2\ 5\cos(20^\circ) = 29 - 20(0.9397) \approx 10.206$

$b \approx 3.19$

$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad 2bc\cos\alpha = b^2 + c^2 - a^2 \quad \rightarrow \quad \cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos\alpha = \dfrac{3.19^2 + 5^2 - 2^2}{2(3.19)(5)} \approx 0.9773 \rightarrow \alpha \approx 12.2^\circ$

$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\cos\gamma = \dfrac{2^2 + 3.19^2 - 5^2}{2(2)(3.19)} \approx -0.8483 \rightarrow \gamma = 148.0^\circ$

5. $a = 6,\ b = 5,\ c = 8$

$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos\alpha = \dfrac{5^2 + 8^2 - 6^2}{2(5)(8)} \approx 0.6625 \rightarrow \alpha \approx 48.5^\circ$

$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\cos\beta = \dfrac{6^2 + 8^2 - 5^2}{2(6)(8)} \approx 0.7813 \rightarrow \beta = 38.6^\circ$

$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\cos\gamma = \dfrac{6^2 + 5^2 - 8^2}{2(6)(5)} \approx -0.0500 \rightarrow \gamma = 92.9^\circ$

6. $a = 8,\ b = 5,\ c = 4$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\alpha = \frac{5^2 + 4^2 - 8^2}{2(5)(4)} \approx -0.5750 \rightarrow \alpha \approx 125.1^\circ$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos\beta = \frac{8^2 + 4^2 - 5^2}{2(8)(4)} \approx 0.8594 \rightarrow \beta \approx 30.8^\circ$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\gamma = \frac{8^2 + 5^2 - 4^2}{2(8)(5)} \approx 0.9125 \rightarrow \gamma = 24.1^\circ$$

7. $a = 9, \ b = 6, \ c = 4$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\alpha = \frac{6^2 + 4^2 - 9^2}{2(6)(4)} \approx -0.6042 \rightarrow \alpha \approx 127.2^\circ$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos\beta = \frac{9^2 + 4^2 - 6^2}{2(9)(4)} \approx 0.8472 \rightarrow \beta = 32.1^\circ$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\gamma = \frac{9^2 + 6^2 - 4^2}{2(9)(6)} \approx 0.9352 \rightarrow \gamma = 20.7^\circ$$

8. $a = 4, \ b = 3, \ c = 4$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\alpha = \frac{3^2 + 4^2 - 4^2}{2(3)(4)} \approx 0.3750 \rightarrow \alpha \approx 68.0^\circ$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos\beta = \frac{4^2 + 4^2 - 3^2}{2(4)(4)} \approx 0.7188 \rightarrow \beta = 44.0^\circ$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\gamma = \frac{4^2 + 3^2 - 4^2}{2(4)(3)} \approx 0.3750 \rightarrow \gamma = 68.0^\circ$$

9. $a = 3, \; b = 4, \; \gamma = 40°$

$c^2 = a^2 + b^2 - 2ab\cos\gamma$

$c^2 = 3^2 + 4^2 - 2 \cdot 3 \cdot 4\cos(40°) \approx 6.6149$

$c \approx 2.57$

$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos\alpha = \dfrac{4^2 + 2.57^2 - 3^2}{2(4)(2.57)} \approx 0.6617 \rightarrow \alpha \approx 48.6°$

$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\cos\beta = \dfrac{3^2 + 2.57^2 - 4^2}{2(3)(2.57)} \approx -0.0256 \rightarrow \beta = 91.5°$

10. $a = 2, \; c = 1, \; \beta = 10°$

$b^2 = a^2 + c^2 - 2ac\cos\beta$

$b^2 = 2^2 + 1^2 - 2 \cdot 2 \cdot 1\cos(10°) \approx 1.0608$

$b \approx 1.03$

$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos\alpha = \dfrac{1.03^2 + 1^2 - 2^2}{2(1.03)(1)} \approx -0.9413 \rightarrow \alpha \approx 160.3°$

$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\cos\gamma = \dfrac{2^2 + 1.03^2 - 1^2}{2(2)(1.03)} \approx 0.9857 \rightarrow \gamma = 9.7°$

11. $b = 1, \; c = 3, \; \alpha = 80°$

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

$a^2 = 1^2 + 3^2 - 2 \cdot 1 \cdot 3\cos(80°) \approx 8.9581$

$a \approx 2.99$

$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\cos\gamma = \dfrac{2.99^2 + 1^2 - 3^2}{2(2.99)(1)} \approx 0.1572 \rightarrow \gamma \approx 81.0°$

$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\cos\beta = \dfrac{2.99^2 + 3^2 - 1^2}{2(2.99)(3)} \approx 0.9443 \rightarrow \beta = 19.2°$

12. $a = 6, \; b = 4, \; \gamma = 60°$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$
$$c^2 = 6^2 + 4^2 - 2 \cdot 6 \cdot 4\cos(60^\circ) = 28$$
$$c \approx 5.29$$
$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \to \quad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos\alpha = \frac{4^2 + 5.29^2 - 6^2}{2(4)(5.29)} \approx 0.1887 \to \alpha \approx 79.1^\circ$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \to \quad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos\beta = \frac{6^2 + 5.29^2 - 4^2}{2(6)(5.29)} \approx 0.7559 \to \beta = 40.9^\circ$$

13. $a = 3, \ c = 2, \ \beta = 110^\circ$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$b^2 = 3^2 + 2^2 - 2 \cdot 3 \cdot 2\cos(110^\circ) \approx 17.1042$$
$$b \approx 4.14$$
$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \to \quad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos\alpha = \frac{4.14^2 + 2^2 - 3^2}{2(4.14)(2)} \approx 0.7331 \to \alpha \approx 42.9^\circ$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \to \quad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\cos\gamma = \frac{3^2 + 4.14^2 - 2^2}{2(3)(4.14)} \approx 0.8913 \to \gamma = 27.0^\circ$$

14. $b = 4, \ c = 1, \ \alpha = 120^\circ$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$a^2 = 4^2 + 1^2 - 2 \cdot 4 \cdot 1\cos(120^\circ) = 21$$
$$a \approx 4.58$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \to \quad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\cos\gamma = \frac{4.58^2 + 4^2 - 1^2}{2(4.58)(4)} \approx 0.9819 \to \gamma \approx 10.9^\circ$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \to \quad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos\beta = \frac{4.58^2 + 1^2 - 4^2}{2(4.58)(1)} \approx 0.6524 \to \beta = 49.3^\circ$$

15. $a = 2, \ b = 2, \ \gamma = 50^\circ$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$
$$c^2 = 2^2 + 2^2 - 2 \;\; 2 \;\; 2\cos(50^\circ) \approx 2.8577$$
$$c \approx 1.69$$
$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos\alpha = \frac{2^2 + 1.69^2 - 2^2}{2(2)(1.69)} \approx 0.4225 \rightarrow \alpha \approx 65.0^\circ$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos\beta = \frac{2^2 + 1.69^2 - 2^2}{2(2)(1.69)} \approx 0.4225 \rightarrow \beta \approx 65.0^\circ$$

16. $a = 3, \quad c = 2, \quad \beta = 90^\circ$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$b^2 = 3^2 + 2^2 - 2 \;\; 3 \;\; 2\cos(90^\circ) = 13$$
$$b \approx 3.61$$
$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos\alpha = \frac{3.61^2 + 2^2 - 3^2}{2(3.61)(2)} \approx 0.5562 \rightarrow \alpha \approx 56.2^\circ$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\cos\gamma = \frac{3^2 + 3.61^2 - 2^2}{2(3)(3.61)} \approx 0.8325 \rightarrow \gamma \approx 33.6^\circ$$

17. $a = 12, \quad b = 13, \quad c = 5$
$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos\alpha = \frac{13^2 + 5^2 - 12^2}{2(13)(5)} \approx 0.3846 \rightarrow \alpha \approx 67.4^\circ$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos\beta = \frac{12^2 + 5^2 - 13^2}{2(12)(5)} = 0 \rightarrow \beta = 90^\circ$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\cos\gamma = \frac{12^2 + 13^2 - 5^2}{2(12)(13)} \approx 0.9231 \rightarrow \gamma = 22.6^\circ$$

18. $a = 4, \quad b = 5, \quad c = 3$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\alpha = \frac{5^2 + 3^2 - 4^2}{2(5)(3)} = 0.6 \rightarrow \alpha = 53.1^\circ$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos\beta = \frac{4^2 + 3^2 - 5^2}{2(4)(3)} = 0 \rightarrow \beta = 90^\circ$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\gamma = \frac{4^2 + 5^2 - 3^2}{2(4)(5)} = 0.8 \rightarrow \gamma = 36.9^\circ$$

19. $a = 2,\ b = 2,\ c = 2$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\alpha = \frac{2^2 + 2^2 - 2^2}{2(2)(2)} = 0.5 \rightarrow \alpha = 60^\circ$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos\beta = \frac{2^2 + 2^2 - 2^2}{2(2)(2)} = 0.5 \rightarrow \beta = 60^\circ$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\gamma = \frac{2^2 + 2^2 - 2^2}{2(2)(2)} = 0.5 \rightarrow \gamma = 60^\circ$$

20. $a = 3,\ b = 3,\ c = 2$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\alpha = \frac{3^2 + 2^2 - 3^2}{2(3)(2)} \approx 0.3333 \rightarrow \alpha = 70.5^\circ$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos\beta = \frac{3^2 + 2^2 - 3^2}{2(3)(2)} \approx 0.3333 \rightarrow \beta = 70.5^\circ$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\gamma = \frac{3^2 + 3^2 - 2^2}{2(3)(3)} \approx 0.7778 \rightarrow \gamma = 38.9^\circ$$

21. $a = 5, \quad b = 8, \quad c = 9$

$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos\alpha = \dfrac{8^2 + 9^2 - 5^2}{2(8)(9)} \approx 0.8333$

$\alpha \approx 33.6^{\circ}$

$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\cos\beta = \dfrac{5^2 + 9^2 - 8^2}{2\ \ 5\ \ 9} \approx 0.4667 \rightarrow \beta \approx 62.2^{\circ}$

$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\cos\gamma = \dfrac{5^2 + 9^2 - 8^2}{2\ \ 5\ \ 9} \approx 0.1000 \rightarrow \gamma \approx 84.3^{\circ}$

22. $a = 4, \quad b = 3, \quad c = 6$

$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos\alpha = \dfrac{3^2 + 6^2 - 4^2}{2(3)(6)} \approx 0.8056 \rightarrow \alpha = 36.3^{\circ}$

$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\cos\beta = \dfrac{4^2 + 6^2 - 3^2}{2(4)(6)} \approx 0.8958 \rightarrow \beta = 26.4^{\circ}$

$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\cos\gamma = \dfrac{4^2 + 3^2 - 6^2}{2(4)(3)} \approx -0.4583 \rightarrow \gamma = 117.3^{\circ}$

23. $a = 10, \quad b = 8, \quad c = 5$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\alpha = \frac{8^2 + 5^2 - 10^2}{2(8)(5)} \approx -0.1375 \rightarrow \alpha \approx 97.9^{\circ}$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos\beta = \frac{10^2 + 5^2 - 8^2}{2(10)(5)} \approx 0.6100 \rightarrow \beta \approx 52.4^{\circ}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\gamma = \frac{10^2 + 5^2 - 8^2}{2(10)(8)} \approx 0.8688 \rightarrow \gamma \approx 29.7^{\circ}\ ^{\circ}$$

24. $a = 9, \;\; b = 7, \;\; c = 10$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\alpha = \frac{7^2 + 10^2 - 9^2}{2(7)(10)} \approx 0.4857 \rightarrow \alpha \approx 60.9^{\circ}$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \rightarrow \quad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos\beta = \frac{9^2 + 10^2 - 7^2}{2(9)(10)} \approx 0.7333 \rightarrow \beta \approx 42.8^{\circ}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \quad \rightarrow \quad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\gamma = \frac{9^2 + 7^2 - 10^2}{2(9)(7)} \approx 0.2381 \rightarrow \gamma = 76.2^{\circ}$$

25. Find the third side of the triangle using the Law of Cosines:

$a = 50, \;\; b = 70, \;\; \gamma = 70^{\circ}$

$c^2 = a^2 + b^2 - 2ab\cos\gamma$

$c^2 = 50^2 + 70^2 - 2 \;\; 50 \;\; 70\cos(70^{\circ}) \approx 5005.86 \rightarrow c \approx 70.75$

The houses are approximately 70.75 feet apart.

26. (a) The angle inside the triangle at Sarasota is 180˚ – 50˚ = 130˚. Use the Law of Cosines to find the third side:

$a = 150, \;\; b = 100, \;\; \gamma = 130^{\circ}$

$c^2 = a^2 + b^2 - 2ab\cos\gamma$

$c^2 = 150^2 + 100^2 - 2 \;\; 150 \;\; 100\cos(130^{\circ}) \approx 51783.63$

$c \approx 227.6$ miles

(b) Use the Law of Sines to find the angle inside the triangle at Orlando:

$$\frac{\sin\alpha}{150}=\frac{\sin(130^\circ)}{227.6}\rightarrow \sin\alpha=\frac{150\sin(130^\circ)}{227.6}\approx 0.5049\rightarrow \alpha\approx 30.3^\circ$$

Since the angle of the triangle is 30.3˚, the angle through which the pilot must turn is 180˚ – 30.3˚ = 149.7˚.

27. (a) After 15 minutes, the plane would have flown 220(0.25) = 55 miles.
Find the third side of the triangle:

$a=55,\quad b=330,\quad \gamma=10^\circ$

$c^2=a^2+b^2-2ab\cos\gamma$

$c^2=55^2+330^2-2\quad 55\quad 330\cos(10^\circ)\approx 76176.48\rightarrow c\approx 276$

Find the measure of the angle opposite the 330 side:

$$\cos\beta=\frac{a^2+c^2-b^2}{2ac}$$

$$\cos\beta=\frac{55^2+276^2-330^2}{2(55)(276)}\approx -0.9782\rightarrow \beta\approx 168^\circ$$

The pilot should turn through an angle of 180˚ – 168˚ = 12˚.

(b) If the total trip is to be done in 90 minutes, and 15 minutes were used already, then there are 75 minutes or 1.25 hours to complete the trip. The plane must travel 276 miles in 1.25 hours.

$$r=\frac{276}{1.25}=220.8\text{ miles / hour}$$

The pilot must maintain a speed of 220.8 mi/hr to complete the whole trip in 90 minutes.

28. After 10 hours the ship will have traveled 150 nautical miles along its altered course. Use the Law of Cosines to find the distance from Barbados on the new course.

$a=600,\quad b=150,\quad \gamma=20^\circ$

$c^2=a^2+b^2-2ab\cos\gamma$

$c^2=600^2+150^2-2\quad 600\quad 150\cos 20^\circ\approx 213355.33\rightarrow c\approx 461.9$ nautical miles

(a) Use the Law of Cosines to find the angle opposite the side of 600:

$$\cos\alpha=\frac{b^2+c^2-a^2}{2bc}$$

$$\cos\beta=\frac{150^2+461.9^2-600^2}{2(150)(461.9)}\approx -0.8959\rightarrow \beta\approx 153.6^\circ$$

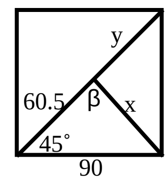
The captain needs to turn the ship through an angle of 180˚ – 153.6˚ = 26.4˚.

(b) $t=\dfrac{461.9}{15}\approx 30.8$ hours

30.8 hours are required for the second leg of the trip. The total time for the trip will be 40.8 hours.

29. (a) Find x in the figure:

$x^2 = 60.5^2 + 90^2 - 2(60.5)90\cos 45° \approx 4059.86$
$x \approx 63.7$ feet
It is about 63.7 feet from the pitching rubber to first base.



(b) Use the Pythagorean Theorem to find y in the figure:

$$90^2 + 90^2 = (60.5 + y)^2 \rightarrow 8100 + 8100 = (60.5 + y)^2$$
$$16200 = (60.5 + y)^2 \rightarrow 60.5 + y \approx 127.3 \rightarrow y \approx 66.8 \text{ feet}$$

It is about 66.8 feet from the pitching rubber to second base.

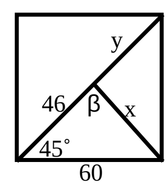
(c) Find β in the figure by using the Law of Cosines:

$$\cos\beta = \frac{60.5^2 + 63.7^2 - 90^2}{2(60.5)(63.7)} \approx -0.0496 \rightarrow \beta \approx 92.8°$$

The pitcher needs to turn through an angle of 92.8° to face first base.

30. (a) Find x in the figure:

$x^2 = 46^2 + 60^2 - 2(46)60\cos 45° \approx 1812.77$
$x \approx 42.6$ feet
It is about 42.6 feet from the pitching rubber to first base.



(b) Use the Pythagorean Theorem to find y in the figure:

$$60^2 + 60^2 = (46 + y)^2 \rightarrow 3600 + 3600 = (46 + y)^2$$
$$7200 = (46 + y)^2 \rightarrow 46 + y \approx 84.9 \rightarrow y \approx 38.9 \text{ feet}$$

It is about 38.9 feet from the pitching rubber to second base.

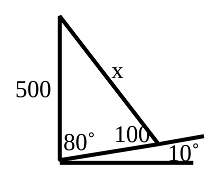
(c) Find β in the figure by using the Law of Cosines:

$$\cos\beta = \frac{46^2 + 42.6^2 - 60^2}{2(46)(42.6)} \approx 0.0844 \rightarrow \beta \approx 85.2°$$

The pitcher needs to turn through an angle of 85.2° to face first base.

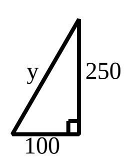
31. (a) Find x by using the Law of Cosines:

$x^2 = 500^2 + 100^2 - 2(500)100\cos 80° \approx 242{,}635$
$x \approx 492.6$ feet
The guy wire needs to be about 492.6 feet long.



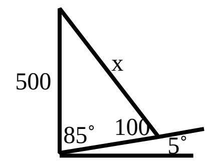
(b) Use the Pythagorean Theorem to find the value of y:

$y^2 = 100^2 + 250^2 = 72500$
$y = 269.3$ feet
The guy wire needs to be about 269.3 feet long.

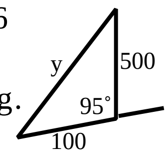


32. Find x by using the Law of Cosines:

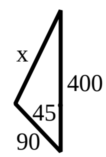
$x^2 = 500^2 + 100^2 - 2(500)100\cos 85° \approx 251{,}284$
$x \approx 501.3$ feet
The guy wire needs to be about 501.3 feet long.



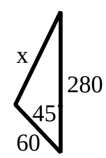
Find y by using the Law of Cosines:
$y^2 = 500^2 + 100^2 - 2(500)100\cos 95° \approx 268{,}716$
$y \approx 518.4$ feet
The guy wire needs to be about 518.4 feet long.



33. Find x by using the Law of Cosines:
$x^2 = 400^2 + 90^2 - 2(400)90\cos 45° \approx 117{,}188.3$
$x \approx 342.3$ feet
It is approximately 342.3 feet from dead center to third base.

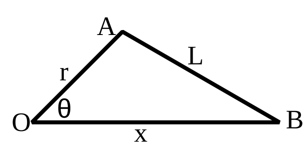


34. Find x by using the Law of Cosines:
$x^2 = 280^2 + 60^2 - 2(280)60\cos 45° \approx 58{,}241.2$
$x \approx 241.3$ feet
It is approximately 241.3 feet from dead center to third base.



35. Use the Law of Cosines:

$$L^2 = x^2 + r^2 - 2xr\cos\theta$$
$$x^2 - 2xr\cos\theta + r^2 - L^2 = 0$$
$$x = \frac{2r\cos\theta + \sqrt{(2r\cos\theta)^2 - 4(1)(r^2 - L^2)}}{2(1)}$$
$$x = \frac{2r\cos\theta + \sqrt{4r^2\cos^2\theta - 4\left(r^2 - L^2\right)}}{2}$$
$$x = r\cos\theta + \sqrt{r^2\cos^2\theta + L^2 - r^2}$$



36. Use the Law of Cosines to find the length of side d:

$$d^2 = r^2 + r^2 - 2\,r\,r\,\cos\theta = 2r^2 - 2r^2\cos\theta = 2r^2(1 - \cos\theta) = 4r^2\left(\frac{1-\cos\theta}{2}\right)$$
$$d = 2r\sqrt{\frac{1-\cos\theta}{2}} = 2r\sin\frac{\theta}{2}$$

Since $d < s$, and $s = r\theta$, we have $2r\sin\frac{\theta}{2} < r\theta$ or $\sin\frac{\theta}{2} < \frac{\theta}{2}$.
Therefore, $\sin\theta < \theta$ for any angle θ.

37. $$\cos\left(\frac{\gamma}{2}\right) = \sqrt{\frac{1+\cos\gamma}{2}} = \sqrt{\frac{\left(1 + \frac{a^2+b^2-c^2}{2ab}\right)}{2}} = \sqrt{\frac{2ab + a^2 + b^2 - c^2}{4ab}} = \sqrt{\frac{(a+b)^2 - c^2}{4ab}}$$

$$= \sqrt{\frac{(a+b+c)(a+b-c)}{4ab}} = \sqrt{\frac{2s(2s-c-c)}{4ab}} = \sqrt{\frac{4s(s-c)}{4ab}} = \sqrt{\frac{s(s-c)}{ab}}$$

38. $$\sin\left(\frac{\gamma}{2}\right) = \sqrt{\frac{1-\cos\gamma}{2}} = \sqrt{\frac{\left(1-\frac{a^2+b^2-c^2}{2ab}\right)}{2}} = \sqrt{\frac{2ab-a^2-b^2+c^2}{4ab}}$$
$$= \sqrt{\frac{-(a^2-2ab+b^2-c^2)}{4ab}} = \sqrt{\frac{-\left((a-b)^2-c^2\right)}{4ab}} = \sqrt{\frac{-(a-b+c)(a-b-c)}{4ab}}$$
$$= \sqrt{\frac{(a-b+c)(b+c-a)}{4ab}} = \sqrt{\frac{(2s-2b)(2s-2a)}{4ab}} = \sqrt{\frac{4(s-b)(s-a)}{4ab}}$$
$$= \sqrt{\frac{(s-a)(s-b)}{ab}}$$

39. $$\frac{\cos\alpha}{a} + \frac{\cos\beta}{b} + \frac{\cos\gamma}{c} = \frac{b^2+c^2-a^2}{2bca} + \frac{a^2+c^2-b^2}{2acb} + \frac{a^2+b^2-c^2}{2abc}$$
$$= \frac{b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2}{2abc} = \frac{a^2+b^2+c^2}{2abc}$$